On the Almost Surely Convergence of the Specific Sequence on $D_{p,q}$ Metric

H. R. Moradi [a,*] and E. Deiri[a]

[a]*Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran.*

Abstract. In this paper, we will discuss the concept of almost sure convergence for specific groups of fuzzy random variables. For this purpose, we use the type of generalized Chebyshev inequalities. Moreover, we show the concept of almost sure convergence of weighted average pairwise NQD of fuzzy random variables.

Index to information contained in this paper

1 **Introduction**
2 **Preliminaries**
3 **Main Result**
4 **Conclusion**

1. Introduction

Zadeh in 1965 introduced the concept of fuzzy set by extending the characteristic function [11]. The occurrence of fuzzy random variable makes the combination of randomness and fuzziness more persuasive, since the probability theory can be used to model uncertainty and the fuzzy sets theory can be used to model imprecision Deiri et al. [3]. Fuzzy random variables are useful in several applied fields. The concept of negative quadrant dependent random variables was introduced by Lehmann [8]. A stronger notion of negative dependent of the theory of multi-variant probability inequality, Dev and Proschan [4], Szeki [10] is that of negative association.

*Corresponding author. Email: hamid_moradi_68@yahoo.com

2. Preliminaries

Definition 2.1 The sequence $\{X_n,\ n \geq 1\}$ is said to converge almost surely to X (and this denoted as $X_n \overset{a.s.}{\to} X$ as $n \to \infty$) if

$$P\left(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) \tag{2.1}$$

Definition 2.2 (Chebyshev inequality)

Let X be a random variable with finite expected value μ and finite non-zero variance σ^2. Then for any real number $k > 0$,

$$Pr\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2} \tag{2.2}$$

Lemma 2.3 (Borel-Cantelli Lemma)

Suppose $A_n \in A$ and $\sum\limits_{n=1}^{\infty} P(A_n) < \infty$. Then,

$$P(A_n \text{ , infinitely often}) = 0 \tag{2.3}$$

In this section, we describe some basic concepts of fuzzy numbers and fuzzy random variables.

Definition 2.4 Let E be a universal set, then a fuzzy set of E is defined by its membership function $\widetilde{A} : E \to [0,1]$, where $\widetilde{A}(x)$ is the membership grade of x to $\widetilde{A}$.

Definition 2.5 $\widetilde{A}_\alpha$ is called the $\alpha - level\ (cut)$ set of $\widetilde{A}$, defined by $\widetilde{A}_\alpha = \left\{x \in E : \widetilde{A}(x) \geq \alpha\right\}$. According to the decomposition theorem of fuzzy sets, we have, $\widetilde{A}(x) = sep\left\{\alpha I_{\widetilde{A}_\alpha} : \alpha \in [0,1]\right\}$. Where $I_{\widetilde{A}_\alpha}$ is the indicator function of ordinary set $\widetilde{A}_\alpha$.

Definition 2.6 A mapping $\widetilde{X} : \Omega \to \Im(\Re)$ is said to be a fuzzy random variable associated with (Ω, x) if and only if

$$\left\{(\Omega, x) : x \in \widetilde{X}_\alpha(\omega)\right\} \in A \times B \ . \tag{2.4}$$

Where B denote $\sigma - field$ of Borel set in $\Re$.

For a fuzzy random variable $\widetilde{X}$ and $\omega \in \Omega$, let $\widetilde{X}(\omega)$ be a fuzzy set with the membership function $\widetilde{X}(\omega)(x)$.

Definition 2.7 Let $D : \Im(\Re) \times \Im(\Re) \to [0,\infty)$

$$D_{p,q}\left(\widetilde{A}, \widetilde{B}\right) = \begin{cases} \left(\int\limits_0^1 \left|q\left(A_\alpha^+ - B_\alpha^+\right) + (1-q)\left(A_\alpha^- - B_\alpha^-\right)\right|^p d\alpha\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sup\limits_{0 \leq \alpha \leq 1} \left|q\left(A_\alpha^+ - B_\alpha^+\right) + (1-q)\left(A_\alpha^- - B_\alpha^-\right)\right| & p = \infty \end{cases} .$$

The analytical properties of $D_{p,q}$ depend on the first parameter p, while the second parameter q of $D_{p,q}$ characterizing the subjective weight attributed to the sides of the fuzzy numbers

