An Adaptive Wavelet Solution to Generalized Stokes Problem

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Abstract. In this paper we will present an adaptive wavelet scheme to solve the generalized Stokes problem. Using divergence free wavelets, the problem is transformed into an equivalent matrix vector system, that leads to a positive definite system of reduced size for the velocity. This system is solved iteratively, where the application of the infinite stiffness matrix, that is sufficiently compressible, is replaced by an adaptive approximation. Finally we prove that this adaptive method has optimal computational complexity, that is it recovers an approximate solution with desired accuracy at a computational expense that stays proportional to the number of terms in a corresponding wavelet-best N-term approximation.

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1.   Introduction

An important example for a system of partial differential equation is a simple model for viscous incompressible flow which is given by

$$
\begin{cases}
-\alpha \triangle \overrightarrow{u} + \beta \overrightarrow{u} + \nabla p = \overrightarrow{f} & \text{in } \Omega, \\[2mm]
\quad\quad \text{div} \overrightarrow{u} = 0 & \text{in } \Omega, \\[2mm]
\quad\quad\quad \overrightarrow{u} = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1}
$$

where the vector valued function $\overrightarrow{u}$ and the scalar field p represent velocity and pressure of the fluid, respectively and α, β are positive constant numbers. Obviously, one has to factor the constants from p, for instance, by requiring $\int_\Omega p(x)dx = 0$. The solution of (1) exist and is unique for any bounded and connected $\Omega \subseteq \Re^n$ with Lipschitz boundary condition [3]. In special case that $\alpha = 1$, $\beta = 0$ the problem is solved by Yingchun Jiang (see [12]). By mixed weak formulation we change our problem to a saddle point problem and hence to indefinite system [6, 8, 10]. Also by using divergence-free weak formulation we obtain a positive definite linear system of reduced size involving only velocity. Note that the pressure can be obtained by means of a postprocessing [11]. By adaptive wavelet methods we obtain an approximation solution for this system of equations, that is a nonlinear approximation [2]. However we can see a linear approximation in [14].

The adaptive method that we use in this paper to solve (1) takes the following form:

Suppose that we have a wavelet basis$\{\psi_\lambda\}_{\lambda\in\Lambda}$ to be used for numerically resolving the equation (1). Our adaptive scheme will iteratively produce finite sets $\Lambda_j \subseteq \Lambda, j = 1, 2, ...$ and the Galerkin approximation u_{Λ_j} to u from the space $S_{\Lambda j} := span(\{\psi_\lambda\}_{\lambda\in\Lambda_j})$.

The function u_{Λ_j} is a linear combination of $N_j := \#\Lambda_j$ wavelets, where $\#\Lambda_j$ is the cardinal of Λ_j. Thus the adaptive method can be viewed as a particular form of nonlinear N-term wavelet approximation and a benchmark for the performance of such an adaptive method is provided by comparison with best N-term approximation (in the energy norm) when full knowledge of u is available.

2.   Idea

Suppose Λ is a countable set and

$$
\Psi = \{\psi_\lambda : \lambda \in \Lambda\}
$$

is a basis for Hilbert space H. Every $v \in H$ has an expansion in terms of elements of Ψ

$$
v := \sum_{\lambda\in\Lambda} d_\lambda \psi_\lambda.
\tag{2}
$$

The coefficients d_λ can be expressed via the dual basis, this is a collection of functionals

$$
\widetilde{\Psi} = \{\widetilde{\psi}_\lambda : \lambda \in \Lambda\}
$$

such that

$$\langle \psi_\lambda, \widetilde{\psi}_\lambda \rangle = \delta_{\lambda,\lambda'} \quad \lambda, \lambda' \in \Lambda \tag{3}$$

where $\langle .,. \rangle$ denotes the inner product on H. In fact the coefficients d_λ in (2) are given by

$$d_\lambda = \langle v, \widetilde{\psi}_\lambda \rangle.$$

To simplify further expansion we introduce some compact notations:
Let us consider a given countable collection of functions Φ in H as a column vector, so that an expansion with coefficients $c_\phi, \phi \in \Phi$ can be formally treated as an inner product

$$C^T \Phi := \sum_{\phi \in \Phi} c_\phi \phi,$$

where C^T denotes the transpose of C. Likewise for any $v \in H$ the quantities $\langle \Phi, v \rangle$ and $\langle v, \Phi \rangle$ denote the column and row vector of coefficients $\langle \phi, v \rangle$ and $\langle \phi, v \rangle$, $\phi \in \Phi$, respectively.
Therefore (2) can be written briefly as $d^T \Psi$. Also for two countable collections Φ, Ψ of functions, we consider the matrix

$$\langle \Phi, \Psi \rangle = (\langle \phi, \psi \rangle)_{\phi \in \Phi, \psi \in \Psi}.$$

