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Analysis of Discrete-Time Machine Repair Problem with two Removable Servers under Triadic Policy

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Abstract. This paper analyzes a controllable discrete-time machine repair problem with L operating machines and two technicians. The number of working servers can be adjusted depending on the number of failed machines in the system one at a time at machine's failure or at service completion epochs. Analytical closed-form solutions of the stationary probabilities of the number of failed machines in the system are obtained. We develop the total expected cost function per machine per unit time and obtain the optimal operating policy and the optimal service rate at minimum cost using quadratic fit search method and simulated annealing method. Various performance measures along with numerical results to illustrate the influence of various parameters on the buffer behavior are also presented.

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1. Introduction

During the last few decades, discrete-time queueing systems have been considerably investigated and have been extensively employed to various fields, such as telecommunication systems, communication networks, production systems, manufacturing systems, etc. These queueing systems are more accurate and effective

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than their continuous-time counterparts to study and design slotted digital transmitting communication systems. Extensive study of discrete-time queueing models have been reported in [1-2].

Many researchers have studied queueing systems with threshold policy in which the working servers can be adjusted one at a time at any arrival or service completion epochs depending on the number of customers present in the system. Tadj and Choudhury [15] reported a survey of continuous-time queue with single threshold. The $M/M/1$ queue with N policy has been first studied by Yadin and Naor [21], where the server turns on whenever N or more customers are present in the system, and turns the server off when the system becomes empty. When the server is turned off, the server can not work till N customers are present in the system.

The service resources can be better utilized with the two service rate control to enhance the operational economy. The optimal operation of an $M/M/2$ queue with two removable servers has been investigated by Bell [1]. Rhee and Sivazlian [13] presented the busy period distribution in the controllable $M/M/2$ queue operating under the triadic $(0, K, N, M)$ policy. The $M/M/2$ queue, the N policy $M/M/1$ queue, and the $M/M/1$ queue are the special cases of the controllable $M/M/2$ queue operating under the triadic $(0, Q, N, M)$ policy. The optimal control of an $M/M/2$ queue with finite capacity L operating machines under the triadic policy has been examined by Wang and Wang [16]. The optimal operating policy for a controllable queueing model in which cost elements, arrival rate and service rate are all fuzzy numbers have been discussed by Lin and Ke [8]. Discrete-time $GI/D\text{-}MSP/1/K$ queue with N threshold policy has been studied by Goswami and Vijaya Laxmi [3]. In continuous-time, the analysis of optimal thresholds of an infinite buffer two severs system with triadic policy has been investigated by Lin and Ke [9]. An infinite buffer discrete-time two severs system with triadic policy has been studied by Goswami and Mund [2].

In many manufacturing systems, machine interference is a significant problem. A machine may fail due to some unpredicted fault and thus requires a repair facility, after which it can again begin functioning properly. Extensive research of a machine repair problems have been reported in [5, 14]. Wang [17] presented the steady-state analytic solutions of an $M/M/1$ machine repair problem with a single service station subject to breakdowns. The machine repair problem with R unreliable service stations has been developed by Wang [18]. The $M/E_k/1$ machine repair problem with an unreliable server was considered by Wang and Kuo [19]. The reliability characteristics of a repairable system with warm standbys and server breakdowns was investigated by Wang et al. [20]. Sensitivity analysis of machine repair problems in manufacturing systems with service interruptions can be found in Ke and Lin [6]. Lv et al. [11] discussed unreliable multi-server machine repairable system with variable breakdown rates. Ke et al. [7] studied a machine repair problem with warm standbys, imperfect coverage, service pressure coefficient, and unreliable multi-technicians.

Many practical and highly automated manufacturing processes, production or machines facilities are subject to failures leading to significant loss of production output which in succession affects the company's revenue. The theory of machine repairable systems with spares has many applications, such as power stations, hospitals, manufacturing systems and industrial systems, where standby equipments are needed. Generally, there is a trade-off between the magnitude of machine interference and the operator staffing level. For a practical view, machine interference problems cannot simply take the relationship between machines and the operator but also impacts upon many other components such as the part arrival rate, existence of external operations and machine failures.

In many real life congestion setting of machine repair problem, the system efficiency can be enhanced by rendering sufficient spare part back-up in case of machine failure. In a multi-processor computer with a shared memory, the processors work for a time before they need data from the memory and enter the memory queue for servicing. As soon as the shared memory responds, they resume operation. Liou et al. [10] studied the controllable $M/M/2$ machine repair problem with L operating machines operating under the triadic policy. Many continuous-time queueing systems with machine repair problems have been studied, but in the literature their discrete-time counterparts have got very little attention. In computer and digital telecommunication systems, discrete-time queueing systems are more suitable than their continuous-time counterparts to determine system performance measures, due to clock-driven procedure of those systems.

