

Coupled Fixed Point Theorems for Generalized φ -mappings satisfying Contractive Condition of Integral Type on Cone Metric Spaces

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Abstract. In this paper, we unify, extend and generalize some results on coupled fixed point theorems of generalized φ - mappings with some applications to fixed points of integral type mappings in cone metric spaces.

Keywords: Generalized φ -pair mappings, Coupled fixed point, Complete cone metric spaces.

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1. Introduction

Huang and Zhang [4] generalized the concept of metric spaces by considering vector-valued metrics (cone metrics) with values in ordered real Banach spaces. Since then, several Fixed point theorems have been proved in the context of cone metric spaces (for example, see [1-4,7]). The concept of coupled fixed point was recently introduced by T. G. Bhaskar and V. Lakshmikantham [2]. Recently, F. Sabetghadam, H. P. Masiha and A. H. Sanatpour [10] proved some coupled fixed point theorems in cone metric spaces. In this paper, we unify, extend and generalize the results in [10] with some applications to integral type.

DEFINITION 1.1 ([4]). A cone P is a subset of a real Banach space E such that

- P is closed, nonempty and $P \neq \{0\}$;
- if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;
- $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, the partial ordering \leq with respect to P is defined by

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$x \leq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P . Also, we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P is called normal if there exist a constant $M > 0$ such that for every $x, y \in E$, if $0 \leq x \leq y$; then $\|x\| \leq M\|y\|$. The least positive number satisfying this inequality is called the normal constant of P .

In this paper, we suppose that E is a real Banach space, $P \subseteq E$ is a cone with $\text{int}P \neq \emptyset$ and \leq is partial ordering with respect to P . We also note that the relation $P + \text{int}P \subseteq \text{int}P$ and $\lambda \text{int}P \subseteq \text{int}P$ ($\lambda > 0$) always hold true.

DEFINITION 1.2 Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering \leq with respect to the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies the following conditions [4]:

- $d_1)$ $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- $d_2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d_3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space.

DEFINITION 1.3 Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X [4]. Then

- $i)$ $\{x_n\}_{n \geq 1}$ converges to x ; denoted by $\lim_{n \rightarrow \infty} x_n = x$, if for every $c \in E$ with $0 \ll c$ there exist a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$;
- $ii)$ $\{x_n\}_{n \geq 1}$ is a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X .

DEFINITION 1.4 Let f and g be self-maps of a set X (i.e., $f, g : X \rightarrow X$). If $w = fx = gx$ for some x in X , then x is a coincidence point of f and g , and w is called a point of coincidence of f and g . Self-maps f and g are said to be weakly compatible if they commute at their coincidence point, that is, if $fx = gx$ for some x in X , then $fgx = gfx$ [6].

DEFINITION 1.5 Let (X, d) be a cone metric space. An element (x, y) in $X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$ [10].

Recently, F. Sabetghadam, H. P. Masiha and A. H. Sanatpour [10] proved the existence of unique coupled

fixed point for the following contractive conditions in a cone metric space:

$$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v), \quad (1)$$

where k, l are nonnegative constants with $k + l < 1$.

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), x) + ld(F(u, v), u), \quad (2)$$

where k, l are nonnegative constants with $k + l < 1$.

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), u) + ld(F(u, v), x), \quad (3)$$

where k, l are nonnegative constants with $k + l < 1$.

In this research paper, we unify, extend and generalize the contractive conditions (1), (2) and (3). Furthermore, we also prove some other coupled fixed point theorems in cone metric spaces.

DEFINITION 1.6 *Let P be a cone [11]. A non-decreasing mapping $\varphi : P \rightarrow P$ is called a φ -mapping if*

- $\varphi_1)$ $\varphi(0) = 0$ and $0 < \varphi(\omega)$ for $\omega \in P - \{0\}$;
- $\varphi_2)$ $\omega - \varphi(\omega) \in \text{int}(P)$ for every $\omega \in \text{int}(P)$;
- $\varphi_3)$ $\lim_{n \rightarrow \infty} \varphi^n(\omega) = 0$ for every $\omega \in P - \{0\}$;
- $\varphi_4)$ $\varphi(cu) \leq c\varphi(u)$, where $c > 0$.