Specifically, the above biorthogonality relation (3) becomes

$$\langle \Psi, \widetilde{\Psi} \rangle = I$$

where I denotes the identity matrix.
Now suppose that Ł is a bounded linear bijection that maps H into H^*, dual space of H, that is

$$\|Łv\|_{H^*} \sim \|v\|_H \quad , v \in H$$

where a $\sim$ b means that,there are constants c_1, c_2 such that

$$c_1 a \leqslant b \leqslant c_2 a.$$

Note that in this case, by Riesz representation theorem, the equation

$$Łu = f \tag{4}$$

has a unique solution $u \in H$ for every $f \in H^*$.
The basic idea is to transform the equation (4) into an infinite discrete system of equations. This can be done with the aid of suitable bases for the underlaying spaces. Given such bases seeking the solution u of (4) is equivalent to finding the expansion of the sequence d such that $u = d^T \Psi$. Inserting this into (4) yields $(Ł\Psi)^T d = f$. This gives the the following system of equations

$$\langle Ł\Psi, \Psi\rangle^T d = \langle f, \Psi\rangle^T. \tag{5}$$

When Ψ is a wavelet basis, the solution to (5) gives the wavelet coefficients of the solution u to (4).

An advantage of wavelet bases is that they allow for trivial preconditioning of the linear system (5) [4]. This preconditioning is given by the matrix D, where D is a fixed positive diagonal matrix such that

$$\|D^{-1}d\|_{\ell_2(\Lambda)} \sim \|d^T\Psi\|_H. \tag{6}$$

The result is

$$D\langle Ł\Psi, \Psi\rangle^T D D^{-1} d = D\langle f, \Psi\rangle^T,$$

or

$$AU = F, \tag{7}$$

where

$$A := D\langle Ł\Psi, \Psi\rangle^T D, U := D^{-1}d, F := D\langle f, \Psi\rangle^T \in \ell_2(\Lambda). \tag{8}$$

Now assume that Ł is positive definite and selfadjoint, that is, the bilinear form a defined on H×H by

$$a(u, v) := \langle Łu, v\rangle,$$

is symmetric. Also assume that Ł is elliptic in the sense that

$$a(v, v) \sim \|v\|_H^2, v \in H.$$

It follows that H is also a Hilbert space with respect to the inner product a and this inner product induces an equivalent norm, called the energy norm, on H by

$$\|.\|_a^2 := a(.,.).$$

Combining this with (6), we obtain that $\|.\|_a$ and $\|.\|_{\ell_2(\Lambda)}$ are equivalent, thus there exist constants $c_1, c_2 > 0$ such that,

$$c_1\|.\|_{\ell_2(\Lambda)}^2 \leqslant \|.\|_a^2 \leqslant c_2\|.\|_{\ell_2(\Lambda)}^2. \tag{9}$$

It follows that the unique solution of (4) is also the unique solution of the variational equation

$$a(u, v) = \langle f, v\rangle, v \in H.$$

The typical examples satisfying in the above assumptions are Poisson's, Helmholtz or the biharmonic equations on bounded domains in $\Re^n$. In these examples H is a Sobolev space; [7, 9].

3. Divergence Free-Wavelets

In this section we present a divergence-free wavelet basis for the space

$$\overrightarrow{H}(\Re^N) := \{\overrightarrow{f} \in \overrightarrow{H}(div; \Re^n) : div\, \overrightarrow{f} = 0\},$$

of divergence-free vector fields in $L^2(\Re^n)^n$, where

$$\overrightarrow{H}(div; \Re^n) := \{\overrightarrow{f} \in L^2(\Re^n)^n : div\, \overrightarrow{f} \in L^2(\Re^n)\},$$

which will be used later,(for more information see [13]).

Assume that ϕ is an a-refinable function in the sense that

$$\phi(x) = \sum_{k \in \mathbb{Z}^n} a_k \phi(2x - k).$$

The Laurant series

$$a(z) := \sum_{k \in \mathbb{Z}^n} a_k z^k$$

with $z = e^{-i\xi}, \xi \in \Re^n$ is called the symbol of ϕ.

For abbreviation we use the following notations:

$$E := \{(e_1, e_2, ..., e_n)^T : e_i \in \{0, 1\}\}, E^* := E \backslash \{0\}.$$

For $e \in E$ we define

$$\phi_e(.) := \phi(2. - e).$$

Since $\mathbb{Z}^n = \bigcup_{e \in E} \{e + 2k : k \in \mathbb{Z}^n\}$ and $\phi_e(x - k) = \phi(2x - (e + 2k))$ then

$$\sum_{k \in \mathbb{Z}^n} a_k \phi(2x - k) = \phi(x) = \sum_{e \in E} \sum_{k \in \mathbb{Z}^n} a_{e+2k} \phi_e(x - k)$$

where the sequences $\{a_{e+2k}\}_{k \in \mathbb{Z}^n}$ determine the subsymbols

$$a_e(z) := \sum_{k \in \mathbb{Z}^n} a_{e+2k} z^k$$

of the symbol $a(z)$ of ϕ. Since

$$a(z) = \sum_{e \in E} z^e a_e(z),$$

it is clear that $a(z)$ is determined by its subsymbols.