This paper analyzes an optimal thresholds of a discrete-time machine repair problem with two removable servers operating under the triadic policy. In the triadic policy, whenever there are no failed machines in the system, both servers are temporarily inactive until certain specified conditions arise. First of all, we assume that both the servers are turned off. By using the recursive method the stationary probabilities and some performance measures with numerical results have been presented. The results found in this paper are appropriate for implementation in software packages developed for various management control systems where an automatic threshold control limits alert can be employed as a cautionary mechanism.

The rest of the paper is organized as follows. Section 2 presents model description and analysis of the queueing model for the stationary probabilities at arbitrary epoch. System characteristics have been discussed in Section 3. The cost analysis and optimization investigation is carried out in Section 4. Numerical results in the form of tables and graphs to study the parameter effect on the system performance are presented in Section 5. Section 6 concludes the paper.

2. Model Description

We consider a discrete-time machine repair model with L identical operating machines that are served by two removable servers which are subject to breakdowns under the triadic policy. In the triadic policy, initially, we assume that both the servers remain inactive until certain specified conditions develop. When the number of failed machines waiting for service reaches a specific number, say N, one of the servers becomes active instantly. When the number of failed machines waiting for service increases to another specified level, say M $(M > N)$, then the left server also turns active instantly. If both the servers are active and the number of failed machines in the system decreases to Q, where $3 \leq Q < N$, the server just finishing a service becomes inactive at that time. In addition, if the number of failed machines drops down to zero while one server is active, the server becomes inactive.

In discrete-time queues, number of failed machines and their onward service may occur simultaneously around slot boundaries. Their occurrence order may be taken care of by either early arrival system (EAS) or late arrival system with delayed access (LAS-DA), which are also known as departure-first (DF) or arrival-first (AF) policies. Here we discuss the model with LAS-DA and therefore, a potential machine fails in $(t-, t)$ and a potential service occurs in $(t, t+)$, for $t = 0, 1, 2, \ldots$. For more details on this topic, see [4**?**]. The inter-arrival times of failed machines are independent and geometrically distributed with probability mass function (p.m.f.) $a_n = \bar{\lambda}^{n-1}\lambda,\ 0 < \lambda < 1,\ n \geq 1$. When an operating machine fails, it is instantly sent to a server and is served in order of its breakdown and the service times of both the servers are assumed to be independent and geometrically distributed with p.m.f. $s_n = \bar{\mu}^{n-1}\mu,\ 0 < \mu < 1,\ n \geq 1$, where for any real number $x \in [0, 1]$, we denote $\bar{x} = 1 - x$. The mean failure rate λ_n is given by $\lambda_n = (L - n)\lambda$ and $\bar{\lambda}_n = (1 - (L - n)\lambda)$ for $1 \leq n \leq L - 1$.

Let $P_{n,0}(t)$ denote the probability that the server is turned off and there are n $(0 \leq n \leq N - 1)$ failed machines in the system. Let $P_{n,1}(t)$ denote the probability that one of the servers is turned on and active and there are n $(1 \leq n \leq M - 1)$ failed machines in the system. Further, let $P_{n,2}(t)$ be the probability that both the servers are turned on and active and there are n $(Q + 1 \leq n \leq L)$ failed machines in the system.In steady-state, let us define

$$
\begin{aligned}
&P_{n,0} = \lim_{t\to\infty} P_{n,0}(t),\ 0 \leq n \leq N;\ P_{n,1} = \lim_{t\to\infty} P_{n,1}(t),\ 1 \leq n \leq M - 1;\\
&P_{n,2} = \lim_{t\to\infty} P_{n,2}(t),\ Q + 1 \leq n \leq L.
\end{aligned}
$$

2.1 *Analysis of the Model*

In order to get the stationary probabilities, we construct the difference equations by relating the states of the system at two consecutive time epochs $t-$ and $(t+1)-$, where for the sake of simplicity, we use the symbol t instead of $t-$, we obtain