DEFINITION 1.7 *Let P be a cone and let $\{\omega_n\}$ be a sequence in P . One says that $\omega_n \rightarrow 0$ if for every $\epsilon \in P$ with $0 \ll \epsilon$ there exists $N > 0$ such that $\omega_n \ll \epsilon$ for all $n \geq N$ [11].*

DEFINITION 1.8 *For a non-decreasing mapping $T : P \rightarrow P$, we define the following conditions which will be used in the sequel [11]:*

- $T_1)$ For every $\omega_n \in P$, $\omega_n \rightarrow 0$ if and only if $T\omega_n \rightarrow 0$;
- $T_2)$ For every $\omega_1, \omega_2 \in P$, $T(\alpha\omega_1 + \beta\omega_2) \leq \alpha T(\omega_1) + \beta T(\omega_2)$, $\alpha, \beta \geq 0$.

2. Main Results

We will now consider the following theorems.

THEOREM 2.1 *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition*

$$T(d(f(x, y), F(u, v))) \leq \varphi(T(j)), \tag{4}$$

for all $x, y, u, v \in X$, where

$$j \in \left\{ \frac{d(x, u) + d(y, v)}{2}, d(F(x, y), x), d(F(u, v), u), \frac{d(F(x, y), u), d(F(u, v), x)}{2}, \frac{d(F(x, y), x), d(F(u, v), u)}{2} \right\}$$

$\varphi : P \rightarrow P$ is a nondecreasing mapping satisfying $(\varphi_1) - (\varphi_4)$, and $T : P \rightarrow P$ is a nondecreasing mapping satisfying $(T_1) - (T_3)$. Then F has a unique coupled fixed point.

Proof Choose $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), \dots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n).$$

By (4), we have that

$$\begin{aligned}
 T(d(x_n, x_{n+1})) &= T(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\
 &\leq \varphi\left(T\left(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2}, d(F(x_{n-1}, y_{n-1}), x_{n-1}), \right. \right. \\
 &\quad \left. \left. d(F(x_n, y_n), x_n), \frac{d(F(x_{n-1}, y_{n-1}), x_n) + d(F(x_n, y_n), x_{n-1})}{2}, \right. \right. \\
 &\quad \left. \left. \frac{d(F(x_{n-1}, y_{n-1}), x_{n-1}) + d(F(x_n, y_n), x_n)}{2}\right)\right). \\
 &= \varphi\left(T\left(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2}, d(x_n, x_{n-1}), \right. \right. \\
 &\quad \left. \left. d(x_{n+1}, x_n), \frac{d(x_{n+1}, x_{n-1})}{2}, \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2}\right)\right).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 T(d(y_n, y_{n+1})) &\leq \varphi\left(T\left(\frac{d(F(y_{n-1}, y_n), d(x_{n-1}, x_n))}{2}, d(y_n, y_{n-1}), d(y_{n+1}, y_n), \right. \right. \\
 &\quad \left. \left. \frac{d(y_{n+1}, y_{n-1})}{2}, \frac{d(y_n, y_{n-1}) + d(y_{n+1}, y_n)}{2}\right)\right).
 \end{aligned}$$

Set $d_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$ then we must first show that

$$\begin{aligned}
 T(d_n) &\leq \varphi(T(dx_n, x_{n-1}) + d(y_n, y_{n-1})), \\
 &= \varphi(T(d_{n-1})), \\
 &\leq \varphi^n(T(d_0)), \quad \forall n \geq 1.
 \end{aligned} \tag{5}$$

It is sufficient to consider the following cases. ■

Case 1. If $j = d(x_n, x_{n-1}) + s(y_n, y_{n-1})$,

$$T(d_n) = T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \leq \varphi(T(d(x_n, x_{n-1}) + d(y_n, y_{n-1}))).$$

This implies that $T(d_n) \leq \varphi(T(d_{n-1})) \leq \varphi^n(T(d_0))$ and (5) is satisfied.

