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Solving Singular ODEs in Unbounded Domains with Sinc-Collocation Method

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Abstract. Spectral approximations for ODEs in unbounded domains have only received limited attention. In many applicable problems, singular initial value problems arise. In solving these problems, most of numerical methods have difficulties and often could not pass the singular point successfully. In this paper, we apply the sinc-collocation method for solving singular initial value problems. The ability of the sinc-collocation method in overcoming the singular points difficulties makes it an efficient method in dealing with these equations. We use numerical examples to highlight efficiency of sinc-collocation method in problems with singularity in equations.

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1. Introduction

While spectral approximations for ordinary differential equations (ODEs) in bounded domains have achieved great success; and popularity in recent years, spectral approximations for ODEs in unbounded domains have only received limited attention. Several spectral methods for treating unbounded domains have been proposed by different researchers. Direct approaches using Laguerre polynomials were investigated by Maday et al. [14], Funaro [4] and [8, 19]. Indirect approaches, e.g. Guo [5–7] have proposed a method that proceeds by mapping the

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original problem in an unbounded domain to a problem in a bounded domain, and then using suitable Jacobi polynomials to approximate the resulting problems.

Sinc methods for the numerical solution of ODEs and PDEs have been extensively studied. This method is a very effective technique, particularly for problems with singular solutions and those on unbounded domains. Sinc methods have many applications in mathematics and engineering which include heat transfer [10, 16, 18], population growth [1], fluid mechanics [28], optimal control [22], inverse problems [12, 13, 15, 20] and medical imaging [24].

Sinc functions were first analyzed in [27] and [21]. The books [22] and [11] provide excellent overviews of existing methods based on sinc functions for solving ODEs, PDEs, and integral equations. For solving differential equations, two methods based on sinc approximation were presented by sinc-Galerkin and by sinc-collocation. The first sinc-Galerkin method was presented in [21] to solve two-point boundary-value problems for second-order differential equations with Dirichlet boundary conditions. Ref. [2] applied the sinc-Galerkin method to solve a certain class of singular two-point boundary value problems and expressed the exact solution of the differential equations via the use of Greens functions as an integral type. Sinc-collocation procedure for the numerical solution of the initial value problem is developed in [3] and its proven that sinc procedure converges to the solution at an exponential rate. In [22], it is shown that both sinc-collocation and sinc-Galerkin converge with exponential rate.

In this article, we apply the sinc-collocation method to solve initial value problems:

$$\begin{aligned} &\mathcal{L}(y) = p(x)y" + q(x)y' + u(x)y = f(x, y), \\ &y(0) = a_0, y'(0) = b_0, \end{aligned} \tag{1}$$

where $p(x), q(x), u(x)$ and $f(x, y)$, are analytic functions that in examples $p(x)$ has a zero in $(0, \infty)$ then differential equation has singular point. We demonstrate some problems which the sinc-collocation method is applicable.

2. Preliminaries

Sinc function is defined on $-\infty < x < \infty$ by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2}$$

This function is translated with evenly spaced nodes are given as [3, 23]

$$S(k, h)(x) = sinc(\frac{x - kh}{h}), \; k = 0, \pm 1, \pm 2, \ldots, h > 0. \tag{3}$$

If $f(z)$ is analytic on a strip domain

$$|Imz| < d, \tag{4}$$

in the z-plane and $|f(z)| \to 0$ as $z \to \pm\infty$ then, the series

$$\mathcal{C}(f,h) = \sum_{k=-\infty}^{\infty} f(kh) sinc(\frac{z-kh}{h}), \tag{5}$$

converges, we call it whittaker cardinal expansion [26, 27].From [25] we can write

$$f(z) = \mathcal{C}(f,h) + E_{sinc}, \qquad E_{sinc}(h) = O\left(\exp\left(-\frac{\pi d}{h}\right)\right), \tag{6}$$

where d is half width of strip domain (4).