$$
\begin{aligned}
P_{0,0}(t+1) &= \bar{\lambda}_0 P_{0,0}(t) + \bar{\lambda}_1 \mu P_{1,1}(t),\\
P_{n,0}(t+1) &= \bar{\lambda}_n P_{n,0}(t) + \lambda_{n-1} P_{n-1,0}(t), \quad 1 \leq n \leq N - 1,\\
P_{n,1}(t+1) &= \left(\bar{\lambda}_n \bar{\mu} + \lambda_n \mu\right) P_{n,1}(t) + \bar{\lambda}_{n+1} \mu P_{n+1,1}(t)\\
&\quad + \Phi(2 \leq n \leq Q - 2)\lambda_{n-1}\bar{\mu} P_{n-1,1}(t), \quad 1 \leq n \leq Q - 2,\\
P_{n,1}(t+1) &= \left(\bar{\lambda}_n \bar{\mu} + \lambda_n \mu\right) P_{n,1}(t) + \bar{\lambda}_{n+1} \mu P_{n+1,1}(t) + \lambda_{n-1}\bar{\mu} P_{n-1,1}(t)\\
&\quad + \Phi(n = Q)\left(2\bar{\lambda}_{n+1}\mu\bar{\mu} + \lambda_{n+1}\mu^2\right) P_{n+1,2}(t)\\
&\quad + \Phi(Q - 1 \leq n \leq Q)\bar{\lambda}_{n+2}\mu^2 P_{n+2,2},\ Q - 1 \leq n \leq N - 1,\\
P_{N,1}(t+1) &= \left(\bar{\lambda}_N \bar{\mu} + \lambda_N \mu\right) P_{N,1}(t) + \bar{\lambda}_{N+1} \mu P_{N+1,1}(t)\\
&\quad + \lambda_{N-1}\bar{\mu} P_{N-1,1}(t) + \lambda_{N-1} P_{N-1,0}(t),\\
P_{n,1}(t+1) &= \Phi(N + 1 \leq n \leq M - 2)\bar{\lambda}_{n+1}\mu P_{n+1,1} + \lambda_{n-1}\bar{\mu} P_{n-1,1}\\
&\quad + \left(\bar{\lambda}_n \bar{\mu} + \lambda_n \mu\right) P_{n,1}(t),\ N + 1 \leq n \leq M - 1,\\
P_{n,2}(t+1) &= \left(\bar{\lambda}_n \bar{\mu}^2 + 2\lambda_n \mu\bar{\mu}\right) P_{n,2}(t) + \left(2\bar{\lambda}_{n+1}\mu\bar{\mu} + \lambda_{n+1}\mu^2\right) P_{n+1,2}(t)\\
&\quad + \Phi(Q + 2 \leq n \leq L - 3)\lambda_{n-1}\bar{\mu}^2 P_{n-1,2}(t) + \bar{\lambda}_{n+2}\mu^2\\
&\quad P_{n+2,2}(t) + \Phi(n = M)\lambda\bar{\mu} P_{n-1,1}(t),\ Q + 1 \leq n \leq L - 3,\\
P_{L-2,2}(t+1) &= \left(\bar{\lambda}_{L-2}\bar{\mu}^2 + 2\lambda_{L-2}\mu\bar{\mu}\right) P_{L-2,2}(t) + \lambda_{L-3}\bar{\mu}^2 P_{L-3,2}(t)(t)\\
&\quad + \left(2\bar{\lambda}_{L-1}\mu\bar{\mu} + \lambda_{L-1}\mu^2\right) P_{L-1,2} + \mu^2 P_{L,2}(t),\\
P_{L-1,2}(t+1) &= \left(\bar{\lambda}_{L-1}\bar{\mu}^2 + 2\lambda_{L-1}\mu\bar{\mu}\right) P_{L-1,2}(t) + \lambda_{L-2}\bar{\mu}^2 P_{L-2,2}(t)\\
&\quad + 2\mu\bar{\mu} P_{L,2}(t),\\
P_{L,2}(t+1) &= \bar{\mu}^2 P_{L,2}(t) + \lambda_{L-1}\bar{\mu}^2 P_{L-1,2}(t).
\end{aligned}
$$

where $\Phi(\Omega)$ is equal to 1 when the expression Ω is satisfied; otherwise, its value is 0. In the steady-state, above equations reduce to

$$\lambda_0 P_{0,0} = \bar{\lambda}_1 \mu P_{1,1}, \tag{1}$$

$$\lambda_n P_{n,0} = \lambda_{n-1} P_{n-1,0}, \quad 1 \leq n \leq N-1, \tag{2}$$

$$\begin{aligned}(\bar{\lambda}_n \mu + \lambda_n \bar{\mu}) P_{n,1} &= \bar{\lambda}_{n+1} \mu P_{n+1,1} + \Phi(2 \leq n \leq Q-2) \lambda_{n-1} \bar{\mu} P_{n-1,1}, \\ &\quad 1 \leq n \leq Q-2,\end{aligned} \tag{3}$$