Definition 2.8 A pair of fuzzy random variables $\left(\widetilde{X}, \widetilde{Y}\right)$ is called negatively quadrant dependent (NQD) if

$$P\left[\widetilde{X}_\alpha^{(\cdot)} \leq x, \widetilde{Y}_\alpha^{(\cdot)} \leq y\right] \leq P\left[\widetilde{X}_\alpha^{(\cdot)} \leq x\right] P\left[\widetilde{Y}_\alpha^{(\cdot)} \leq y\right] \ \forall \alpha \in [0,1], \forall x, y \in \Re \quad (2.5)$$

Definition 2.9 A set of fuzzy random variables is called NQD for all pairs $\left(\widetilde{X}, \widetilde{Y}\right)$ in the sets $\widetilde{X}$ and $\widetilde{Y}$ satisfied in (2.5).

Definition 2.10 A pair of fuzzy random variables $\left(\widetilde{X}, \widetilde{Y}\right)$ are said to be negatively associated (N A) if

$$Cov\left\{f\left(\widetilde{X}\right)_\alpha^{(\cdot)}, g\left(\widetilde{Y}\right)_\alpha^{(\cdot)}\right\} \leq 0 \ \forall \alpha \in [0,1] \ . \quad (2.6)$$

Theorem 2.1 Let $\{\omega_n\}_{n \geq 1}$ be a non-negative sequence and $W_n = \sum_{i=1}^{n} \omega_i$ such that $W_n \to \infty$, and $\frac{\omega_n}{W_n} \to 0$ when $n \to \infty$.

Let $\left\{\widetilde{X}_n : n \geq 1\right\}$ be a sequence of negatively quadrant dependence of non-negative fuzzy random variables. Put $\widetilde{S}_n = \oplus_{i=1}^{n}\left(\omega_i \odot \widetilde{X}_n\right)$. If

  I. $\sup\limits_{i \geq 1} E\left[D_{2,q}\left(\widetilde{X}_i, \widetilde{0}\right)\right] < \infty$

  II. $\sum\limits_{i=1}^{\infty} \frac{\omega_i^2 D \operatorname{var}(\widetilde{X}_i)}{W_i^2} < \infty$

  Then, as $n \to \infty$ for $1 \leq p \leq 2$ we have,

$$\frac{D_{p,q}\left(\widetilde{S}_n, E\widetilde{S}_n\right)}{W_n} \overset{a.s.}{\to} 0$$

Proof Let $a > 1$ and for each $k \geq 1$ put,

$$n_k = \inf\left\{n : W_n \geq a^k\right\}$$

Since $\frac{W_n}{W_{n+1}} \to 0$ as $n \to \infty$ it follows that $W_n \sim a^k$ for all large k. Therefore, for some $c > 0$ and every $i = 1, 2, \ldots$ by the Theorem 1 in [6],

$$\{k : n_k \geq i\} \subset \{k : W_{n_k} \geq W_i\} \subset \left\{k : ca^k > W_i\right\} \quad (2.7)$$

By Chebyshev inequality and by Theorem 1 in [5],

$$\sum_{k=1}^{\infty} P\left\{\frac{1}{W_{n_k}} D_{2,q}\left(\widetilde{S}_{n_k}, E\widetilde{S}_{n_k}\right) \geq \varepsilon\right\} \leq \sum_{k=1}^{\infty} \sum_{i=1}^{nk} \frac{\omega_i^2 D \operatorname{var}\left(\widetilde{X}_i\right)}{\varepsilon^2 \alpha^{2k}} \leq \sum_{k=1}^{\infty} \frac{\omega_i^2 D \operatorname{var}\left(\widetilde{X}_i\right)}{\varepsilon^2 W_i^2}$$

$$(2.8)$$

For every $\varepsilon > 0$ since $Cov\left(X_{ia}^{(.)}, X_{ja}^{(.)}\right) \leq 0$; $i \neq j$. Thus, by the Borel-Cantelli lemma as $k \to \infty$,

$$\frac{1}{W_{n_k}} D_{2,q}\left(\widetilde{S}_{n_k}, E\widetilde{S}_{n_k}\right) \to 0 \tag{2.9}$$