If $f(x)$ be a real function, sinc expansion (5) is defined on $-\infty < x < \infty$, while the equation that we want to solve is defined $a < x < \infty$, and hence we need some transformation which the given interval transform on to $-\infty < x < \infty$. In many of applications, the sinc method transformation

$$\phi(z) = ln(sinh(z)), \tag{7}$$

has been used. The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + iy : \left|arg\left(sinh(z)\right)\right| < d < \frac{\pi}{2}\right\}, \tag{8}$$

on to

$$D_d = \{\zeta = \xi + i\eta : |\eta| < d < \pi/2\}. \tag{9}$$

Define h by

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \qquad 0 < \alpha \leqslant 1. \tag{10}$$

The h is the mesh size in D_d for the uniform grids $\{kh\}$, $-\infty < k < \infty$. The base functions on (a,∞) are given by

$$S(j,h)o\phi(x) = sinc\left(\frac{\phi(x) - jh}{h}\right). \tag{11}$$

The sinc grid points $z \in (a,\infty)$ in D_E will be denoted by x because they are real. The inverse images of the equispaced grids in the SE transformation are

$$x = \phi^{-1}(t) = \psi(t) = ln(e^{kh} + \sqrt{e^{2kh}+1}). \tag{12}$$

If $y(x)$ be a solution of (1) and we use sinc expansion for numerical solution over (a,∞) and $y(\psi(t))$ is analytic in a strip domain $|Imt| < d$, then from (5) and (6) we have

$$y(x) = \sum_{j=-\infty}^{\infty} y(x_j) S(j,h)(\psi^{-1}(x)) + E_{sinc}(h).$$
$$x_j = \psi(jh), \qquad E_{sinc}(h) = O\left(\exp\left(-\frac{\pi d}{h}\right)\right). \tag{13}$$

In (13) $x_j = \psi(jh)$, $j = 0, \pm 1, \pm 2, \ldots$ are called sinc points. By replacing $y(x_j)$ with its approximation y_j, we obtain

$$\tilde{y}_h(x) = \sum_{j=-\infty}^{\infty} y_j S(j,h)(\psi^{-1}(x)). \tag{14}$$

In practical calculations we use finite terms of (14). Suppose that we truncate sum (14) at $j = N$ from [25], we have

$$|y(x) - \sum_{j=-N}^{N} y_j S(j,h)(\psi^{-1}(x))| \leqslant c\sqrt{N}exp(-\sqrt{\pi d\alpha N}), \tag{15}$$

that c depends only on f, d, α.

For solving problem (1) with sinc methods, we need following lemma.

Lemma 2.1 *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E to D_d Given by (8) and (9). Then*

$$\delta_{jk}^{(0)} = [S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{16}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{17}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k, \end{cases} \tag{18}$$

Proof Ref [22]. ∎

3. Sinc-Collocation Method

For initial conditions in (1), the sinc basis functions in (11) does not have derivative when x tends to zero. Thus, we modify the sinc basis functions as

$$xS_j(x). \tag{19}$$

Now the derivative of the modified sinc basis functions are defined as x approaches to zero and they are equal to zero [17]. In order to approximate the solution of (1) with initial conditions, we construct a polynomial p(x) that satisfies (1). This polynomial is given by

$$p(x) = \lambda x^2 + b_0 x + a_0. \tag{20}$$

The approximate solution for y(x), in (1) with initial conditions in (1) is represented by

$$y_N(x) = u_N(x) + p(x). \tag{21}$$

where

$$u_N(x) = \sum_{j=-N}^{N} c_j x S_j(x). \tag{22}$$

In (20), λ is constant to be determined. It is noted that the approximate solution $y_N(x)$, satisfy initial conditions in (1), since $u_N(0) = u'_N(0) = 0$. Furthermore, we have $p(0) = \alpha_0$ and $p'(0) = b_0$. The $2N+1$ coefficients $\{c_j\}_{j=-N}^{N}$ and the unknown λ are determined by substituting $y_N(x)$ into (1) and evaluating the result at the Sinc points

$$x_j = ln(e^{jh} + \sqrt{e^{2jh}+1}), j = -N-1, \ldots, N. \tag{23}$$

Obviously by using (16)-(18) and (21), we have

$$u_N(x_{-N-1}) = 0, \tag{24}$$

$$u_N(x_j) = c_j x_j, \quad j = -N, \ldots, N, \tag{25}$$

$$u'_N(x_{-N-1}) = \sum_{k=-N}^{N} c_k x_{-N-1} \phi'(x_{-N-1}) \delta^{(1)}_{k(-N-1)}, \tag{26}$$

$$u'_N(x_j) = \sum_{k=-N}^{N} c_k \{x_j \phi'(x_j) \delta^{(1)}_{kj} + \delta^{(0)}_{kj}\}, \quad j = -N, \ldots, N, \tag{27}$$

$$u''_N(x_j) = \sum_{k=-N}^{N} c_k \{2\phi'(x_j)\delta^{(1)}_{kj} + x_j \phi''(x_j)\delta^{(1)}_{kj} + x_j \phi'^2(x_j)\delta^{(2)}_{kj}\}, \quad j = -N, \ldots, N, \tag{28}$$

Substituting (21), (23)-(28) in (1), we obtain

$$p(x_j)y''(x_j) + q(x_j)y'(x_j) + u(x_j)y(x_j) = f(x_j, y(x_j)), \; j = -N-1, \ldots, N \tag{29}$$

If (1) is linear then (29) gives $2N+2$ linear algebraic equation. If (1) is nonlinear then (29) gives a nonlinear algebraic equation which can be solved for the unknown coefficients c_k and λ by using the Newton's method. Consequently, $y(x)$ given in (1) can be calculated.