We will define functions ψ_e for $e \in E^*$, such that this system is ℓ^2-stable. To this end, we choose symbols $a^e(z), (e \in E^*)$, such that the functions ψ_e are defined by the subsymbols of $a^e(z)$:

$$\psi_e(x) := \sum_{\widetilde{e} \in E} \sum_{k \in \mathbb{Z}^n} a^e_{\widetilde{e}+2k} \phi_{\widetilde{e}}(x - k).$$

Also, define a new system $\psi_e^{(\nu)}$ via its symbol $a^{(\nu),e}(z)$ which is given by:

$$a^{\nu,e}(z) := a^e(z) \begin{cases} \frac{2}{1+z_\nu} & \text{if } e_\nu = 0, \\ \\ \frac{1-z_\nu}{2} & \text{if } e_\nu = 1. \end{cases}$$

DEFINITION 3.1   *Let $e \in E^*$, i_e be an index $(1 \leqslant i_e \leqslant n)$ with $e_{i_e} = 1$. For $i \neq i_e$ we define:*

$$(\overrightarrow{\psi}_{e,i}^{\Lambda})_j := \begin{cases} 0 & \text{if } j \notin \{i, i_e\} , \\ \\ \psi_e^{((i))} & \text{if } j = i, \\ \\ -\frac{1}{4}(\frac{\partial}{\partial x_i}\psi_e^{((i,i_e))}) & \text{if } j = i_e , \end{cases} \tag{10}$$

where $\psi_e^{((i))} := \psi_e^{(1,\dots,i-1,i+1,\dots,n)}$, $\psi_e^{((0))} := \psi_e^{(1,\dots,n)}$, and $\psi^{(i,j)} = (\psi^{(i)})^{(j)}$.

It is clear that i_e is not uniquely determined, but this is not important for finding a basis for $\overrightarrow{H}(\Re^n)$ [13].

THEOREM 3.2   *Let $\varphi \in L^2(\Re^n)$ be ℓ^2-stable, compactly supported and a^*-refinable such that its Fourier transform $(\hat{\varphi})$ satisfies*

$$|\hat{\varphi}(\xi)| \leqslant C(1 + \|\xi\|)^{-\epsilon-n/2}.$$

Let $\phi \in L^2(\Re^n)$ be the a-refinable function defined by the symbol

$$a(z) = a^*(z) \prod_{\nu=1}^{n} \frac{1 + z_\nu}{2}.$$

Moreover, suppose $\{\psi_e : e \in E^*\}$ is a pre-wavelet system of compactly supported functions induced by ϕ such that the corresponding symbols $a^e(z)$ are divisible by $1 + z_\nu$, if $e_\nu = 0$ and that the resulting symbols are finite. Then the system

$$\{\overrightarrow{\psi}_{e,i,j,k}^{\Lambda} : (e, j, k) \in E^* \times \mathbb{Z} \times \mathbb{Z}^n, i \neq i_e\} \tag{11}$$

defined by (10) forms a Riesz-basis for $\overrightarrow{H}(\Re^n)$.

Proof See [13].   ∎

4.   Basic Facts

Let $\mathbb{Z}_+$ be the set of non-negative integers. For any $m \in \mathbb{Z}_+$, $C^m(\Omega)$ is the space of functions which, together with their derivatives of order less than or equal to m, are continuous on Ω that is,

$$C^m(\Omega) = \{v \in C(\Omega) \mid \partial^\alpha v \in C(\Omega) \text{ for } |\alpha| \leqslant m\}.$$

We set

$$C^\infty(\Omega) = \{v \in C(\Omega) | \; v \in C^m(\Omega) \; \forall m \in \mathbb{Z}_+\}.$$

Also we say that $v \in C(\Omega)$ has a *compact support* if $supp(v) = \overline{\{x \in \Omega| \; v(x) \neq 0\}}$ is a proper compact subset of Ω. Later on, we need the space

$$C_0^\infty(\Omega) = \{v \in C^\infty(\Omega) | \; supp(v) \; \text{is a proper subset of} \; \Omega\}.$$

In order to give an adaptive wavelet solution to (1) we consider the space $H := \{\overrightarrow{f} \in H_0^1(\Omega)^n : div\,\overrightarrow{f} = 0\}$ where $H^1(\Omega)$ is a Sobolev space and $H_0^1(\Omega)$ is the completeness of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. It is obvious that $H \subseteq \overrightarrow{H}(\Omega) \bigcap \overrightarrow{H}_0(div; \Omega)$ where $\overrightarrow{H}_0(div; \Omega) := clos_{\overrightarrow{H}(div;\Omega)} C_0^\infty(\Omega)^n$ and $\overrightarrow{H}(\Omega)$ is defined as in the first paragraph of section 3. Now assume that Λ^{df} is a countable set such that

$$\Psi^{df} := \{\overrightarrow{\psi}_\lambda^{df} : \lambda \in \Lambda^{df}\}$$

satisfy in (10) forms a Riesz basis for H. We use $|\lambda| := j$ to denote the level of the wavelets.It was shown in [15] that

$$\| \sum_{\lambda \in \Lambda^{df}} c_\lambda \overrightarrow{\psi}_\lambda^{df} \|_{H^1(\Omega)^n} \sim \sum_{\lambda \in \Lambda^{df}} 2^{2|\lambda|} |c_\lambda|^2. \tag{12}$$