Case 2. If $j = d(x_n, x_{n-1} + d(y_n, y_{n-1}))$, then

$$T(d_n) = T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \leq \varphi(T(d(x_n, x_{n-1}) + d(y_n, y_{n-1}))).$$

So that $T(d_n) \leq \varphi(T(d_{n-1})) \leq \varphi^n(t(d_0))$ and (5) is proved.

Case 3. If $j = \frac{d(x_{n+1}, x_{n-1})}{2} + \frac{d(y_{n+1}, y_{n-1})}{2}$, then

$$\begin{aligned}
 T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) &\leq \varphi\left(T\left(\frac{d(x_{n+1}, x_{n-1})}{2} + \frac{d(y_{n+1}, y_{n-1})}{2}\right)\right) \\
 &\leq \varphi\left(T\left(\frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2} + \frac{d(y_{n+1}, y_n) + d(y_n, y_{n-1})}{2}\right)\right) \\
 &\leq \varphi\left(T\left(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\right)\right) + \\
 &\quad \left(T\frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)}{2}\right).
 \end{aligned}
 \tag{6}$$

Then from (6) and (φ_4) , we have

$$t(d_n) \leq \varphi(T(d_{n-1})) \leq \varphi^n(T(d_0)). \tag{7}$$

Hence (5) is satisfied.

Case 4. If $j = \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2} + \frac{d(y_n, y_{n-1}) + d(y_{n+1}, y_n)}{2}$, then

$$\begin{aligned}
 T(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) &\leq \varphi\left(T\left(\frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2} + \frac{d(y_n, y_{n-1}) + d(y_{n+1}, y_n)}{2}\right)\right) \\
 &\leq \varphi\left(T\left(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\right)\right) + \\
 &\quad T\left(\frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)}{2}\right) +
 \end{aligned}
 \tag{8}$$

Then from (8) and (φ_4) , we have

$$\begin{aligned}
 T(d_n) &\leq \varphi(T(d_{n-1})), \\
 &\leq \varphi^n(T(d_0)), \quad \forall n \geq 1.
 \end{aligned}
 \tag{9}$$

Hence (5) is satisfied.

If $d_0 = 0$, then (x_0, y_0) is a coupled fixed point of F. Now, let $d_0 > 0$, for each $n \geq m$. by (9) and φ_1 , we have:

$$\begin{aligned}
 T(d(x_n, x_m) + d(y_n, y_m)) &\leq T(d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(x_{n-1}, x_{n-2}) \\
 &\quad + d(y_{n-1}, y_{n-2}) + \dots + d(x_{m+1}, x_m) \\
 &\quad + d(y_{m+1}, y_m)) \\
 &\leq \varphi(T(d_{n-1} + d_{n-2} + \dots + d_m)) \\
 &< T(d_{n-1} + T(d_{n-2}) + \dots + T(d_m)) \\
 &\leq \varphi^{n-1}(T(d_0)) + \varphi^{n-2}(T(d_0)) \\
 &\quad + \varphi^{n-3}(T(d_0)) + \dots + \varphi^m(T(d_0)).
 \end{aligned}$$

which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X , and there exist $x^*, y^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$ and $\lim_{n \rightarrow \infty} y_n = y^*$. Let $c \in E$ with $0 \ll c$. For every $n \in N$ there exists $N \in N$ such that $T(d(x_n, x^*)) \ll \frac{c}{2}$ and $T(d(y_n, y^*)) \ll \frac{c}{2}$ for all $n \geq N$. Thus by (2.1)

