The Application of the Variational Homotopy Perturbation Method on the Generalized Fisher's Equation

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Abstract. In this paper, we consider the variational homotopy perturbation method (VHPM) to obtain an approximate series solution for the generalized Fisher's equation which converges to the exact solution in the region of convergence. Comparisons are made among the variational iteration method (VIM), the exact solutions and the proposed method. The results reveal that the proposed method is very effective and simple and can be applied for other nonlinear problems in mathematical.

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1. Introduction

In this paper, we consider the generalized Fisher's equation

$$u_t = u_{xx} + u\,(1 - u^\alpha), \tag{1}$$

where $u_t = \frac{\partial u}{\partial t}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$. Fisher proposed equation of $u_t = u_{xx} + u\,(1 - u)$ as a model for the propagation of a mutantgene, with u denoting the density of an advantageous. This equation is encountered in chemical kinetics and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat and neutron population in a nuclear reaction and branching. Moreover, the same equation occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions, and branching Brownian motion processes [1]. In this work, we solve the generalized Fisher's equation via VHPM.

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2. Variational Homotopy Perturbation Method

To convey the basic idea of the variational homotopy perturbation method, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{2}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. According to variational iteration method [2–4, 6–11], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau) \left\{ Lu_n + N\tilde{u}_n - g(\tau) \right\} \, d\tau, \tag{3}$$

where $\lambda(\tau)$ is a Lagrange multiplier [2–4, 6–11] which can be identified optimally via the variational iteration method. The subscripts n denote the n th approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta \tilde{u}_n = 0$ and (3) is called a correct functional. Now, we apply the homotopy perturbation method;

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p \int_0^x \lambda(\tau) \left\{ N \left(\sum_{i=0}^{\infty} p^i u_i \right) \right\} \, d\tau - \int_0^x \lambda(\tau) g(\tau) \, d\tau, \tag{4}$$

which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian's polynomials.
The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [12–18]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [12–18] to obtain

$$f = \sum_{i=0}^{\infty} p^i u_i = u_0 + p \, u_1 + p^2 u_2 + \cdots . \tag{5}$$

If $p \to 1$, then (5) becomes the approximate solution of the form

$$u = \lim_{p \to 1} f = u_0 + u_1 + u_2 + \cdots . \tag{6}$$

A comparison of like powers of p gives solutions of various orders.

3. Implementation of VHPM

At first, in this section we consider special case of the generalized Fisher's equation and then take into account a general form.

Case 1: We consider the generalized Fisher's equation (1) with fixed value $\alpha = 3$ as follows:

$$u_t = u_{xx} + u \left(1 - u^3 \right), \tag{7}$$

subject to the initial condition $u_0(x,t) = \phi^2(x)$, where $\phi(x) = \frac{1}{(1+e^{\frac{3}{\sqrt{10}}x})^{\frac{1}{3}}}$.

To solve Eq. (7) by means of VHPM, we consider

$$L(u) = u_t, \tag{8}$$

$$N(u) = -u_{xx} - u + u^4, \tag{9}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [2–4, 6–11], we can construct a correct functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) \left\{ u_{n_\tau} - \tilde{u}_{n_{xx}} - \tilde{u}_n(1 - \tilde{u}_n^3) \right\} d\tau, \tag{10}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ u_{n_\tau} - u_{n_{xx}} - u_n(1 - u_n^3) \right\} d\tau. \tag{11}$$

Applying the variational homotopy perturbation method, we have:

$$u_0 + p\, u_1 + p^2 u_2 + \cdots = \phi^2(x) + p \int_0^t (u_{0_{xx}} + p\, u_{1_{xx}} + p^2 u_{2_{xx}} + \cdots)\, d\tau$$

$$+ p \int_0^t (u_0 + p\, u_1 + p^2 u_2 + \cdots)\, d\tau$$

$$- p \int_0^t (u_0 + p\, u_1 + p^2 u_2 + \cdots)^4\, d\tau. \tag{12}$$

Comparing the coefficient of like powers of p, we have:

$u_0(x,t) = \phi^2(x),$

$u_1(x,t) = \dfrac{7}{5} \phi^5(x)\, e^{\frac{3}{\sqrt{10}}x}\, t,$

$u_2(x,t) = \dfrac{49}{50} \phi^8(x)\, e^{\frac{3}{\sqrt{10}}x} \left(2e^{\frac{3}{\sqrt{10}}x} - 3 \right) \dfrac{t^2}{2!},$