According to section 2 we can discretize the problem (1) for $\alpha, \beta = 1$ (the other cases are similar). The weak formulation of the problem (1) is

$$\int_\Omega (\nabla u \nabla v + uv) dx = \int (fv) dx, \;\; \forall v.$$

Thus the bilinear form $a(.,.)$ in section 2 has the following form

$$a(u, v) = \langle \nabla u, \nabla v \rangle + \langle u, v \rangle.$$

For $\overrightarrow{u} \in H$ we have $\overrightarrow{u} = \sum_{\lambda \in \Lambda^{df}} d_\lambda \psi_\lambda^{df}$, thus (1) is equivalent to

$$a(\overrightarrow{u}, \overrightarrow{\psi}_{\lambda'}^{df}) = \langle \overrightarrow{f}, \overrightarrow{\psi}_{\lambda'}^{df} \rangle \;\;\;\; \forall \lambda' \in \Lambda^{df},$$

and in turns, it is equivalent to

$$a(\sum_{\lambda \in \Lambda^{df}} d_\lambda \overrightarrow{\psi}_\lambda^{df}, \overrightarrow{\psi}_{\lambda'}^{df}) = \langle \overrightarrow{f}, \psi_{\lambda'}^{df} \rangle,$$

or

$$\sum_{\lambda \in \Lambda^{df}} d_\lambda a(\overrightarrow{\psi}_\lambda^{df}, \psi_{\lambda'}^{df}) = \langle \overrightarrow{f}, \psi_{\lambda'}^{df} \rangle.$$

Now let $D := diag(2^{-|\lambda|})_{\Lambda^{df} \times \Lambda^{df}}$ [4], thus by (7) our problem is equivalent to

$$AU = F, \tag{13}$$

where A, U, F, satisfy (8).It is easy to show that the matrix A is quasi-sparse in the sense that if $A = (a_{\lambda,\lambda'})_{\lambda,\lambda' \in \Lambda^{df}}$ then there is a constant C_A such that for some $\sigma > \frac{n}{2}$ and $\beta > n$

$$|a_{\lambda,\lambda'}| \leqslant C_A 2^{-||\lambda|-|\lambda'||\sigma}(1 + d(\lambda, \lambda')^{-\beta})$$

where $d(\lambda, \lambda') := 2^{min\{|\lambda|,|\lambda'|\}} dist(supp(\psi_\lambda), supp(\psi_{\lambda'}))$ [1, 9]. Also it was shown in [5] that, this quasi-sparse matrix defines a bounded operator on $\ell^2(\Lambda^{df})$.

Note that the bilinear form a is symmetric and bounded on $H_0^1(\Omega)$, also by Sobolev inequality we have

$$a(\overrightarrow{u}, \overrightarrow{u}) \sim \|\overrightarrow{u}\|_{H_0^1(\Omega)^n}^2, \quad \forall \overrightarrow{u} \in H_0^1(\Omega)^n.$$

Because of boundedness of A on $\ell^2(\Lambda^{df})$ we have

$$\|AU\|_{\ell^2(\Lambda^{df})} \leqslant C\|U\|_{\ell^2(\Lambda^{df})}, \tag{14}$$

also $a(\overrightarrow{u}, \overrightarrow{v}) = \langle AU, V \rangle$ and (12) imply that

$$\langle AU, U \rangle = a(\overrightarrow{u}, \overrightarrow{u}) \geqslant c_1\|\overrightarrow{u}\|_{H_0^1(\Omega)^n}^2 \geqslant c_A\|U\|_{\ell^2(\Lambda^{df})}^2, \tag{15}$$

that means, A is positive definite.Also, since

$$c_A\|U\|_{\ell_2(\Lambda^{df})}^2 \leqslant \langle AU, U \rangle \leqslant \|AU\|_{\ell_2(\Lambda^{df})}\|U\|_{\ell_2(\Lambda^{df})},$$

we have

$$c_A\|U\|_{\ell_2(\Lambda^{df})} \leqslant \|AU\|_{\ell_2(\Lambda^{df})} \leqslant C_A\|U\|_{\ell_2(\Lambda^{df})}. \tag{16}$$

By the above we have the following theorem:

THEOREM 4.1 *The matrix A in (13) is quasi-sparse, bounded, positive definite, symmetric and also*

$$\|AU\|_{\ell^2(\Lambda^{df})} \sim \|U\|_{\ell^2(\Lambda^{df})}, \quad \forall U = D^{-1}d, \tag{17}$$

where $\overrightarrow{u} = d\Psi^{df}$.