$$\begin{aligned} T(d(F(x^*, y^*), x^*)) &\leq T(d(x_N, F(x^*, y^*)) + d(x_N, X^*)) \\ &= T(d(F(x_{N-1}, y_{N-1}), F(x^*, y^*)) + d(x_N, x^*)) \\ &\leq \varphi\left(T\left(\frac{d(x_{N-1}, x^*) + d(y_{N-1}, y^*)}{2}, d(F(x_{N-1}, y_{N-1}), x_{N-1}), \right. \right. \\ &\quad \left. \left. \frac{d(F(x_{N-1}, y_{N-1}), x_{N-1}) + d(F(x^*, y^*), x^*)}{2} + d(x_N, x^*)\right)\right) \\ &= \varphi\left(T\left(\frac{d(x_{N-1}, x^*) + d(y_{N-1}, y^*)}{2}, d(x_N, x_{N-1}), \right. \right. \\ &\quad \left. \left. d(F(x^*, y^*), x^*), \frac{d(x_N, x^*) + d(F(x^*, y^*), x_{N-1})}{2}, \right. \right. \\ &\quad \left. \left. \frac{d(x_N, x_{N-1}) + d(F(x^*, y^*), x^*)}{2} + d(x_N, x^*)\right)\right). \end{aligned}$$

Case 1*. If $j = \frac{d(x_{N-1}, x^*) + d(y_{N-1}, y^*)}{2} + d(x_N, x^*)$. Then, we have that:

$$T(d(F(x^*, y^*), x^*)) \leq \varphi\left(T\left(\frac{d(x_{N-1}, x^*) + d(y_{N-1}, y^*)}{2} + d(x_N, x^*)\right)\right). \quad (10)$$

choose

$$\begin{aligned} T(d(x_{N-1}, x^*)) &\ll \frac{c}{2}, \\ T(d(y_{N-1}, y^*)) &\ll \frac{c}{2}, \end{aligned}$$

and

$$T(d(x_N, x^*)) \ll \frac{c}{2}.$$

Then, by (φ_1) and (10), we have

$$T(d(F(x^*, y^*), x^*)) \ll c.$$

Case 2*. If $j = d(x_N, x_{N-1}) + d(x_N, x^*)$, Then

$$\begin{aligned} T(d(F(x^*, y^*), x^*)) &\leq \varphi\left(T(d(x_N, x_{N-1}) + d(x_N, x^*) + d(x_N, x^*))\right) \\ &\leq \varphi\left(T(d(x_N, x^*) + d(x^*, x_{N-1}) + d(x_N, x^*))\right) \end{aligned} \quad (11)$$

choose

$$T(d(x_N, x^*)) \ll \frac{c}{4},$$

and

$$T(d(x_{N-1}, x^*)) \ll \frac{c}{2}.$$

Hence, by (φ_1) and (11), we obtain

$$T(d(F(x^*, y^*), x^*)) \ll c.$$

Case 3*. If $j = \frac{d(x_N, x^*) + d(F(x^*, y^*), x_{N-1})}{2} + d(x_N, x^*)$, we have that

$$\begin{aligned} T(d(F(x^*, y^*), x^*)) &\leq \varphi(T(\frac{d(x_N, x^*) + d(F(x^*, y^*), x_{N-1})}{2} + d(x_N, x^*))), \\ &\leq \varphi(T(\frac{d(x_N, x^*) + d(F(x^*, y^*), x^*) + d(x^*, x_{N-1})}{2} + d(x_N, x^*))), \\ &\leq \varphi(T(\frac{d(x_N, x^*) + d(x^*, x_{N-1})}{2} + d(x_N, x^*))), \\ &\quad + T(\frac{d(F(x^*, y^*), x^*)}{2}). \end{aligned} \tag{12}$$