$u_3(x,t) = -\dfrac{343}{500} \phi^{11}(x)\, e^{\frac{3}{\sqrt{10}}x} \left(-9 + 27e^{\frac{3}{\sqrt{10}}x} - 4e^{\frac{6}{\sqrt{10}}x} \right) \dfrac{t^3}{3!},$

Thus the components which constitute $u(x,t)$ are written like this

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots = \phi^2(x)$$
$$+ \frac{7}{5} \phi^5(x) \, e^{\frac{3}{\sqrt{10}}x} \, t$$
$$+ \frac{49}{50} \phi^8(x) \, e^{\frac{3}{\sqrt{10}}x} \left(2 e^{\frac{3}{\sqrt{10}}x} - 3 \right) \frac{t^2}{2!}$$
$$- \frac{343}{500} \phi^{11}(x) \, e^{\frac{3}{\sqrt{10}}x} \left(-9 + 27 \, e^{\frac{3}{\sqrt{10}}x} - 4 \, e^{\frac{6}{\sqrt{10}}x} \right) \frac{t^3}{3!}$$
$$+ \cdots,$$

The exact solution by Wang [5] is given by

$$u(x,t) = \left\{ \frac{1}{2} \tanh \left[-\frac{3}{2\sqrt{10}} \left(x - \frac{7}{\sqrt{10}} t \right) \right] + \frac{1}{2} \right\}^{\frac{2}{3}}. \tag{13}$$

Case 2: Now, we consider the generalized Fisher's equation

$$u_t = u_{xx} + u\,(1 - u^\alpha), \tag{14}$$

subject to the initial condition

$u(x,0) = \phi^2(x;\alpha)$, where $\phi(x;\alpha) = \dfrac{1}{\left(1 + e^{\frac{\alpha}{\sqrt{2\alpha+4}}x}\right)^{\frac{1}{\alpha}}}$.

To solve Eq. (14) by means of VHPM, we consider

$$L(u) = u_t, \tag{15}$$

$$N(u) = -u_{xx} - u + u^{\alpha+1}, \tag{16}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [2–4, 6–11], we can construct a correct functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) \left\{ u_{n_\tau} - \tilde{u}_{n_{xx}} - \tilde{u}_n(1 - \tilde{u}_n^\alpha) \right\} \, d\tau, \tag{17}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ u_{n_\tau} - u_{n_{xx}} - u_n(1 - u_n^\alpha) \right\} \, d\tau. \tag{18}$$

Applying the variational homotopy perturbation method, we have:

$$u_0 + p\,u_1 + p^2 u_2 + \cdots = \phi^2(x;\alpha) + p\int_0^t (u_{0_{xx}} + p\,u_{1_{xx}} + p^2 u_{2_{xx}} + \cdots)\,d\tau$$

$$+ p\int_0^t (u_0 + p\,u_1 + p^2 u_2 + \cdots)\,d\tau$$

$$- p\int_0^t (u_0 + p\,u_1 + p^2 u_2 + \cdots)^{\alpha+1}\,d\tau. \quad (19)$$

Comparing the coefficient of like powers of p, we have:

$$u_0(x,t) = \phi^2(x;\alpha),$$

$$u_1(x,t) = \frac{\alpha+4}{\alpha+2}\,\phi^{2+\alpha}(x;\alpha)\,e^{\frac{\alpha}{\sqrt{2\alpha+4}}x}\,t,$$

$$u_2(x,t) = \frac{1}{2}\frac{(\alpha+4)^2}{(\alpha+2)^2}\,\phi^{2+2\alpha}(x;\alpha)\,e^{\frac{\alpha}{\sqrt{2\alpha+4}}x}\left(2e^{\frac{\alpha}{\sqrt{2\alpha+4}}x} - \alpha\right)\frac{t^2}{2!},$$

$$u_3(x,t) = -\frac{1}{4}\frac{(\alpha+4)^3}{(\alpha+2)^3}\,\phi^{2+3\alpha}(x;\alpha)\,e^{\frac{\alpha}{\sqrt{2\alpha+4}}x}\left(-\alpha^2 + (6\alpha+\alpha^2)\,e^{\frac{\alpha}{\sqrt{2\alpha+4}}x} - 4e^{\frac{2\alpha}{\sqrt{2\alpha+4}}x}\right)\frac{t^3}{3!},$$

