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Numerical Results on Finite p-Groups of Exponent p^2

B. Ahmadi and H. Doostie*

Mathematics Department, Science and Research Branch, Islamic Azad University, Tehran P.O. Box14515/1775, Iran.

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Abstract. The Fibonacci lengths of the finite p-groups have been studied by R. Dikici and co-authors since 1992. All of the considered groups are of exponent p, and the lengths depend on the celebrated Wall number $k(p)$. The study of p-groups of nilpotency class 3 and exponent p has been done in 2004 by R. Dikici as well. In this paper we study all of the p-groups of nilpotency class 3 and exponent p^2. This completes the study of Fibonacci length of all p-groups of order p^4, proving that the Fibonacci length is $k(p^2)$.

Keywords: Fibonacci length, p-groups, Nilpotency class 3.

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1. Introduction

Let $s = (s_i)$ be the 2-step Fibonacci sequence of numbers defined by $s_0 = 0$, $s_1 = 1$, $s_i = s_{i-2} + s_{i-1}$, for $i \geqslant 2$. We may extend the sequence backwards to obtain a bi-infinite sequence. The fundamental period or Wall number (see [10]) of this sequence is denoted by $k(s, p^n)$, where the sequence reduced modulo p^n, for a positive integer n and a prime p. Since now on, we denote $k(s, p^n)$ by $k(p^n)$.

A 2-step general Fibonacci sequence in a finite non-abelian 2-generated group $G = \langle a, b\rangle$ is defined by $x_0 = a$, $x_1 = b$, $x_i = x_{i-2}^m x_{i-1}^l$, for $i \geqslant 2$ and the integers m and l. If $m = l = 1$, the least period of this sequence is called the Fibonacci length of G and denoted by $k(G)$. Since 1990, the Fibonacci length has been studied and calculated for certain classes of finite groups. For instance, see [2], [3], [8], and [7].

There are only five classes of p-groups of order p^4 and nilpotency class 3 (see [9]), i.e; the groups

*Corresponding author. Email: doostih@gmail.com

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$$H = \langle a, b, c, d | a^p = b^p = c^p = d^p = 1,\ [a, b] = [a, c] = [a, d] = 1, [b, d] = 1, [c, d] = b\rangle,\ p \neq 3,$$
$$K = \langle a, b, c | a^9 = b^3 = c^3 = 1,\ [a, b] = 1, [a, c] = b, [c, b^{-1}] = a^{-3},$$
$$L_\alpha = \langle a, b, c | a^{p2} = b^p = 1, c^p = a^{\alpha p}, [a, b] = a^p, [a, c] = b, [b, c] = 1\rangle$$

where $\alpha = 0, 1$, or a non-residue modulo p.

The first group is of exponent p and studied by R. Dikici [4]. Other remained groups are of exponent p^2. First of all we attempt to give a power-commutator presentation for the groups (see Johnson [6]) and by investigating their nilpotency class we will go to the computation of Fibonacci length.

Theorem 1.1 Main*Suppose that p is a prime and $p \neq 2$, and let G be a p-group of nilpotency class 3 and of order p^4 which is of exponent p^2. Then $k(G) = k(p^2)$.*

2. The Group *K*

Let $G = K$. Since a^{-3} is a non-identity element of $[G, G']$, it is clear that G has nilpotency class 3. Hence $[G, G'] \leqslant Z(G)$. Therefore a^3 is a central element. The following series is a central series for G such that G_{i-1}/G_i are cyclic of order p:

$$1 = G_4 \leqslant G_3 \leqslant G_2 \leqslant G_1 \leqslant G_0 = G,$$

where

$$G_3 = \langle a^3\rangle, G_2 = \langle a^3, b\rangle, G_1 = \langle a^3, b, c\rangle.$$

Hence a power-commutator presentation of G may be given as follows:

$$G = \langle x, y, z | x^3 = y^3 = z^3 = 1, w^3 = x,\ [x, y] = [x, z] = [x, w] = 1, [z, y] = x, [w, y] = 1, [w, z] = y\rangle$$

Note that in the new presentation, the group G be generated by w and z. Moreover x is a central element. Also, each element of G can be uniquely represented as $x^a y^b z^c w^d$, where a, b, c reduced modulo 3 and d reduced modulo 9. First we give some elementary results.