$$\begin{aligned}(\bar{\lambda}_n \mu + \lambda_n \bar{\mu}) P_{n,1} &= \bar{\lambda}_{n+1} \mu P_{n+1,1} + \Phi(Q-1 \leq n \leq Q) \bar{\lambda}_{n+2} \mu^2 P_{n+2,2} \\ &\quad + \lambda_{n-1} \bar{\mu} P_{n-1,1} + \Phi(n = Q) \left(2\bar{\lambda}_{n+1} \mu \bar{\mu} + \lambda \mu^2\right) P_{n+1,2}, \\ &\quad Q-1 \leq n \leq N-1,\end{aligned} \tag{4}$$

$$(\bar{\lambda}_N \mu + \lambda_N \bar{\mu}) P_{N,1} = \bar{\lambda}_{N+1} \mu P_{N+1,1} + \lambda_{N-1} \bar{\mu} P_{N-1,1} + \lambda_{N-1} P_{N-1,0}, \tag{5}$$

$$\begin{aligned}(\bar{\lambda}_n \mu + \lambda_n \bar{\mu}) P_{n,1} &= \Phi(N+1 \leq n \leq M-2) \bar{\lambda}_{n+1} \mu P_{n+1,1} + \lambda_{n-1} \bar{\mu} P_{n-1,1}, \\ &\quad N+1 \leq n \leq M-1,\end{aligned} \tag{6}$$

$$\begin{aligned}\psi_n P_{n,2} &= \left(2\bar{\lambda}_{n+1} \mu \bar{\mu} + \lambda_{n+1} \mu^2\right) P_{n+1,2} + \Phi(n = M) \lambda_{n-1} \bar{\mu} P_{n-1,1} \\ &\quad + \bar{\lambda}_{n+2} \mu^2 P_{n+2,2} + \Phi(Q+2 \leq n \leq L-3) \lambda_{n-1} \bar{\mu}^2 P_{n-1,2}, \\ &\quad Q+1 \leq n \leq L-3,\end{aligned} \tag{7}$$

$$\begin{aligned}\psi_{L-2} P_{L-2,2} &= \left(2\bar{\lambda}_{L-1} \mu \bar{\mu} + \lambda_{L-1} \mu^2\right) P_{L-1,2} + \lambda_{L-3} \bar{\mu}^2 P_{L-3,2} \\ &\quad + \mu^2 P_{L,2},\end{aligned} \tag{8}$$

$$\psi_{L-1} P_{L-1,2} = \lambda_{L-2} \bar{\mu}^2 P_{L-2,2} + 2\mu\bar{\mu} P_{L,2}, \tag{9}$$

$$\left(1 - \bar{\mu}^2\right) P_{L,2} = \lambda_{L-1} \bar{\mu}^2 P_{L-1,2}, \tag{10}$$

where $\psi_n = \left(1 - \bar{\lambda}_n \bar{\mu}^2 - 2\lambda_n \mu \bar{\mu}\right),\ 1 \leq n \leq L-1$. The steady-state probabilities $P_{n,0}$ $(0 \leq n \leq N-1)$, $P_{n,1}$ $(1 \leq n \leq M-1)$ and $P_{n,2}$, $(Q+1 \leq n \leq L)$ are computed by solving the equations (1) - (10). From (2), we get

$$P_{n,0} = \left(\frac{L}{L-n}\right) P_{0,0}, \quad 1 \leq n \leq N-1. \tag{11}$$

Using (1) and (3) recursively, we obtain

$$P_{n,1} = h_n P_{0,0}, \quad 1 \leq n \leq Q-1, \tag{12}$$

where

$$h_1 = \frac{\lambda_0}{\bar{\lambda}_1 \mu}, h_2 = \left(\frac{\bar{\lambda}_1 \mu + \lambda_1 \bar{\mu}}{\bar{\lambda}_2 \mu}\right) h_1 \quad \text{and}$$

$$h_n = \frac{\left(\bar{\lambda}_{n-1} \mu + \lambda_{n-1} \bar{\mu}\right) h_{n-1} - \left(\lambda_{n-2} \bar{\mu}\right) h_{n-2}}{\bar{\lambda}_n \mu},\ 1 \leq n \leq Q-1.$$

From (6),

$$P_{n,1} = h_n P_{M-1,1},\ n = M-1, M-2, \ldots, N, \tag{13}$$

where

$$h_{M-1} = 1,\ h_{M-2} = \frac{\bar{\lambda}_{M-1}\mu + \lambda_{M-1}\bar{\mu}}{\lambda_{M-2}\bar{\mu}} \quad \text{and}$$
$$h_n = \frac{\left(\bar{\lambda}_{n+1}\mu + \lambda_{n+1}\bar{\mu}\right) h_{n+1} - \left(\bar{\lambda}_{n+2}\mu\right) h_{n+2}}{\lambda_n\bar{\mu}}, \quad n = M-3, \ldots, N.$$