5. Adaptive Scheme

In this section we shall present an adaptive algorithm in order to resolve the solution to (13). This algorithm generates step by step an ascending sequence of (nested) Λ_j and approximate solutions U_{Λ_j} supported in Λ_j so that on the one hand the cardinality of $(\Lambda_j \backslash \Lambda_{j-1})$ which is denoted by $\#(\Lambda_j \backslash \Lambda_{j-1})$ stays as small as possible, while on the other hand the error estimate satisfy the following property

$$\|U - U_{\Lambda_j}\|_{\ell^2(\Lambda^{df})} \leqslant \epsilon_j := 2^{-j}\epsilon_0,$$

where ϵ_0 is an upper bound for $\|U\|_{\ell^2(\Lambda^{df})}$.
Note that the matrix A is quasi-sparse and it is compressible of order s for $s > min\{\frac{\sigma}{n} - \frac{1}{2}, \frac{\beta}{n} - 1\}$. Thus there are two positive sequences $(\alpha_j)_{j \geqslant 0}$ and $(\beta_j)_{j \geqslant 0}$ that

are both summable and for every $j \geqslant 0$ there exists a matrix A_j with at most $2^j \alpha_j$ nonzero entries per row and per column such that

$$\|A - A_j\| \leqslant 2^{-js}\beta_j. \tag{18}$$

Now let $V \in \ell^2(\Lambda^{df})$, for each $n \geqslant 1$ let v_n^* be the n-th largest of the number $|v_\lambda|$ and let $V^* := (v_n^*)_{n=1}^\infty$. For each $0 < \tau < 2$ we let $\ell_\tau^\omega(\Lambda^{df})$ denote the collection of all vectors $V \in \ell^2(\Lambda^{df})$ for which

$$|V|_{\ell_\tau^\omega(\Lambda^{df})} := sup_{n \geqslant 1} n^{\frac{1}{\tau}} v_n^*$$

is finite. This expression defines a quasi norm for $\ell_\tau^\omega(\Lambda^{df})$ and we define

$$\|V\|_{\ell_\tau^\omega(\Lambda^{df})} := |V|_{\ell_\tau^\omega(\Lambda^{df})} + \|V\|_{\ell^2(\Lambda^{df})}.$$

Hence we have a norm for $\ell_\tau^\omega(\Lambda^{df})$. It is shown in [5], That the matrix A maps $\ell_\tau^\omega(\Lambda^{df})$ boundedly into itself for $\tau = (\frac{1}{2} + s)^{-1}$ that is, for every $V \in \ell_\tau^\omega(\Lambda^{df})$, we have

$$\|AV\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant C\|V\|_{\ell_\tau^\omega(\Lambda^{df})}.$$

Following [5], for an accuracy η, we assume that we have the following routines at our disposal.

COARSE $[W, \eta] \rightarrow (\Lambda, \bar{W})$
(i) Define $N := \#(supp \ W)$ and sort the nonzero entries of W into decreasing order in module and obtain the vector $\lambda^* := (\lambda_1, ..., \lambda_N)$ of indices which gives the decreasing rearrangement $W^* = (|w_{\lambda_1}|, ..., |w_{\lambda_N}|)$ of nonzero entries of W; then compute $\|W\|_{\ell^2(\Lambda^{df})} = \sum_{i=1}^N |w_{\lambda_i}|^2$.

(ii) For $k = 1, 2, ...$, form the sum $\sum_{j=1}^k |w_{\lambda_j}|^2$ in order to find the smallest value k such that this sum exceeds $\|W\|_{\ell_2(\Lambda^{df})}^2 - \eta^2$. For this k define $K = k$ and set $\Lambda := \{\lambda_j : j = 1, ..., K\}$; define $\bar{W}$ by $\bar{w}_\lambda = w_\lambda$ for $\lambda \in \Lambda$ and $\bar{w}_\lambda = 0$ for $\lambda \notin \Lambda$.

Now, let $0 < \eta < \|V\|_{\ell_2(\Lambda^{df})}$ and W be a finitely supported approximation to V such that $\|V - W\|_{\ell_2(\Lambda^{df})} \leqslant d\eta$ for some $d < 1$, then it is obvious that the **COARSE** $[W, (1 - d)\eta]$ produces $\bar{W}$ supported on Λ which $\|V - \bar{W}\|_{\ell_2(\Lambda^{df})} \leqslant \eta$. (Note that the output $\bar{W}$ of **COARSE**, by construction, satisfies $\|W - \bar{W}\|_{\ell_2(\Lambda^{df})} \leqslant \eta$). Moreover, we have the following lemma [5]:

LEMMA 5.1 *If* $V \in \ell_\tau^\omega(\Lambda^{df}), \tau = (s + \frac{1}{2})^{-1}$, *for some* $s > 0$ *then the outputs* $\bar{W}$, Λ *of* **COARSE** *requires at most* $2N$ *arithmetic operations and* $NlogN$ *sorts,where* $N = \#supp(W)$. *Moreover,*

$$\|\bar{W}\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant C\|V\|_{\ell_\tau^\omega(\Lambda^{df})} \tag{19}$$

and Λ (*the cardinality of* $supp(\bar{W})$) *is bounded by*

$$\#(\Lambda) \leqslant C\|V\|_{\ell_\tau^\omega(\Lambda^{df})}^{\frac{1}{s}} \eta^{\frac{1}{s}}. \tag{20}$$

COROLLARY 5.2 *If $\bar{F}$ is an optimal N-term approximation of the data F with accuracy η, then for $\widetilde{\eta} \geqslant \eta$ **COARSE** $[\bar{F}, \widetilde{\eta} - \eta]$ produces an approximation G to F with support $\Lambda_{\widetilde{\eta}}$, such that $\|G - F\|_{\ell_2(\Lambda^{df})} \leqslant \widetilde{\eta}$. Moreover, if $F \in \ell_\tau^\omega(\Lambda^{df})$, then*

$$\#(\Lambda_{\widetilde{\eta}}) \leqslant C\widetilde{\eta}^{\frac{-1}{s}} \|F\|_{\ell_\tau^\omega(\Lambda^{df})}^{\frac{1}{s}},$$

with C depending only on s.