Lemma 2.1 *For every positive integers m and n,*

(i) $z^m y^n = x^{mn} y^n z^m$.

(ii) $w^m z^n = x^{m\binom{n+1}{2}} y^{mn} z^n w^m$.

Proof *Since x is a central element of G, then (i) may be proved by the induction method. To prove (ii) we may use (i) and the relation $[w, y] = 1$.*

■

Lemma 2.2 *let $x^a y^b z^c w^d$ and $x^{a'} y^{b'} z^{c'} w^{d'}$ be elements of G. Then*

$$(x^a y^b z^c w^d)(x^{a'} y^{b'} z^{c'} w^{d'}) = x^{a+a'+cb'+cdc'+d\binom{c'+1}{2}} y^{b+b'+dc'} z^{c+c'} w^{d+d'}$$

Proof *By using Lemma 2.1, we have:*

$$\begin{aligned}(x^ay^bz^cw^d)(x^{a'}y^{b'}z^{c'}w^{d'}) &= x^{a+a'}y^bz^cw^dy^{b'}z^{c'}w^{d'}\\ &= x^{a+a'}y^bz^cy^{b'}w^dz^{c'}w^{d'}\\ &= x^{a+a'}y^bz^cy^{b'}x^{d\binom{c'+1}{2}}y^{dc'}z^{c'}w^dw^{d'}\\ &= x^{a+a'+d\binom{c'+1}{2}}y^bz^cy^{b'+dc'}z^{c'}w^{d+d'}\\ &= x^{a+a'+d\binom{c'+1}{2}}y^bx^{c(b'+dc')}y^{b'+dc'}z^cz^{c'}w^{d+d'}\\ &= x^{a+a'+d\binom{c'+1}{2}+c(b'+dc')}y^{b+b'+dc'}z^{c+c'}w^{d+d'}.\end{aligned}$$

■

LEMMA 2.3 *Let* $x6ay^bz^cw^d$ *and* $x^{a'}y^{b'}z^{c'}w^{d'}$ *be elements of* G *and* m *and* l *be positive integers. Then*

(i) $(x^ay^vz^cw^d)^m = x^{ma+\binom{m}{2}bc+\binom{m}{2}\binom{c+1}{2}d+\frac{(m-1)m(2m-1)}{6}c^2d}y^{mb+\binom{m}{2}cd}z^{mc}w^{md}$.

(ii) $(x^ay^vz^cw^d)^m(x^{a'}y^{b'}z^{c'}w^{d'})^l = x^{a''}y^{b''}z^{c''}w^{d''}$,

where

$$\begin{aligned}a'' &= ma+\binom{m}{2}bc+\binom{m}{2}\binom{c+1}{2}d+\frac{(m-1)m(2m-1)}{6}c^2d\\ &\quad+la'+\binom{l}{2}b'c'+\binom{l}{2}\binom{c'+1}{2}d'+\frac{(l-1)(2l-1)}{6}c'^2d'\\ &\quad+mlcb'+m\binom{l}{2}cc'd'+m^2lcdc'+md\binom{lc'+1}{2}\\ b'' &= mb+\binom{m}{2}cd+lb'+\binom{l}{2}c'd'+mldc'\\ c'' &= mc+lc'\\ d'' &= md+ld'.\end{aligned}$$

Proof *(i) By induction on* m. *(ii) By using (i) and Lemma 2.2.* ■

Now by using Lemma 2.2, we can obtain Fibonacci sequence in the group G. We shall use vector notation to calculate the sequence and define an infinite sequence $r_i = (a_i, b_i, c_i, d_i)$ via the 2-step recurrence and initial data $r_0 = (0, 0, 0, 1)$ which corresponds to w, and $r_1 = (0, 0, 1, 0)$ which corresponds to z.

PROPOSITION 2.4 *For the group* G, $K(G) = k(9) = 24$.