4. Numerical Examples

In this section, we present some examples to show the efficiency and capability of the sinc-collocation method. The problems are solved with Matlab on a personal computer. In these examples, the exact value and approximate value of solutions is given at sinc points with $d = \pi/3, \alpha = 1/2$.

Table 1. Using sinc-collocation method for 4.1

	Exact solution	Present method
0.1	0.9983375	0.9983366
0.5	0.9607689	0.9607069
1	0.8660254	0.8663028
5	0.3273268	0.3273718
10	0.1706640	0.1709176

Table 2. Using sinc-collocation method for 4.2

x_i	Exact solution	Present method	error
0.000000000013133	0.000038792791897	0.000038792778765	0.000038792778881
0.000000003279900	0.000706495379473	0.000706492099573	0.000706492804839
0.000000273783368	0.006227587696481	0.006227861479849	0.006228344665708
0.000000106522446	0.018479282597956	0.018479389120402	0.018492027362223
0.000001267283816	0.154195725467932	0.154194458184116	0.162319802471533
0.000338713134380	0.352604792701003	0.352266079566623	0.649184213505005
0.016201212959128	0.200536409538704	0.184335196579576	2.513526255765731
0.012623280863273	0.109797656089339	0.097174375226066	5.046307039667219
0.035729808572526	0.104297628562742	0.068567819990216	7.222825424320359

Example 4.1 Consider standard Lane-Emden equation

$$y" + \frac{2}{x}y' + y^m = 0, \quad x > 0,$$
$$y(0) = 0, y'(0) = 1,$$

with $m = 5$. Table 1 shows the approximations of $y(x)$ for solving our problem with the sinc-collocation method with $N = 19$. We can find exact solution from [9].
This problem is a nonlinear second order initial value problem that has singularity at $x = 0$.

Example 4.2 Consider

$$x(2x^2+1)^3y" + 2(2x^2+1)^3y' - 2(x^2+1)^3y = -2(4x^5+4x^3+6x^2+x-1),$$
$$y(0) = 0, y'(0) = 1,$$

with exact solution

$$y = \frac{x}{2x^2+1}.$$

This problem is a linear second order initial value problem that has singularity at $x = 0$. Table 2 shows the approximations of $y(x)$ for solving our problem with the sinc-collocation method with $N = 100$.

Example 4.3 Consider the equation

$$(x-1)(x^2+1)^3y" + (x^2+1)^3y' - 2(x^2+1)^3y^2 = -2(x^4+2x^3+8x^2-6x-1),$$
$$y(0) = 0, y'(0) = 1,$$

with exact solution

$$y = \frac{x}{x^2+1}.$$

Table 3. Using sinc-collocation method for 4.3

x_i	Exact solution	Present method	error
0.000000000954633	0.000341994963467	0.000341994008834	0.000341994048834
0.000000007929786	0.003014968367572	0.003014960437786	0.003014987844234
0.000000062417924	0.026558108480370	0.026558046062446	0.026576804715466
0.000000346256007	0.220349834576572	0.220349488320565	0.232233460718585
0.000000184284696	0.383345610995369	0.383345795280065	0.466921020663844
0.000009935624534	0.456718729640720	0.456708794016186	0.649184213505005
0.000032009651565	0.496008042067412	0.496040051718977	0.881373587019543
0.000543787132348	0.494032603977756	0.494576391110104	1.159467339645706
0.000921987149662	0.463823446840111	0.464745433989774	1.472682704370908
0.009983343628579	0.194219033394465	0.204202377023045	4.683591269426189

This problem is a nonlinear second order initial value problem that has singularity at $x = 1$. Table 3 shows the approximations of $y(x)$ for solving our problem with the sinc-collocation method with $N = 75$

5. Conclusion

In this article we cannot use sinc bases functions for our initial value problems. Thus, we modified the sinc basis functions in our research. Then we applied sinc-collocation method for solving singular initial value problems. Numerical examples highlight the efficiency of sinc-collocation method in problems with singularity in equations.

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