From (4), using (13) and simplifying, we obtain

$$P_{n,1} = h_n P_{M-1,1} + f_n P_{0,0},\ n = N-1, N-2, \ldots, Q, \tag{14}$$

where $f_{N-1} = \frac{-L}{(L-N+1)\bar{\mu}},\ \ f_{N-2} = \frac{(\bar{\lambda}_{N-1}\mu + \lambda_{N-1}\bar{\mu})f_{N-1}}{\lambda_{N-2}\bar{\mu}}$ and

$$f_n = \frac{\left(\bar{\lambda}_{n+1}\mu + \lambda_{n+1}\bar{\mu}\right) f_{n+1} - (\bar{\lambda}_{n+2}\mu) f_{n+2}}{\lambda_n\bar{\mu}},\ n = N-3, N-4, \ldots, Q,$$
$$h_n = \frac{\left(\bar{\lambda}_{n+1}\mu + \lambda_{n+1}\bar{\mu}\right) h_{n+1} - (\bar{\lambda}_{n+2}\mu) h_{n+2}}{\lambda_n\bar{\mu}},\ n = N-1, N-2, \ldots, Q.$$

Using (7) - (10), we obtain

$$P_{n,2} = d_n P_{L,2},\ n = L, L-1, \ldots, M, \tag{15}$$

where

$$d_L = 1,\ d_{L-1} = \frac{2\mu - \mu^2}{\lambda_{L-1}\bar{\mu}^2},$$
$$d_{L-2} = \frac{\psi_{L-1}(2\mu - \mu^2)}{(\lambda_{L-1}\bar{\mu}^2)(\lambda_{L-2}\bar{\mu}^2)} - \frac{2\mu\bar{\mu}}{\lambda_{L-2}\bar{\mu}^2},$$

$$d_{L-3} = \frac{\psi_{L-2}d_{L-2} - (2\bar{\lambda}_{L-1}\mu\bar{\mu} + \lambda_{L-1}\mu^2)d_{L-1} - \mu^2 d_L}{\lambda_{L-3}\bar{\mu}^2},$$
$$d_n = \frac{\psi_{n+1}d_{n+1} - (2\bar{\lambda}_{n+2}\mu\bar{\mu} + \lambda_{n+2}\mu^2)d_{n+2} - \bar{\lambda}_{n+3}\mu^2 d_{n+3}}{\lambda_n\bar{\mu}^2},$$
$$n = L-4, \ldots, Q.$$

From (6), we obtain

$$P_{n,2} = d_n P_{L,2} + e_n P_{M-1,1},\ n = M-1, \ldots, Q+1, \tag{16}$$

where

$$
\begin{aligned}
e_{M-1} &= \frac{-1}{\bar{\mu}}, \ e_{M-2} = \frac{\psi_{M-1} e_{M-1}}{\lambda_{M-2}\bar{\mu}^2}, \\
e_{M-3} &= \frac{\psi_{M-2} e_{M-2} - \left(2\bar{\lambda}_{M-1}\mu\bar{\mu} + \lambda_{M-1}\mu^2\right) e_{M-1}}{\lambda_{M-3}\bar{\mu}^2}, \\
e_n &= \frac{\psi_{n+1} e_{n+1} - \left(2\bar{\lambda}_{n+2}\mu\bar{\mu} + \lambda_{n+2}\mu^2\right) e_{n+2} - \bar{\lambda}_{n+3}\mu^2 e_{n+3}}{\lambda_n \bar{\mu}^2}, \\
& n = M-4, \ldots, Q.
\end{aligned}
$$

Setting $n = Q$ in (5), we get after simplification

$$P_{L,2} = -\left(\frac{e_Q}{d_Q}\right) P_{M-1,1}. \tag{17}$$

Putting $n = Q - 1$ in (4), we get

$$P_{0,0} = K_1 P_{M-1,1}, \tag{18}$$

where

$$K_1 = \left[\frac{\bar{\lambda}_Q \mu h_Q + \bar{\lambda}_{Q+1}\mu^2\left(e_{Q+1}d_Q - e_Q d_{Q+1}\right)}{d_Q\left\{(\bar{\lambda}_{Q-1}\mu + \lambda_{Q-1}\bar{\mu})h_{Q-1} - \lambda_{Q-2}\bar{\mu}h_{Q-2} - \bar{\lambda}_Q \mu f_Q\right\}}\right].$$

Now using the normalization condition, we obtain

$$
\begin{aligned}
P_{M-1,1} = &\left[K_1 \sum_{n=0}^{N}\left(\frac{L}{L-n}\right) + K_1 \sum_{n=Q}^{N-1} f_n + \sum_{n=Q+1}^{L} d_n + K_1 \sum_{n=1}^{Q-1} h_n \right. \\
&\left. + \sum_{n=Q}^{M-1} h_n + \sum_{n=Q+1}^{M-1} e_n \right]^{-1}.
\end{aligned} \tag{19}
$$