Proof *We obtain the following loop (Note that* a_i, b_i, c_i *reduced modulo 3 and* d_i

reduced modulo 9):

$$\begin{array}{llll} r_0 = (0,0,0,1), & r_6 = (1,1,2,5), & r_{12} = (1,1,0,8), & r_{18} = (2,1,1,4), \\ r_1 = (0,0,1,0), & r_7 = (1,0,1,8), & r_{13} = (1,2,2,0), & r_{19} = (0,1,2,1), \\ r_2 = (1,1,1,1), & r_8 = (2,0,0,4), & r_{14} = (2,1,2,8), & r_{20} = (2,1,0,5), \\ r_3 = (2,1,2,1), & r_9 = (0,0,1,3), & r_{15} = (2,0,1,8), & r_{21} = (1,2,2,6), \\ r_4 = (0,1,0,2), & r_{10} = (0,1,1,7), & r_{16} = (1,0,0,7), & r_{22} = (0,1,2,2), \\ r_5 = (1,2,2,3), & r_{11} = (1,1,2,1), & r_{17} = (0,0,1,6), & r_{23} = (0,0,1,8). \end{array}$$

and $r_{24} = (0,0,0,1)$, $r_{25} = (0,0,1,0)$. *Hence* $k(G) = k(9) = 24$. ■

3. The Group L_α

The Case $\alpha = 0$: Let $G = L_\alpha$, where $\alpha = 0$. Then $G = \langle a,b,c | a^{p^2} = b^p = c^p = 1, [a,b] = a^p, [a,c] = b, [b,c] = 1\rangle$. By the relations of group, $a^p \in [G,G']$. Therefore, G has nilpotency class 3 and $[G,G'] \leqslant Z(G)$. Hence a^p is a central element of G. A power-commutator presentation of G may be given as follows:

$$G = \langle x,y,z,w | x^p = y^p = z^p = 1, w^p = x, [x,y] = [x,z] = [x,w] = 1,$$
$$[z,y] = 1, [w,y] = x, [w,z] = y\rangle.$$

The Case $\alpha = 1$: Let $G = L_\alpha$, where $\alpha = 1$. Then $G = \langle a,b,c | a^{p^2} = b^p = 1, c^p = a^p, [a,b] = a^p, [a,c] = b, [b,c] = 1\rangle$. We may show that G has the following power-commutator presentation:

$$G = \langle x,y,z,w | x^p = y^p = 1, z^p = w^p = x, [x,y] = [x,z] = [x,w] = 1,$$
$$[z,y] = 1, [w,y] = x, [w,z] = y\rangle.$$

The case where α **is a non-residue modulo** p**:** Let $G = L_\alpha$, where α is a non-residue modulo p. Then $G = \langle a,b,c | a^{p^2} = b^p = 1, c^p = a^{\alpha p}, [a,b] = a^p, [a,c] = b, [b,c] = 1\rangle$. We may show that G has the following power-commutator presentation:

$$G = \langle x,y,z,w | x^p = y^p = 1, z^p = x^\alpha, w^p = x, [x,y] = [x,z] = [x,w] = 1,$$
$$[z,y] = 1, [w,y] = x, [w,z] = y\rangle.$$

Note that in the new presentations, the group G is generated by w and z. Moreover, x is a central element. Also, each element of G can be uniquely represented as $x^a y^b z^c w^d$, where in the first case a, b, c reduced modulo p and d reduced modulo p^2 and in the second and third cases a and b reduced modulo p and c and d reduced modulo p^2. From now on we suppose that $G = L_\alpha$, where $\alpha = 0, 1$, or a non-residue modulo p. First we prove some elementary results.

Lemma 3.1 *For every positive integers m and n,*

(i) $w^m y^n = x^{mn} y^n w^m$.
(ii) $w^m z^n = x^{\binom{m+1}{2} n} y^{mn} z^n w^m$.

Proof *Since x is a central element of G, then (i) may be proved by the induction method. To prove (ii) we may use (i) and the relation $[z,y]=1$.* ∎

Lemma 3.2 *Let $x^ay^bz^cw^d$ and $x^{a'}y^{b'}z^{c'}w^{d'}$ be elements of G. Then*

$$(x^ay^bz^cw^d)(x^{a'}y^{b'}z^{c'}w^{d'}) = x^{a+a'+db'+\binom{d+1}{2}c'}y^{b+b'+dc'}z^{c+c'}w^{d+d'}.$$