By (φ_4) and (12), we have

$$\begin{aligned} T(d(F(x^*, y^*), x^*)) &\leq \varphi(T(d(x_N, x^*) + d(x^*, x_{N-1})) \\ &\quad + 2d(x_N, x^*))). \end{aligned} \tag{13}$$

choose

$$T(d(x_N, x^*)) \ll \frac{c}{4},$$

and

$$T(d(x^*, x_{N-1})) \ll \frac{c}{4}.$$

Then, by (φ_1) and (13), we have

$$T(d(F(x^*, y^*), x^*)) \ll c.$$

Case 4*. If $j = \frac{d(x_N, x_{N-1}) + d(F(x^*, y^*), x^*)}{2} + d(x_N, x^*)$, we have that

$$\begin{aligned} T(d(F(x^*, y^*), x^*)) &\leq \varphi(T(\frac{d(x_N, x_{N-1}) + d(F(x^*, y^*), x^*)}{2} + d(x_N, x^*))), \\ &\leq \varphi(T(\frac{d(x_N, x^*) + d(x^*, x_{N-1}) + d(F(x^*, y^*), x^*)}{2} + d(x_N, x^*))), \\ &\leq \varphi(T(\frac{d(x_N, x^*) + d(x^*, x_{N-1})}{2} + d(x_N, x^*))), \\ &\quad + T(\frac{d(F(x^*, y^*), x^*)}{2}). \end{aligned} \tag{14}$$

By (φ_4) and (14), we have

$$T(d(F(x^*, y^*), x^*)) \leq \varphi(T(d(x_N, x^*) + d(x^*, x_{N-1})) + 2d(x_N, x^*)). \quad (15)$$

choose

$$T(d(x_N, x^*)) \ll \frac{c}{4},$$

and

$$T(d(x^*, x_{N-1})) \ll \frac{c}{4}.$$

Then, by (φ_1) and (15), we have

$$T(d(F(x^*, y^*), x^*)) \ll c.$$

In all cases, $T(d(F(x^*, y^*), x^*)) = 0$ or equivalently $F(x^*, y^*) = x^*$. Similarly, we can show that $F(y^*, x^*) = y^*$. Hence, we have shown that (x^*, y^*) is a coupled fixed point of F .

Next, we show that (x^*, y^*) is a unique coupled fixed point of F . Assume that (x', y') is another coupled Fixed point of F , then by Case 1-4, we have that

$$\begin{aligned} T(d(x', x^*)) &= T(d(F(x', y'), F(x^*, y^*))) \\ &\leq \varphi(T(d(F(x', y'), x') + d(F(x^*, y^*), x^*))) \\ &= \varphi(T(d(x', x') + d(x^*, x^*))) = 0. \end{aligned}$$

Hence, $x' = x^*$. We can similarly show that $y' = y^*$, hence $(x', y') = (x^*, y^*)$.

Theorem 2.1 lead to the following Corollaries:

COROLLARY 2.2 *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition*

$$T(d(F(x, y), F(u, v))) \leq \varphi(T(j)), \quad (16)$$

for all $x, y, u, v \in X$, where

$j \in \left\{ \frac{d(x, u) + d(y, v)}{2}, d(F(x, y), x), d(F(u, v), u), \frac{d(F(x, y), u) + d(F(u, v), x)}{2} \right\}$, $\varphi : P \rightarrow P$ is a nondecreasing mapping satisfying $(\varphi_1) - (\varphi_4)$ and $T : P \rightarrow P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

COROLLARY 2.3 *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition*

$$T(d(F(x, y), F(u, v))) \leq \varphi(T(j)), \quad (17)$$

for all $x, y, u, v \in X$, where

$j \in \left\{ \frac{d(x,u)+d(y,v)}{2}, d(F(x,y),x), d(F(u,v),u), \frac{d(F(x,y),u)+d(F(u,v),x)}{2} \right\}$, $\varphi : P \rightarrow P$ is a nondecreasing mapping satisfying $(\varphi_1) - (\varphi_4)$ and $T : P \rightarrow P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

Remark 2.4. Theorem 2.2 [10], Theorem 2.5 [10] and Theorem 2.6 [10] are special cases of Corollary 2.3.