$$\vdots$$

Thus the components which constitute $u(x,t)$ are written like this

$$u(x,t) = u_0 + u_1 + u_2 + \cdots = \phi^2(x;\alpha)$$

$$+ \frac{\alpha+4}{\alpha+2}\,\phi^{2+\alpha}(x;\alpha)\,e^{\frac{\alpha}{\sqrt{2\alpha+4}}x}\,t$$

$$+ \frac{1}{2}\frac{(\alpha+4)^2}{(\alpha+2)^2}\,\phi^{2+2\alpha}(x;\alpha)\,e^{\frac{\alpha}{\sqrt{2\alpha+4}}x}\left(2\,e^{\frac{\alpha}{\sqrt{2\alpha+4}}x} - \alpha\right)\frac{t^2}{2!}$$

$$+ \cdots,$$

The exact solution by Wang [5] is given by

$$u(x,t) = \left\{\frac{1}{2}\tanh\left[-\frac{\alpha}{2\sqrt{2\alpha+4}}\left(x - \frac{\alpha+4}{\sqrt{2\alpha+4}}t\right)\right] + \frac{1}{2}\right\}^{\frac{2}{3}}. \quad (20)$$

For later numerical computation, we let the expression

$$\varphi_n = \sum_{i=0}^n u_i(x,t), \quad (21)$$

to denote the n-term approximation to u(x,t).

In what follows, we present the absolute errors between φ_{2VHPM} and the exact solution and the absolute errors between the 2-iterate of VIM ($u_{2_{VIM}}$) and the exact solution for the values of $t = 0.1$, $x = 0(0.1)0.5$ and $\alpha = 3$.

Table 1.   The numerical results for $\varphi_{2_{VHPM}}$ and $u_{2_{VIM}}$
in comparison with the exact solution of u.

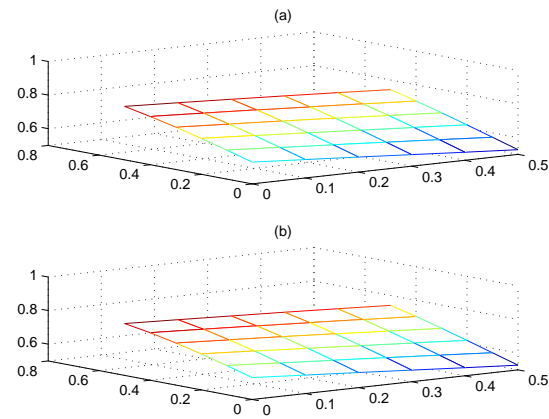| x	$|u - \varphi_{2_{VHPM}}|$	$|u - u_{2_{VIM}}|$
0.0	$1.2212e - 004$	$3.7695e - 005$
0.1	$1.2847e - 004$	$2.5820e - 005$
0.2	$1.3283e - 004$	$1.4565e - 005$
0.3	$1.3511e - 004$	$4.2358e - 006$
0.4	$1.3530e - 004$	$4.9027e - 006$
0.5	$1.3342e - 004$	$1.2642e - 005$

Figure 1.   The exact solution (a) $u(x,t)$ and the approximate solution (b) $\varphi_{2_{VHPM}}$ with fixed value $\alpha = 3$ at $x = 0(0.1)0.5$, t=0(0.1)0.5.

The numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy.
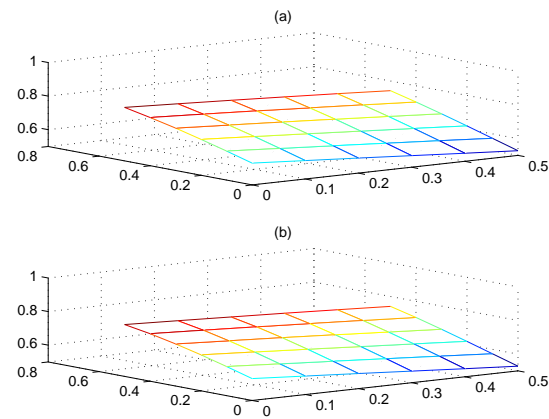

Figure 2.   The exact solution (a) $u(x,t)$ and the 2-iterate of VIM (b) $u_{2_{VIM}}$ with fixed value $\alpha = 3$ at $x = 0(0.1)0.5$, t=0(0.1)0.5.

4.   Conclusion

In this paper, varational homotopy perturbation method is proposed for solving the generalized Fisher's equation. The small amount of computation compared to that required in other methods such as the variational iteration method and the rapid convergence show that the method is reliable and provides a significant improvement in solving partial differential equations over existing methods. The computations in this paper are done by MATLAB software.

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