Proof *By using Lemma 3.1, we have:*

$$\begin{aligned}(x^ay^bz^cw^d)(x^{a'}y^{b'}z^{c'}w^{d'}) &= x^{a+a'}y^bz^cw^dy^{b'}z^{c'}w^{d'}\\ &= x^{a+a'}y^bz^cx^{db'}y^{b'}w^dz^{c'}w^{d'}\\ &= x^{a+a'+db'}y^{b+b'}z^cw^dz^{c'}w^{d'}\\ &= x^{a+a'+db'}y^{b+b'}z^cx^{\binom{d+1}{2}c'}y^{dc'}z^{c'}w^dw^{d'}\\ &= x^{a+a'+db'+\binom{d+1}{2}c'}y^{b+b'+dc'}z^{c+c'}w^{d+d'}.\end{aligned}$$

∎

Lemma 3.3 *Let $x^ay^bz^cw^d$ and $x^{a'}y^{b'}z^{c'}w^{d'}$ be elements of G and m and l be positive integers. Then*

(i) $(x^ay^bz^cw^d)^m = x^{ma+\binom{m}{2}bd+\binom{m}{2}c\binom{d+1}{2}+\binom{m}{3}cd^2}y^{mb+\binom{m}{2}cd}z^{mc}w^{md}$.

(ii) $(x^ay^bz^cw^d)^m(x^{a'}y^{b'}z^{c'}w^{d'})^l = x^{a''}y^{b''}z^{c''}w^{d''}$,
where

$$\begin{aligned}a'' &= ma+\binom{m}{2}bd+\binom{m}{2}c\binom{d+1}{2}+\binom{m}{3}cd^2\\ &\quad + la'+\binom{l}{2}b'd'+\binom{l}{2}c'\binom{d'+1}{2}+\binom{l}{3}c'd'^2\\ &\quad + mldb'+m\binom{l}{2}dc'd'+\binom{md+1}{2}lc'\\ b'' &= mb+\binom{m}{2}cd+lb'+\binom{l}{2}c'd'+mldc',\\ c'' &= mc+lc',\\ d'' &= md+ld'.\end{aligned}$$

Proof *(i) By induction on m. (ii) By using (i) and Lemma 3.2.* ∎

Lemma 3.4 *Every element of the Fibonacci sequence in the group G may be presented by $t_n = x^{a_n}y^{b_n}z^{s_n}w^{s_{n-1}}$, where the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ are defined as follows:*

$$b_0=0, \quad b_n\sum_{i=0}^{n-1} s_{n-1-i}s_{i-1}s_{i+1}, \quad n\geqslant 1,$$

$$a_0=0, \quad a_n\sum_{i=0}^{n-1} s_{n-1-i}\left(s_{i-1}b_{i+1}+\binom{s_{i-1}+1}{2}s_{i+1}\right), \quad n\geqslant 1.$$

Proof *We use an induction method on n. It is obvious that $t_0 = w = x^{a_0}y^{b_0}z^{s_0}w^{s_{-1}}$ and $t_1 = z = x^{a_1}y^{b_1}z^{s_1}w^{s_0}$, for, $a_1 = b_1 = 0$. Now assume that the result holds for n and $n+1$, where $n \geqslant 0$. Then*

$$\begin{aligned} t_{n+2} &= t_n t_{n+1} \\ &= (x^{a_n}y^{b_n}z^{s_n}w^{s_n-1})(x^{a_{n+1}}y^{b_{n+1}}z^{s_{n+1}}w^{s_n}) \\ &= x^{a_n+a_{n+1}+s_{n-1}b_{n+1}+\binom{s_{n-1}+1}{2}s_{n+1}}y^{b_n+b_{n+1}+s_{n-1}s_{n+1}}z^{s_n+s_{n+1}}w^{s_{n-1}+s_n} \\ &= x^{a'}y^{b'}z^{s_{n+2}}w^{s_{n+1}}, \end{aligned}$$

where

$$\begin{aligned} a' &= a_n + a_{n+1} + s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1} \\ &= \sum_{i=0}^{n-1} s_{n-1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) \\ &\quad \sum_{i=0}^{n} s_{n-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) + s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1} \\ &= \sum_{i=0}^{n} s_{n-1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) \\ &\quad - s_{-1}\left(s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1}\right) \\ &\quad + \sum_{i=0}^{n} s_{n-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) + s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1} \\ &= \sum_{i=0}^{n} s_{n+1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) \\ &= \sum_{i=0}^{n+1} s_{n+1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) \\ &= a_{n+2}, \end{aligned}$$