Now, given positive integers n and a, for $n_k \leq n \leq n_{k+1}$,

$$\frac{1}{w_{n_k}} D_{2,q}\left(\widetilde{S}_{n_k}, E\widetilde{S}_{n_k}\right) \leq \frac{1}{w_{n_{k+1}}} D_{2,q}\left(\widetilde{S}_{n_k}, E\widetilde{S}_{n_{k+1}}\right) \frac{w_{n_{k+1}}}{w_{n_k}} + \frac{1}{w_{n_k}} D_{2,q}\left(\widetilde{S}_{n_{k+1}}, E\widetilde{S}_{n_k}\right) \text{ a.s.}$$

By the monotonicity of $\widetilde{S}_n$ it follows (2.9) and (2.10) and (I) so that for $1 \leq p \leq 2$,

$$\lim \sup_{n \to \infty} \frac{1}{W_n} D_{p,q}\left(\widetilde{S}_n, E\widetilde{S}_n\right) \leq \frac{1}{W_n} D_{2,q}\left(\widetilde{S}_n, E\widetilde{S}_n\right)$$

$$\leq \sup_{i \geq 1} E\left[D_{2,q}\left(\widetilde{X}_i, \widetilde{0}\right)\right](\alpha - 1) \text{ a.s.}$$

For every $a > 1$ wich concludes the proof. ∎

3. Main Result

Theorem 3.1 Let $\{W_n\}_{n \geq 1}$ be a non-negative sequence and $W_n = \sum\limits_{i=1}^{n} \omega_i$ such that $W_n \to \infty$, $\frac{\omega_n}{W_n} \to 0$ when $n \to \infty$.

Let $\left\{\widetilde{X}_n : n \geq 1\right\}$ be a sequence of negatively quadrant dependence of fuzzy random variables. Put $\widetilde{S}_n = \oplus_{i=1}^{n}\left(\omega_i \odot \widetilde{X}_n\right)$. If

I. $\sup\limits_{i \geq 1} E\left[D_{1,q}\left(\widetilde{X}_i, E\widetilde{X}_i\right)\right] < \infty$

II. $\sum\limits_{i=1}^{\infty} \frac{\omega_i^2 D \operatorname{var}\left(\widetilde{X}_i\right)}{W_i^2} < \infty$

Then, as $n \to 0$ for $1 \leq p \leq 2$ we have,

$$\frac{D_{p,q}\left(\widetilde{S}_n, E\widetilde{S}_n\right)}{W_n} \xrightarrow{a.s.} 0$$

Proof It is enough to show that $\frac{D_{2,q}\left(\widetilde{S}_n, E\widetilde{S}_n\right)}{W_n} \xrightarrow{a.s.} 0$. First, note that $\omega_i\widetilde{X}_i : i \leq 1$ is a pairwise sequence of NQD fuzzy random variables $\omega_i \geq 0$. An equivalent condition NQD is

$$Cov\left\{\left(f\left(\widetilde{X}_i\right)\right)_{\alpha}^{(.)}, \left(g\left(\widetilde{X}_i\right)\right)_{\alpha}^{(.)}\right\} \leq 0$$

for all non-decreasing (non-increasing) function such that the covariance exists. Hence the fuzzy random variables $\widetilde{X}_i \Theta_H E\widetilde{X}_i$, $i \geq 1$ are pairwise NQD and for $i \geq 1$ assume $E\widetilde{X}_i = E\widetilde{X}_i^+ \Theta_H E\widetilde{X}_i^- = \widetilde{0}$. We consider the sequence $\left\{\omega_i \odot \widetilde{X}_i^+ : i \geq 1\right\}$ and its corresponding sum $\widetilde{S}_n^* = \oplus_{i=1}^{n}\omega_i \odot \widetilde{X}_i^+$ for real numbers, but clearly it is a non-decreasing function. Thus, according to (2.8) we have

$Cov\left(\omega_i\left(X_{ia}^{(.)}\right)^+, \omega_j\left(X_{ja}^{(.)}\right)^+\right) \leq 0$, i.e, $\left\{\omega_i \odot \widetilde{X}_i^+ : i \geq 1\right\}$ is a non-positively sequence of correlated non-negative fuzzy random variables. Assumptions (I) and (II) together with the Theorem 2.11 imply that as $n \to 0$,

$$\frac{1}{W_n} D_{2,q}\left(\widetilde{S}_n^*, E\widetilde{S}_n^*\right) \overset{a.s.}{\to} 0 \tag{3.1}$$

A similar consideration for negative parts, say $\widetilde{S}_n^{**} = \oplus_{i=1}^n \omega_i \odot \widetilde{X}_i^-$, and the fact that $E\widetilde{S}_n^* \Theta_H E\widetilde{S}_n^{**} = \widetilde{0}$ completes the proof. ∎

Now we consider the almost sure convergence of weighted average for pairwise NQD fuzzy random variables as an application of the Theorem 3.1.