To simplify our notations we will denote the notation **COARSE** $[F, \widetilde{\eta}]$ instead of the output of **COARSE** $[\bar{F}, \widetilde{\eta} - \eta]$.
Now let V be a vector of finite support and $N = \#supp \ V$ and $V_{[j]}$ be the vector that agrees with V in its 2^j largest entries and zero otherwise for $j = 1, ..., log[N]$ and for $j > [logN]$ let $V_{[j]} = V$ (note that this process requires at most $NlogN$ sorts).

APPLY $A[V, \eta] \rightarrow (W, \Lambda)$:
(i) Sort the nonzero entries of the vector V and form the vectors $V_{[0]}, \ V_{[j]} - V_{[j-1]}, \ \ j = 1, ..., [logN]$ with $N := supp \ V$ and define $V_{[j]} := V$ for $j > logN$.

(ii) Compute $\|V\|_{\ell_2(\Lambda^{df})}^2, \ \ \|V_{[0]}\|_{\ell_2(\Lambda^{df})}^2, \ \ \|V_{[j]} - V_{[j-1]}\|_{\ell_2(\Lambda^{df})}^2, \ \ j = 1, ..., [logN].$

(iii) Set $k = 0$
    (a) Compute $R_k \ := \ C_A\|V - V_{[k]}\|_{\ell_2(\Lambda^{df})} + 2^{-ks}\beta_k\|V[0]\|_{\ell_2(\Lambda^{df})} + \sum_{j=0}^{k-1} 2^{-js}\beta_j\|V_{[k-j]} - V_{[k-j-1]}\|_{\ell_2(\Lambda^{df})}$, where C_A and β_j were introduced in (16) and (18).

    (b) If $R_k \leqslant \eta$ stop and output k, otherwise replace k by $k + 1$ and return to (a).

(iv) For the output k of **(iii)** and for $j = 0, 1, ..., k$ compute nonzero entries in the matrices A_{k-j} which have a column index in common with one of the nonzero entries of $V_{[j]} - V_{[j-1]}$.

(v) For the output k of **(iii)** compute W_k as the follow

$$W_k := A_k V_{[0]} + A_{k-1}(V_{[1]} - V_{[0]}) + ... + A_0(V_{[k]} - V_{[k-1]}),$$

and take $W := W_k$ and $\Lambda := suppW$, where A_k are as in (18).

LEMMA 5.3 *The outputs W, Λ of **APPLY** $A[\eta, V]$ have the following properties:*
(i)

$$\|AV - W\|_{\ell_2(\Lambda^{df})} \leqslant \eta. \tag{21}$$

Moreover if $V \in \ell_\tau^\omega(\Lambda^{df})$ with $\tau = (s + \frac{1}{2})^{-1}$ then
(ii)

$$\#(\Lambda) \leqslant C\|V\|_{\ell_\tau^\omega(\Lambda^{df})}^{\frac{1}{s}} \eta^{-\frac{1}{s}} \tag{22}$$

and

$$\|W\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant C\|V\|_{\ell_\tau^\omega(\Lambda^{df})}. \tag{23}$$

(iii) *The number of arithmetic operations needed to compute* (W, η) *bounded by* $C\eta^{-\frac{1}{s}}\|V\|_{\ell_\tau^\omega(\Lambda^{df})}^{\frac{1}{s}}\eta^{-\frac{1}{s}} + 2N$ *with* $n := suppV$ *and the number of sorts needed to compute* W *is bounded by* $CNlogN$.

Now,based on the above routines we are prepared to describe our algorithm in order to construct finitely supported U^j, $j = 0, 1, ...,$ which approximate the solution U of(7).

Assume that ϵ is the target accuracy, α is the real number such that $\|I - \alpha A\| := \rho < 1$, ($\alpha$ exists since A is a positive definite matrix) and $K := min\{l \in N : \rho^{l-1}(2\alpha c_2 l + \rho) \leqslant (\frac{8c_2}{c_1} + 2)^{-1}\}$, where c_1, c_2 are as in (9).

SOLVE $[\epsilon, A, F] \rightarrow (U_\epsilon, \Lambda_\epsilon)$

(i) Set $U^0 = 0, \Lambda_0 = \emptyset, \epsilon_0 := c_A^{-1}c_2\|F\|_{\ell_2(\Lambda^{df})}$.