3. System Characteristics

There are various system features of the controllable discrete-time machine repair queueing system operating under the triadic $(0, Q, N, M)$ policy.
Let the expected number of failed machines in the system when all the servers are turned off (L_0), the expected number of failed machines in the system when one of the servers is turned on and working (L_1) and the expected number of failed machines when both the servers are turned on and working (L_2) are given by

$$L_0 = \sum_{n=0}^{N-1} n P_{n,0}, \quad L_1 = \sum_{n=1}^{M-1} n P_{n,1} \quad L_2 = \sum_{n=Q+1}^{L} n P_{n,2}.$$

The expected number of failed machines in the system (L_s) is given by

$$L_s = \sum_{n=0}^{N-1} nP_{n,0} + \sum_{n=1}^{M-1} nP_{n,1} + \sum_{n=Q+1}^{L} nP_{n,2}.$$

The expected number of operating machines in the system ($E[O]$) is given by $E[O] = L - L_s$.
Let the expected number of idle servers in the system ($E[I]$), the expected number of one busy server ($E[B_1]$), the expected number of two busy servers ($E[B_2]$) and the expected number of busy servers ($E[B]$) in the system, can be obtained as

$$E[I] = \sum_{n=0}^{N-1} 2P_{n,0} + \sum_{n=1}^{M-1} P_{n,1},$$
$$E[B_1] = \sum_{n=1}^{M-1} P_{n,1}, \quad E[B_2] = \sum_{n=Q+1}^{L} 2P_{n,2}, \quad E[B] = E[B_1] + E[B_2].$$

The fraction of total time the machines are working, that is, machine availability (MA) is given by

$$MA = \frac{E[O]}{L}.$$

The fraction of the busy servers, that is, operative utilization (OU) is given by

$$OU = \frac{E[B_1] + E[B_2]}{2}.$$

4. Cost Analysis and Optimization Investigation

The performance measures derived can now be used to optimize the performance of the system. We develop the total expected cost function per unit time for the discussed queueing system, assuming the decision variables as (Q, N, M, μ). Our objective is to determine the optimum values (Q, N, M, μ), say (Q^*, N^*, M^*, μ^*), so that the expected cost function is minimized. Let
$C_h \equiv$ cost per unit time per failed machine in the system;
$C_1 \equiv$ cost incurred per unit time for keeping one busy server;
$C_2 \equiv$ cost incurred per unit time for keeping two busy server;
$C_3 \equiv$ fixed cost for every repair rate of the working server.
Let $F(Q, N, M, \mu)$ be the expected cost per unit time. Using the definitions of each cost element and its corresponding system characteristics, we have

$$F(Q, N, M, \mu) = \frac{C_0 L_s + C_1 E[B_1] + C_2 E[B_2] + C_3 E[I] + C_4 \mu}{L}. \tag{20}$$

It is indeed a difficult job to find the optimal values (Q^*, N^*, M^*, μ^*) analytically from (20), as the expected cost function is complex and non-linear. A direct search optimization algorithm is used over a grid whose boundaries for decision variables (Q, N, M) are selected in order to guarantee the global optimum in the interior region. To search the optimal values (Q^*, N^*, M^*, μ^*), it is necessary to have bounds for decision variables.

$$F(Q^*, N^*, M^*, \mu^*) = \min_{\substack{3 \le Q < N < M < L \\ 0.01 \le \mu \le \mu_0}} F(Q, N, M, \mu).$$

For obtaining the optimum value of the continuous parameter μ, we solve the above stated optimization problem using quadratic fit search method (QFSM). In this method, given a 3-point pattern, we can fit a quadratic function through corresponding functional values that has a unique minimum, $\mu^* = \mu^q$, for the given objective function $F(\mu)$. Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with approximate optimum μ^q. The unique optimum μ^* of the quadratic function agreeing with $F(\mu)$ at 3-point operation (μ^l, μ^m, μ^h) given in [12] occurs at

$$\mu^q \cong \frac{1}{2}\left[\frac{F(\mu^l)[s^m - s^h] + F(\mu^m)[s^h - s^l] + F(\mu^h)[s^l - s^m]}{F(\mu^l)[\mu^m - \mu^h] + F(\mu^m)[\mu^h - \mu^l] + F(\mu^h)[\mu^l - \mu^m]}\right]$$

where $s^l = (\mu^l)^2,\ s^m = (\mu^m)^2,\ s^h = (\mu^h)^2$. The detailed algorithm can be found in Rardin [12]. In order to validate the results obtained from QFSM, we further apply the Simulated Annealing (SA) technique using the standard function in Mathematica software.