Theorem 3.2 Let $\left\{\widetilde{X}_n : n \geq 1\right\}$ be a sequence of pairwise NQD fuzzy random variables with finite second moments and let $\widetilde{S}_n = \oplus_{i=1}^n \widetilde{X}_i$. Assume
   I. $\sup_{i \geq 1} E\left[D_{1,q}\left(\widetilde{X}_i, E\widetilde{X}_i\right)\right] < \infty$
   II. $\sum_{i=1}^{\infty} \frac{D\,\mathrm{var}(\widetilde{X}_i)}{D_{1,q}(\widetilde{S}_i, \widetilde{0})} < \infty$
   Then for $1 \leq p \leq 2$ we have $D_{p,q}\left(\widetilde{S}_n, E\widetilde{S}_n\right) \overset{a.s.}{\to} 0$ $n \to \infty$.

Proof Let $\widetilde{Y}_n = \frac{1}{E\left[D_{1,q}\left(\widetilde{X}_n, \widetilde{0}\right)\right]} \odot \widetilde{X}_n$ and $\omega_n = E\left[D_{1,q}\left(\widetilde{X}_n, \widetilde{0}\right)\right]$ and use Theorem 2.11, then $\{\widetilde{Y}_n : n \geq 1\}$ is a sequence of pairwise NQD fuzzy random variables. Thus, by Theorem 3.1 we have,

$$\sum_{n=1}^{\infty} \frac{W_n^2}{W_n^2} D\,\mathrm{var}\left(\widetilde{Y}\right) = \sum_{n=1}^{\infty} \frac{E^2\left[D_{1,q}\left(\widetilde{X}_n, \widetilde{0}\right)\right]}{\sum_{i=1}^n E^2\left[D_{1,q}\left(\widetilde{X}_n, \widetilde{0}\right)\right]} \times D\,\mathrm{var}\left(\frac{1}{D_{1,q}^2\left(\widetilde{X}_n, \widetilde{0}\right)} \odot \widetilde{X}_n\right)$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{E^2\left[D_{1,q}\left(\widetilde{S}_n, \widetilde{0}\right)\right]} D\,\mathrm{var}\left(\widetilde{X}_i\right) < \infty$$

∎

Theorem 3.3 Let $\left\{\widetilde{X}_n : n \geq 1\right\}$ be a sequence of pairwise NQD fuzzy random variables with finite second moments and let $\widetilde{S}_n = \oplus_{i=1}^n \widetilde{X}_i$. Assume
   I. $\sup_{i \geq 1} E\left[D_{1,q}\left(\widetilde{X}_i, E\widetilde{X}_i\right)\right] < \infty$
   II. $\sum_{i=1}^{\infty} \frac{D\,\mathrm{var}(\widetilde{X}_i)}{E^2\left[D_{1,q}(\widetilde{S}_i, \widetilde{0})\right]\left[\log D_{1,q}(\widetilde{S}_i, \widetilde{0})\right]^2} < \infty$ $a.s$
   Then for $1 \leq p \leq 2$ as $n \to \infty$ we have

$$\frac{1}{\log E\left[D_{1,q}\left(\widetilde{S}_i, \widetilde{0}\right)\right]} D_{p,q}\left(\sum_{n=1}^n \frac{1}{E\left[D_{1,q}\left(\widetilde{S}_i, \widetilde{0}\right)\right]} \odot \widetilde{X}_i, E\sum_{n=1}^n \frac{1}{E\left[D_{1,q}\left(\widetilde{S}_i, \widetilde{0}\right)\right]} \odot \widetilde{X}_i\right) \tag{3.2}$$

converges to zero almost surely.