(ii) While $\epsilon_j > c_1\epsilon$, do

    (ii.1) Set $V^0 := U^j$
    (ii.2) For $l = 0, 1, ..., K - 1$:

        (1) **COARSE** $[F, \rho^l\epsilon_j] \rightarrow (F_l, \Lambda_{l,F})$
        (2) **APPLY** $A[V^l, \rho^l\epsilon_j] \rightarrow (W^l, \Lambda_{l,A})$
        (3) $V^{l+1} := V^l - \alpha(W^l - F_l)$

    (ii.3) **COARSE**$[V^K, \frac{4}{c_1}(\frac{8c_2}{c_1} + 2)^{-1}\epsilon_j] \rightarrow (\Lambda_{j+1}, U^{j+1})$
    (ii.4) Set $\epsilon_{j+1} := \frac{\epsilon_j}{2}, j + 1 \rightarrow j$

(iii) Accept $U_\epsilon := U^j$ as a solution.

In the following theorem we give the error estimation for $\|U - U_\epsilon\|$.

THEOREM 5.4 *Let ϵ be the target accuracy and the solution U of $AU = F$ belongs to $\ell_\tau^\omega(\Lambda^{df})$ for $\tau = (s + \frac{1}{2})^{-1}$ with $0 < s < min\{\frac{\sigma}{n} - \frac{1}{2}, \frac{\beta}{n} - 1\}$ then*

$$\|U - U_\epsilon\|_{\ell_2(\Lambda^{df})} \leqslant \epsilon. \tag{24}$$

Proof By the step **(ii)** in **SOLVE**, it is enough to show that

$$\|U - U^j\|_{\ell_2(\Lambda^{df})} \leqslant c_1^{-1}\epsilon_j. \tag{25}$$

we proceed, by induction. By (9) and (16) we have

$$\|U - U^0\|_{\ell_2(\Lambda^{df})} = \|U\|_{\ell_2(\Lambda^{df})} \leqslant \|A^{-1}\|\|F\|_{\ell_2(\Lambda^{df})} \leqslant c_A^{-1}\|F\|_{\ell_2(\Lambda^{df})} = \epsilon_0 c_1^{-1},$$

therefore (25) is true for $j = 0$.

Now assume that (25) is true for j, that is $\|U - U_j\|_{\ell_2(\Lambda^{df})} \leqslant c_1^{-1}\epsilon_j$.

Let $U^j(V)$ be the exact iterates with initial V, then

$$U^{l+1}(U^j) = U^l(U^j) + \alpha(F - AU^l(U^j)),$$

combining this with definition of V^{l+1}, we obtain

$$V^{l+1} - U^{l+1}(U^j) = V^l + \alpha(F_l - W^l) - U^l(U^j) - \alpha(F - AU^l(U^j))$$

$$= (I - \alpha A)(V^l - U^l(U^j)) + \alpha((F_l - F) + (AV^l - W^l)).$$

Hence by step **(ii)** of **SOLVE** and properties of **COARSE** and **APPLY** (lemma (5.1) and lemma (5.2)) we have

$$\|V^{l+1} - U^{l+1}(U^j)\|_{\ell_2(\Lambda^{df})} \leqslant \rho\|V^l - u^l(U^j)\|_{\ell_2(\Lambda^{df})} + 2\alpha\rho^l\epsilon_j,$$

since $V^0 = U^j$ and $U^0(U^j) = U^j$, then

$$\|V^{l+1} - U^{l+1}(U^j)\|_{\ell_2(\Lambda^{df})} \leqslant \rho^{l+1}\|V^0 - U^0(U^j)\|_{\ell_2(\Lambda^{df})}$$

$$+2\alpha\epsilon_j\sum_{i=0}^{l}\rho^i\rho^{l-i} = 2\alpha\epsilon_j\sum_{i=0}^{l}\rho^l = 2\alpha\epsilon_j(l+1)\rho^l,$$

therefore by setting K instead $l + 1$,

$$\|V^K - U^K(U^j)\|_{\ell_2(\Lambda^{df})} \leqslant 2\alpha\epsilon_j K\rho^{K-1}. \tag{26}$$

Also by induction assumption, (26) and definition of K:

$$\|V^K - U\|_{\ell_2(\Lambda^{df})} \leqslant \|V^K - U^K(U^j)\|_{\ell_2(\Lambda^{df})} + \|U^K(U^j) - U\|_{\ell_2(\Lambda^{df})}$$

$$\leqslant 2\alpha K\epsilon_j\rho^{K-1} + \rho^K\|U - U^j\|_{\ell_2(\Lambda^{df})} \leqslant 2\alpha K\epsilon_j\rho^{K-1} + \rho^K\epsilon_j c_1^{-1}$$

$$= (2\alpha K + \rho c_1^{-1})\rho^K\epsilon_j = c_1^{-1}(2\alpha K c_1 + \rho)\rho^{K-1}\epsilon_j$$

$$\leqslant (\frac{8c_2}{c_1} + 2)^{-1}\epsilon_j c_1^{-1}.$$

This inequality and **(ii.3)** in **SOLVE** yield to:

$$\|U^{j+1} - U\|_{\ell_2(\Lambda^{df})} \leqslant \|U^{j+1} - V^K\|_{\ell_2(\Lambda^{df})} + \|V^K - U\|_{\ell_2(\Lambda^{df})}$$