and

$$\begin{aligned} b' &= b_n + b_{n+1} + s_{n-1}s_{n+1} \\ &= \sum_{i=0}^{n-1} s_{n-1-i}s_{i-1}s_{i+1} + \sum_{i=0}^{n} s_{n-i}s_{i-1}s_{i+1} + s_{n-1}s_{n+1} \\ &= \sum_{i=0}^{n} s_{n-1-i}s_{i-1}s_{i+1} - s_{-1}s_{n-1}s_{n+1} + \sum_{i=0}^{n} s_{n-i}s_{i-1}s_{i+1} + s_{n-1}s_{n+1} \\ &= \sum_{i=0}^{n} s_{n+1}s_{i-1}s_{i+1} \\ &= b_{n+2} \end{aligned}$$

■

From now on we shall be working modulo p^2. Let $k = k(p^2)$. The following equations hold and are easy to see:

$$s_{k-i} = s_{-i} = (-1)^{i+1}s_i,\ \sum_{i=0}^{k-1} s_i = \sum_{i=0}^{k-1} s_{k-i},\ \sum_{i=0}^{k-1} s_{i+a} = \sum_{i=0}^{k-1} s_i\ (a \in \mathbb{Z}).$$

The proofs of the Lemmas 3.5, 3.6 and 3.7 may be found in [2] and [4].

Lemma 3.5 *The following equations hold:*

(i) $\sum_{i=0}^{k-1} s_i = 0.$

(ii) $\sum_{i=0}^{k-1} s_i^2 = 0.$

(iii) $\sum_{i=0}^{k-1} s_i^3 = 0.$

Lemma 3.6 *If* $p > 3$, *then*

(i) $\sum_{i=0}^{k-1} s_i s_{i-1} = 0.$

(ii) $\sum_{i=0}^{k-1} s_{i-1}^2 s_i = \sum_{i=0}^{k-1} s_{i-1} s_i^2 = 0.$

Lemma 3.7 *For every integers a, b, c, d, and e the following equations hold:*

(i) $\sum_{i=0}^{k-1} s_{i+a}s_{i+b}s_{-i+c}s_i = 0.$

(ii) $\sum_{i=0}^{k-1}\sum_{j=0}^{i-1} s_{-i+a}s_{i+b}s_{i-j-d}s_{j+e}s_{i+c} = 0.$

Lemma 3.8 *The following equations hold:*

(i) $\sum_{i=0}^{k-1}(-1)^i s_i^3 = 0.$

(ii) $\sum_{i=0}^{k-1}(-1)^i s_{i-1}^2 s_i = \sum_{i=0}^{k-1}(-1)^i s_{i-1}s_i^2 = 0, \quad p > 3.$

Proof

(i) $\sum_{i=0}^{k-1}(-1)^i s_{i-1}^3 = \sum_{i=0}^{k-1} s_{-(i-1)}^3 = \sum_{i=0}^{k-1} s_{k-(i-1)}^3 = \sum_{i=0}^{k-1} s_i^3 = 0.$

(ii) we may write:

$$0 = \sum_{i=0}^{k-1} s_i^3 = \sum_{i=0}^{k-1}(-1)^i s_{i+1}^3 = \sum_{i=0}^{k-1}(-1)^i(s_i + s_{i-1})^3$$

$$= 3\sum_{i=0}^{k-1}(-1)^i s_{i-1}s_i^2 + 3\sum_{i=0}^{k-1}(-1)^i s_{i-1}^2 s_i \qquad (1)$$

On the other hand,

$$
\begin{aligned}
0 &= \sum_{i=0}^{k-1} s_i^3 = \sum_{i=0}^{k-1} (-1)^{i-1} s_{i-2}^3 = \sum_{i=0}^{k-1} (-1)^i (s_i - s_{i-1})^3 \\
&= 3\sum_{i=0}^{k-1} (-1)^i s_{i-1} s_i^2 - 3\sum_{i=0}^{k-1} (-1)^i s_{i-1}^2 s_i. \qquad (2)
\end{aligned}
$$

Adding (1) and (2) we obtain

$$6\sum_{i=1}^{k-1} s_{i-1} s_i^2 = 0,$$

and subtracting (2) from (1) we obtain

$$6\sum_{i=0}^{k-1} s_{i-1}^2 s_i = 0.$$

Since $p > 3$, *(ii) follows.*

■

Now we are ready to prove the main results.