5. Numerical Results

To study the parameter effect on the system performance, numerical computations have been carried out and some of these are presented in this section in the form of tables and graphs. We have considered the following cost parameters: $C_0 = 10, C_1 = 20, C_2 = 30, C_3 = 40$ and $C_4 = 80$ and the other parameters as $Q = 3, N = 5, M = 7$ and $\lambda = 0.09$, for Tables 1-2 and for figures unless otherwise mentioned in their respective graphs and tables. Table 1 presents the optimum values of μ, the minimum expected cost F^*, using quadratic fit search method, along with the respective performance measures $L_s^*, E[O], E[I], E[B_1], E[B_2], MA$ and OU for various values of λ. From the table, we observe that as λ decreases (i) the optimum values $\mu^*,\ F^*, {L_s}^*,\ E[B]$ and OU decrease and (ii) the remaining measures $E[O],\ E[I]$ and MA increase. It is evident in practice that as the lesser rate of machine failure increases the machine availability and the idle time of the server. Table 2 presents the optimum values μ^* and the minimum cost F^* as obtained from the QFSM and SA methods. It also presents some important performance measures at optimum μ^*. Table 3 shows the sensitivity analysis of the optimum values of (Q, N) for various values of M and λ. As indicated, as M increases the cost is decreasing, the higher values of failure rates (λ) certainly increases the optimum values of cost and (Q, N) values.

Figure 1 shows the effect of service rate μ on the total expected cost. The convex shape of the curve shows that there is a minimum value for μ that helps in starting the iterations in QFSM with the starting three point pattern taken as $(0.2, 0.25, 0.3)$. After five iterations the method yielded the optimum $\mu^* = 0.2692$ at which the expected cost is minimum as mentioned in Table 1.

The effect of μ on the performance measures $E[O],\ E[I],\ E[B_1]$ and $E[B_2]$ is depicted in Figure 2. Clearly, as μ increases there is a monotone increase in expected number of operating machines and obviously, $E[B_1]$ and $E[B_2]$ have reverse nature with $E[B_2]$ tending to zero sharply implying that with greater

Table 1. Optimal values of μ for various values of λ.

λ	μ^*	F^*	L_s^*	$E[O]$	$E[B_1]$	$E[B_2]$	MA	OU
0.09	0.2692	13.4650	5.3095	4.6905	0.4302	1.1378	0.4690	0.7840
0.08	0.2612	13.2227	5.1277	4.8723	0.5026	0.9896	0.4872	0.7461
0.07	0.2545	12.9530	4.9028	5.0971	0.5840	0.8179	0.5097	0.7010
0.06	0.2471	12.6499	4.6383	5.3617	0.6635	0.6383	0.5362	0.6509
0.05	0.2329	12.3108	4.3739	5.6261	0.7193	0.4882	0.5626	0.6038

Table 2. Optimal values of μ for various values of λ from QFSM and SA.

λ	μ^*	F^* (QFSM)	F^* (SA)	L_s^*	$E[O]$	MA	OU
0.09	0.2692	13.4650	13.4650	5.3095	4.6905	0.4690	0.7840
0.08	0.2612	13.2227	13.2227	5.1277	4.8723	0.4872	0.7461
0.07	0.2545	12.9530	12.9530	4.9029	5.0971	0.5097	0.7010
0.06	0.2471	12.6499	12.6499	4.6383	5.3617	0.5362	0.6509
0.05	0.2329	12.3108	12.3108	4.3739	5.6261	0.5626	0.6038

Table 3. Sensitivity analysis of expected cost for various values of λ.

	$M = 11$		$M = 12$		$M = 13$	
λ	(Q^*, N^*)	F^*	(Q^*, N^*)	F^*	(Q^*, N^*)	F^*
0.021	(3,4)	6.0199	(3,4)	5.8314	(3,4)	5.7570
0.023	(3,4)	6.6266	(3,4)	6.2963	(3,4)	6.1455
0.025	(6,7)	7.2669	(3,4)	6.8510	(3,4)	6.5774
0.027	(7,8)	7.7844	(8,9)	7.3680	(3,4)	7.0554
0.029	(7,8)	8.3960	(8,9)	7.7863	(8,9)	7.4888

service rates, the expectation of both the servers being busy is almost zero.