$$\leqslant \frac{4c_2}{c_1^2}(\frac{8c_2}{c_1} + 2)^{-1}\epsilon_j + (\frac{8c_2}{c_1} + 2)^{-1}\epsilon_j c_1^{-1}$$

$$= \frac{\epsilon_j}{c_1}(\frac{8c_2}{c_1} + 2)^{-1}(\frac{4c_2}{c_1} + 1) = \frac{\epsilon j}{2}c_1^{-1}$$

$$= \epsilon_{j+1}c_1^{-1},$$

as we desired. ■

THEOREM 5.5 *The number of arithmetic operations used to compute U_ϵ is bounded by a multiple of $\epsilon^{\frac{-1}{s}}$ and the number of sort operations is bounded by a multiple of $\epsilon^{\frac{-1}{s}}|log\epsilon|$.*

Proof By corollary 5.2 the number of arithmetic operations arising from **COARSE** in step **(ii.2)** from j to j+1 is bounded by

$$C(\sum_{l=0}^{K-1} \rho^{\frac{-l}{s}})\epsilon_j^{\frac{-1}{s}}\|F\|_{\ell_\tau^\omega(\Lambda^{df})}^{\frac{1}{s}},\tag{27}$$

and **APPLY** needs at most

$$C(\sum_{l=0}^{K-1} \rho^{\frac{-l}{s}}\|V^l\|_{\ell_\tau^\omega(\Lambda^{df})}) + 2\sum_{l=0}^{K} N_l\tag{28}$$

arithmetic operations and

$$\sum_{l=0}^{K} N_l \log N_l\tag{29}$$

sorts, where $N_l := suppV^l$.

By lemma 5.3

$$\|W^l\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant C\|V^l\|_{\ell_\tau^\omega(\Lambda^{df})},$$

also by lemma 5.1

$$\|F_l\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant \|F\|_{\ell_\tau^\omega(\Lambda^{df})},$$

but ,according to **SOLVE** we have $V^l = V^{l-1} + \alpha(W^{l-1} - F_{l-1})$, therefore

$$\|V^l\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant C(\|V^{l-1}\|_{\ell_\tau^\omega(\Lambda^{df})} + \|F_{l-1}\|_{\ell_\tau^\omega(\Lambda^{df})}).$$

Consequently,

$$\|V^l\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant C(\|V^0\|_{\ell_\tau^\omega(\Lambda^{df})} + \|F\|_{\ell_\tau^\omega(\Lambda^{df})}).\tag{30}$$

Now assume that $U_\epsilon := U^{j+1}$ for some fixed $j \in N$, then

$$\|V^0\|_{\ell_\tau^\omega(\Lambda^{df})} = \|U^j\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant \|U\|_{\ell_\tau^\omega(\Lambda^{df})},$$

from this and (30), we can conclude that $\|V^l\|_{\ell_\tau^\omega(\Lambda^{df})}$ for $1 \leqslant l \leqslant K$ is bounded, that is

$$\|V^l\|_{\ell_\tau^\omega(\Lambda^{df})} \leqslant C.\tag{31}$$

By definition of V^l, we have

$$\#(suppV^l) \leqslant \#(suppV^{l-1}) + \#(suppF_{l-1}) + \#(suppW^{l-1}),$$

now according to the corollary 5.2, $\#(suppF_{l-1}) \leqslant C\|F\|_{\ell^\omega_\tau(\Lambda^{df})}(\rho^{l-1}\epsilon_j)^{\frac{-1}{s}}$ and by lemma 5.3, $\#(suppW^{l-1}) \leqslant C\|V^{l-1}\|^{\frac{1}{s}}_{\ell^\omega_\tau(\Lambda^{df})}(\rho^{l-1}\epsilon_j)^{\frac{-1}{s}}$. Since $\epsilon_j > \epsilon$ we have

$$\#(suppV^l) \leqslant \#(suppV^{l-1}) + C(\rho^{l-1}\epsilon)^{\frac{-1}{s}}.$$

Also

$$\#(suppV^0) = \#(suppU^j) \leqslant C\epsilon^{\frac{-1}{s}}.$$

Therefore

$$\#(suppV^l) \leqslant C\epsilon^{\frac{-1}{s}}. \tag{32}$$

Now, we note that the number of total arithmetic operations from j to $j+1$ is at most the summation of (27) and (28). On the other hand the number of total sorts is at most (29). Combining these with (31) and (32), give the desired results. ∎

6.   Conclusions

Using divergence free wavelets and mixed weak formulation we change the generalized stokes equation to an positive definite linear system. Then we design an adaptive algorithm in order to give an adaptive approximated solution to the problem. This algorithm generates step by step an ascending sequence of nested Λ_j and approximated solutions U^j supported in Λ_j such that the cardinality of $(\Lambda_j - \Lambda_{j-1})$ stays as small as possible, while the error estimate satisfies

$$\|U - U^j\|_{\ell_2(\Lambda^{df})} \leqslant \epsilon_j := 2^{-j}\epsilon_0,$$

where U is the exact solution of the system and ϵ_0 is an upper bound for $\|U\|_{\ell_2(\Lambda^{df})}$.

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