3. Coupled Fixed Point of Operators Satisfying Contractive Condition of Integral Type

In 2002, Branciari [3] introduced a general contractive condition of integral type as follows.

THEOREM 3.1 *Let (X, d) be a complete metric space, $\alpha \in (0, 1)$, and $f : X \rightarrow X$ is a mapping such that for all $x, y \in X$ [3],*

$$\int_0^{d(f(x),f(y))} \phi(t)d(t) \leq \alpha \int_0^{d(x,y)} \phi(t)d(t),$$

where $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is a nonnegative-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$ such that for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$, then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n \rightarrow \infty} f^n x = a$.

In [8], F. Khojasteh et al. defined a new concept of integral with respect to a cone and introduced the Branciari’s results in cone metric spaces.

In this paper, we study coupled fixed point of operators satisfying contractive condition of integral type mappings in cone metric spaces. Our results is an extension of the results of F. Khojasteh et al. [8] to coupled fixed point, introduced by T. G. Bhaskar and V. Lakshmikantham [2]. We start with some definitions, examples and properties as introduced in [8].

DEFINITION 3.2 *Suppose that P is a normal cone in E . Let $a, b \in E$ and $a < b$. We define [8]*

$$[a, b] := \{x \in E : x = tb + (1 - t)a, \text{ where } t \in [0, 1]\}$$

$$[a, b) := \{x \in E : x = tb + (1 - t)a, \text{ where } t \in [0, 1)\}$$

DEFINITION 3.3 *The set $\{a = x_0, x_1, \dots, x_n = b\}$ is called a partition for $[a, b]$ if and only if the sets $[x_{i-1}, x_i]$, $1 \leq i \leq n$ [8], are pairwise disjoint and*

$$[a, b] = \left\{ \bigcup_{i=1}^n [x_{i-1}, x_i] \right\} \cup \{b\}$$

DEFINITION 3.4 *Suppose that P is a normal cone in E , $\phi : [a, b] \rightarrow P$ a map. ϕ is said to be integrable on $[a, b]$ with respect to cone P (or cone integrable function) iff for all partition Q of $[a, b]$ [8]*

$$\lim_{n \rightarrow \infty} L_n^{Con}(\phi, Q) = S^{Con} = \lim_{n \rightarrow \infty} U_n^{Con}(\phi, Q)$$

where S^{Con} must be unique and:

$$L_n^{Con} = \sum_{i=0}^{n-1} \phi(x_{i+1}) \|x_i - x_{i+1}\| \quad (\text{Cone lower summation})$$

and

$$U_n^{Con} = \sum_{i=0}^{n-1} \phi(x_{i+1}) \|x_i - x_{i+1}\| \quad (\text{Cone upper summation})$$

we note

$$S^{Con} = \int_a^b \phi(x) d_P(x) = \int_a^b \phi d_P$$

The set of all cone integrable functions $\phi : [a, b] \rightarrow P$ is denoted $L^T([a, b], P)$.

DEFINITION 3.5 The function $\phi : P \rightarrow E$ is called subadditive cone integrable function iff $\forall a, b \in P$ [8]

$$\int_0^{a+b} \phi d_P \leq \int_0^a \phi d_P + \int_0^b \phi d_P$$

Example Let $E = X = \mathbf{R}$, $d(x, y) = |x - y|$, $P = [0, +\infty)$ and $\phi(t) = \frac{1}{t+1} \forall t > 0$. Then ϕ is a subadditive cone integral function.