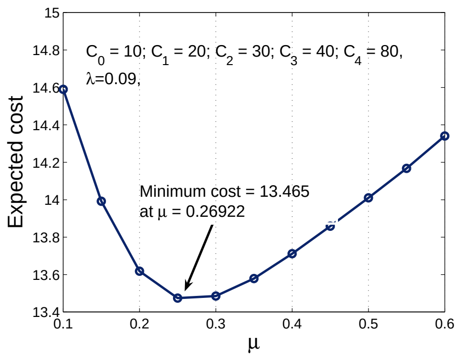

Figure 1. μ vs. cost ($q = 3, N = 5, M = 8, L = 10$)

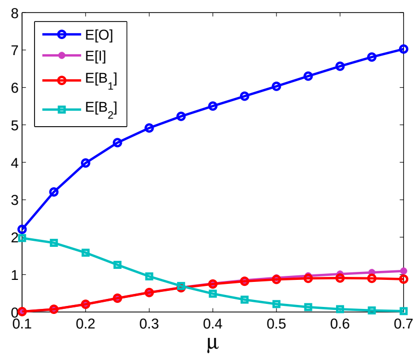

Figure 2. μ vs. $E[O], E[I]$, $E[B_1]$ and $E[B_2]$

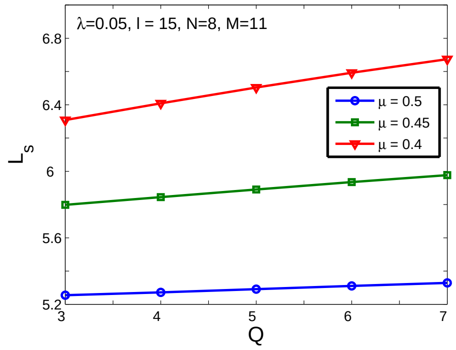

Figure 3. Effect of Q on L_s

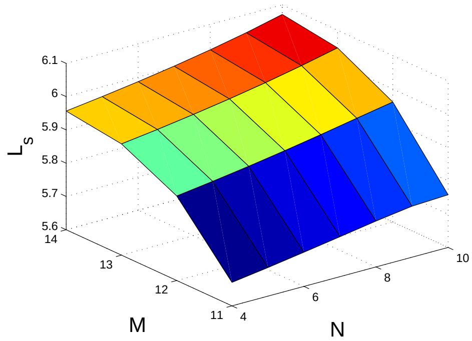

Figure 4. Effect of N and M on L_s

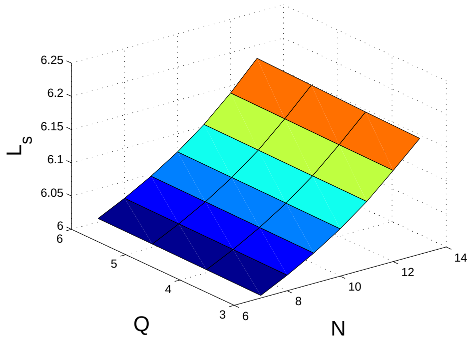

Figure 5. Effect of Q and N on L_s

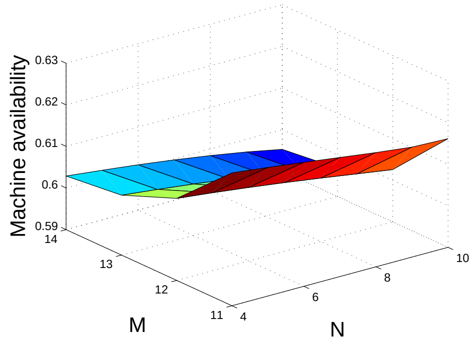

Figure 6. Effect of N and M on MA

Figure 3 shows that the increasing trend of (L_s) with Q for different μ values. For a fixed Q, however, as μ decreases, there is a considerable increase in L_s. Figure 4 provides the effect of M and N on the expected number of failed machines in the system (L_s) for $L = 15, Q = 3, \lambda = 0.055, \mu = 0.5$. The graph showing the increasing trend of L_s with both M and N and more prominently so in case of M and becomes steady at higher values. A similar observation is made in case of Figure 5 where the effect of Q and N on L_s is shown.

For the same parameters as used for Figure 4, Figure 6 shows machine availability as a function of N and M. We can observe that for a fixed M, as N increases there is a decrease in MA and in case of M increasing, MA slightly decreases first and then remains steady.

6. Conclusion

In this paper, we consider a discrete-time machine repair model with L identical operating machines that are subject to breakdowns and are served by two removable servers under the triadic policy. The inter-arrival times of failed machines are independent and geometrically distributed and the service times of both the servers are assumed to be independent and geometrically distributed. Using the recursive method, we have obtained the queue length distributions and further the various performance measures and cost analysis are considered. The parameter effect on the performance of the system is analyzed using some numerical computations. The method of analysis used in this paper can be applied to bulk-arrival and bulk service $Geo/Geo/1$ models under different constraints of service patterns. These topics are left for future investigations.

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