Proof Let $\widetilde{Y}_n = \frac{1}{E[D_{1,q}(\widetilde{X}_n,\widetilde{0})]} \odot \widetilde{X}_n$ and $\omega_n = \frac{E[D_{1,q}(\widetilde{X}_n,\widetilde{0})]}{E[D_{1,q}(\widetilde{S}_n,\widetilde{0})]}$. Clearly $\left\{\widetilde{Y}_n : n \geq 1\right\}$ is a sequence of pairwise NQD fuzzy random variables and $W_n \sim \log E\left[D_{1,q}\left(\widetilde{S}_n,\widetilde{0}\right)\right]$ *a.s.* . Hence by Theorem 3.1 part (II) we have,

$$\sum_{i=1}^{\infty} D\operatorname{var}\left(\frac{1}{E\left[D_{1,q}\left(\widetilde{S}_i,\widetilde{0}\right)\right]} \odot \widetilde{X}_i\right)\frac{1}{\left(\log E\left[D_{1,q}\left(\widetilde{S}_i,\widetilde{0}\right)\right]\right)^2}$$

$$=\sum_{i=1}^{\infty}\frac{D\operatorname{var}\left(\omega_i \odot \widetilde{Y}_i\right)}{W_i^2}=\sum_{i=1}^{\infty}\frac{W_i^2}{W_i^2}D\operatorname{var}\left(\widetilde{Y}_i\right)<\infty$$

Let $\widetilde{T}_n = \oplus_{i=1}^{n}\left(\omega_i \odot \widetilde{Y}_i\right)$. Since

$$\widetilde{T}_n = \oplus_{i=1}^{n}\left(\omega_i \odot \widetilde{Y}_i\right) = \oplus_{i=1}^{n}\left(\frac{1}{E\left[D_{1,q}\left(\widetilde{S}_i,\widetilde{0}\right)\right]} \odot \widetilde{X}_i\right)$$

and

$$\widetilde{Y}_n = \frac{1}{E\left[D_{1,q}\left(\widetilde{X}_n,\widetilde{0}\right)\right]} \odot \widetilde{X}_n$$

then by (3.1) and Theorem 3.1 we have $\frac{1}{W_n}\left(D_{2,q}\,\widetilde{T}_n,E\widetilde{T}_n\right) \to 0$ *a.s.* as $n \to \infty$.
∎

4. Conclusion

In this paper, we discussed the concept of almost sure convergence for specific groups of fuzzy random variables. For this purpose, we used the type of generalized Chebyshev inequalities. Also, generalized some results of NQD random variables with real value, toNQDrandom variables with fuzzy value. For this purpose, proved the sequence$\frac{D_{p,q}\left(\widetilde{S}_n,E\widetilde{S}_n\right)}{W_n}$ is convergent almost sure to zero, as$n \to \infty$.

Acknowledgments

References

[1] M. Amirfakhrian and S. Sajadi, *Fuzzy K-Nearest Neighbor Method toClassify Data in a Closed Area*, International Journal of Mathematical Modelling Computations , **3** (2013) 109–114.

[2] T. Birkel, *A Note on the Strong Law of large numbers for positively dependent random variables*, Statist Probab, **7** (1980) 13–20.

[3] E. Deiri and et al, *Extend the Borel-Cantelli Lemma to Sequences of Non-Independent Random Variables*, Applied Mathematical Sciences, **4 (13)** (2010) 637–642.

[4] K. J. Dev and F. Proschan, *negative association of random variables, with applications*, Ann. Statist, (1983) 286–295.

[5] N. Etemadi, *On the Laws of Large numbers of nonnegative random variables*, j. Multi. Anal, **13** (1983) 187–193.

[6] N. Etemadi, *Stability of sums of weighted nonnegative random variables*, j. Multi. Anal, **13** (1983) 361–365.

[7] H. Inoue, *A Strong Law of Large Numbers for fuzzy random variables*,Fuzzy Sets and Systems, **41** (1991) 285–291.

[8] E. L. Lehmann, concept of dependence, Ann. Math. Statist, **37** (1966) 1137–1153.

[9] M. Matinfar and M. Aminzadeh, *An Iterative Method with Six-Order Convergence for Solving Non-linear quations*, International Journal of Mathematical Modelling Computations , **2** (2012) 45–51.

[10] R. Szekli, *Stochastic Ordering and Dependence in Applied Probability:Lecture Notes in Statistics 97*, Springer Verlag, (1995).

[11] L. A. Zadeh, *fuzzy sets*, Inform And control, **8** (1965) 383–353.