Proof of Main Theorem. By using Lemma 3.4, it is sufficient to show that $a_k = a_{k+1} = b_k = b_{k+1} = 0$. We have:

$$
\begin{aligned}
b_k &= \sum_{i=0}^{k-1} s_{k-1-i} s_{i-1} s_{i+1} = \sum_{i=0}^{k-1} s_{-(i+1)} s_{i-1} s_{i+1} = \sum_{i=0}^{k-1} (-1)^i s_{i-1} s_{i+1}^2 \\
&= \sum_{i=0}^{k-1} (-1)^i s_{i-1} (s_{i-1} + s_i)^2 \\
&= \sum_{i=0}^{k-1} (-1)^i s_{i-1}^3 + \sum_{i=0}^{k-1} (-1)^i s_{i-1} s_i^2 + 2\sum_{i=0}^{k-1} (-1)^i s_{i-1}^2 s_i,
\end{aligned}
$$

and the last three expressions vanish by Lemma 3.8. So $b_k = 0$. Similarly,

$$
\begin{aligned}
b_{k+1} &= \sum_{i=0}^{k} s_{k-i} s_{i-1} s_{i+1} = \sum_{i=0}^{k} s_{-i} s_{i-1} s_{i+1} = \sum_{i=0}^{k} (-1)^{i+1} s_{i-1} s_i s_{i+1} \\
&= \sum_{i=0}^{k-1} (-1)^{i+1} s_{i-1} s_i + s_{i+1} = \sum_{i=0}^{k-1} (-1)^{i+1} s_{i-1} s_i (s_i + s_{i-1}) \\
&= -\left(\sum_{i=0}^{k-1} (-1)^i s_{i-1} s_i^2 + \sum_{i=0}^{k-1} (-1)^i s_{i-1}^2 s_i \right),
\end{aligned}
$$

and the last two sums vanish by Lemma 3.8. On the other hand,

$$\begin{aligned}
a_k &= \sum_{i=0}^{k-1} s_{k-1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right)\\
&= \sum_{i=0}^{k-1} s_{k-(i+1)}\left(s_{i-1}\sum_{j=0}^{i} s_{i-j}s_{j-1}s_{j+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right)\\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{i} s_{-(i+1)}s_{i-1}s_{i-j}s_{j-1}s_{j+1} + \sum_{i=0}^{k-1}\binom{s_{i-1}+1}{2}s_{-(i+1)}s_{i+1}\\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{i-1} s_{-i-1}s_{i-1}s_{i-j}s_{j-1}s_{j+1} + \frac{1}{2}\sum_{i=0}^{k-1}(s_{i-1}+1)s_{i-1}s_{-(i+1)}s_{i+1},
\end{aligned}$$

and the first sum vanishes by Lemma 3.7(ii). For the second sum in the above expression, we have:

$$\begin{aligned}
\sum_{i=0}^{k-1}(s_{i-1}+1)s_{i-1}s_{-(i+1)}s_{i+1} &= \sum_{i=0}^{k-1} s_{i-1}s_{i-1}s_{-(i+1)}s_{i+1} + \sum_{i=0}^{k-1} s_{i-1}s_{-(i+1)}s_{i+1}\\
&= \sum_{i=0}^{k-1} s_{i-2}s_{i-2}s_{-i}s_i + \sum_{i=0}^{k-1}(-1)^i s_{i-1}s_{i+1}^2
\end{aligned}$$

and the first sum vanishes by Lemma 3.7(i) and the second one is equal to bk which is zero. A similar method may be used to prove $a_{k+1} = 0$. This completes the proof showing that $k(G) = k(p^2)$ for all of groups $G = L_\alpha$, where $\alpha = 0, 1$, or non-residue modulo p.

4. Conclusion

The Fibonacci lengths of the finite p-groups had been studied by R. Dikici and co-authors since 1992. All of the considered groups were of exponent p, and the lengths depended on the celebrated Wall number $k(p)$. The study of p-groups of nilpotency class 3 and exponent p had been done in 2004 by R. Dikici as well. In this paper we studied all of the p-groups of nilpotency class 3 and exponent p^2. This completed the study of Fibonacci length of all p-groups of order p^4, proving that the Fibonacci length is $k(p^2)$.

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