THEOREM 3.6 Let (X, d) be a cone metric space and let P a normal cone. Let $\phi : P \rightarrow P$ be a nonvanishing map and a subadditive cone integrable on each $[a, b]$. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition

$$\int_0^{d(F(x,y), F(u,v))} \phi(t) d_P(t) \leq \varphi \left(\int_0^j(x,y,u,v) \phi(t) d_P(t) \right), \quad (18)$$

for all $x, y, u, v \in X$, where

$$j(x, y, u, v) \in \left\{ \frac{d(x, u) + d(y, v)}{2}, d(F(x, y), x), d(F(u, v), u), \frac{d(F(x, y), u) + d(F(u, v), x)}{2}, \frac{d(F(x, y), x) + d(F(u, v), u)}{2} \right\}$$

$\varphi : P \rightarrow P$ is a nondecreasing mapping satisfying $(\varphi_1) - (\varphi_4)$ and $T : P \rightarrow P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

Proof Theorem 3.6 is a Corollary of Theorem 2.1 when $T(j(x, y, u, v)) = \int_0^{j(x,y,u,v)} \phi d_P$. Under this case, T satisfies conditions $(T_1) - (T_2)$. (T_2) results from the subadditivity of ϕ . The condition (T_1) results from the continuity of T and its inverse in 0 . In fact, in a normal cone, if $w_n \rightarrow \ll w$, then w_n converges to w . Now, since T is continuous in 0 , for every sequence w_n converging to 0 , $T(w_n)$ converges to $T(0) = 0$. Since T^{-1} is continuous, given any sequence $T(w_n)$

converging to 0, $T^{-1}(T(w_n)) = w_n$ converges to $T^{-1}(0) = 0$; thus (T_1) is satisfied.

■

Theorem 3.6 lead to the following Corollaries:

COROLLARY 3.7 Let (X, d) be a cone metric space and let P a normal cone. Let $\phi : P \rightarrow P$ be a nonvanishing map and a subadditive cone integrable on each $[a, b]$. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition

$$\int_0^{d(F(x,y),F(u,v))} \phi(t)d_P(t) \leq \varphi \left(\int_0^{j(x,y,u,v)} \phi(t)d_P(t) \right) \tag{19}$$

for all $x, y, u, v \in X$, where

$$j(x, y, u, v) \in \left\{ \frac{d(x, u) + d(y, v)}{2}, d(F(x, y), x), d(F(u, v), u), \frac{d(F(x, y), u) + d(F(u, v), x)}{2} \right\}$$

, $\varphi : P \rightarrow P$ is a nondecreasing mapping satisfying $(\varphi_1) - (\varphi_4)$ and $T : P \rightarrow P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

COROLLARY 3.8 Let (X, d) be a cone metric space and let P a normal cone. Let $\phi : P \rightarrow P$ be a nonvanishing map and a subadditive cone integrable on each $[a, b]$. Suppose that the mapping $F : X \times X \rightarrow X$ satisfies the following contractive condition

$$\int_0^{d(F(x,y),F(u,v))} \phi(t)d_P(t) \leq \varphi \left(\int_0^{j(x,y,u,v)} \phi(t)d_P(t) \right) \tag{20}$$

for all $x, y, u, v \in X$, where

$$j(x, y, u, v) \in \left\{ \frac{d(x, u) + d(y, v)}{2}, d(F(x, y), x), d(F(u, v), u), \frac{d(F(x, y), u) + d(F(u, v), x)}{2} \right\}$$

, $\varphi : P \rightarrow P$ is a nondecreasing mapping satisfying $(\varphi_1) - (\varphi_4)$ and $T : P \rightarrow P$ is a nondecreasing mapping satisfying $(T_1) - (T_2)$. Then F has a unique coupled fixed point.

Remark 3.9.

- (i) Corollary 2.2 gives Corollary 3.7 when $T(j(x, y, u, v)) = \int_0^{j(x,y,u,v)} \phi d_P$ and
- (ii) Corollary 2.3 gives Corollary 3.8 when $T(j(x, y, u, v)) = \int_0^{j(x,y,u,v)} \phi d_P$.

Therefore our Theorem 3.6 is Coupled fixed point of integral type version of Theorem 2.1.

4. Conclusion

In this paper, we unified, extended and generalized some results on coupled fixed point theorems of generalized φ - mappings with some applications to fixed points of integral type mappings in cone metric spaces.

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