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$(\delta, \gamma, 2)$-Bessel Lipschitz Functions in the Space $L_{2,\alpha}(\mathbb{R}^+)$

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Abstract. Using a generalized translation operator, we obtain a generalization of Theorem 5 in [4] for the Bessel transform for functions satisfying the $(\delta, \gamma, 2)$-Bessel Lipschitz condition in $L_{2,\alpha}(\mathbb{R}^+)$.

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1. Introduction and Preliminaries

Integral transforms and their inverses are widely used solve various in calculus, mechanics, mathemtical, physics, and computational mathematics (see, e.g.,[7, 9, 10]).

In [4], we proved theorem related the Bessel transform and (k, γ)-Bessel Lipschitz functions. In this paper, we prove a generalization of this theorem for this transform in the space $L_{2,\alpha}(\mathbb{R}^+)$. For this purpose, we use a generalized translation operator.

Assume that $L_{2,\alpha} = L_{2,\alpha}(\mathbb{R}^+)$, $\alpha > -\frac{1}{2}$, is the Hilbert space of measurable functions $f(x)$ on $\mathbb{R}^+$ with the finite norm

$$\|f\|_{2,\alpha} = \left(\int_0^{\infty} |f(x)|^2 x^{2\alpha+1} dx\right)^{1/2}.$$

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Let

$$B = \frac{d^2}{dt^2} + \frac{(2\alpha+1)}{t}\frac{d}{dt},$$

be the Bessel differential operator.

For $\alpha > -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\Gamma(n+\alpha+1)},$$

where $\Gamma(x)$ is the gamma-function (see [6]).

Lemma 1.1 *[1] The following inequalities are valid for Bessel function* j_α

(1) $|j_\alpha(x)| \leqslant 1$
(2) $1 - j_\alpha(x) = O(x^2);\ 0 \leqslant x \leqslant 1.$

Lemma 1.2 *The following inequality is true*

$$|1 - j_\alpha(x)| \geqslant c,$$

with $x \geqslant 1$*, where* $c > 0$ *is a certain constant.*

Proof Analog of Lemma 2.9 in [3] ■

The Bessel transform of a function $f \in L_{2,\alpha}$ is defined (see [5, 6, 8]) by the formula

$$\widehat{f}(\lambda) = \int_0^\infty f(t)j_\alpha(\lambda t)t^{2\alpha+1}dt;\quad \lambda \in \mathbb{R}^+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^\alpha\Gamma(\alpha+1))^{-2}\int_0^\infty \widehat{f}(\lambda)j_\alpha(\lambda t)\lambda^{2\alpha+1}d\lambda.$$

From [5], we have the Parseval's identity

$$\|\widehat{f}\|_{2,\alpha} = (2^\alpha\Gamma(\alpha+1))\|f\|_{2,\alpha}.$$

In $L_{2,\alpha}$, consider the generalized translation operator T_h defined by

$$T_h f(t) = c_\alpha\int_0^\pi f(\sqrt{t^2+h^2-2th\cos\varphi})\sin^{2\alpha}\varphi d\varphi,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha}\varphi d\varphi\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+\frac{1}{2})}$$

From [2], we have

$$\widehat{(\mathrm{T}_h f)}(\lambda) = j_\alpha(\lambda h)\widehat{f}(\lambda) \tag{1}$$

We note the important property of the Bessel transform: If $f \in \mathrm{L}_{2,\alpha}$ then

$$\widehat{\mathrm{B}f}(\lambda) = (-\lambda^2)\widehat{f}(\lambda). \tag{2}$$

The finite differences of the first and higher orders are defined as follows

$$\Delta_h f(x) = \mathrm{T}_h f(x) - f(x) = (\mathrm{T}_h - \mathrm{E}_{2,\alpha})f(x),$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (\mathrm{T}_h - \mathrm{E}_{2,\alpha})^k f(x) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\mathrm{T}_h^i f(x), \tag{3}$$

where $\mathrm{T}_h^0 f(x) = f(x)$, $\mathrm{T}_h^i f(x) = \mathrm{T}_h(\mathrm{T}_h^{i-1} f(x)); i = 1, 2, ..., k.; k = 1, 2, ...$ and $\mathrm{E}_{2,\alpha}$ is a unit operator in $\mathrm{L}_{2,\alpha}$.

Let $\mathrm{W}_{2,\alpha}^k$ be the Sobolev space constructed by the Bessel operator B, i.e.,

$$\mathrm{W}_{2,\alpha}^k = \{f \in \mathrm{L}_{2,\alpha},\ \mathrm{B}^j f \in \mathrm{L}_{2,\alpha};\ j = 1, 2, ..., k\}.$$

In [4], we have the following result

Theorem 1.3 *Let $f \in \mathrm{L}_{2,\alpha}$. Then the following are equivalents*

(1) $f \in Lip(k, \gamma, 2),\ 0 < k < 1,\ \gamma \geqslant 0,$
(2) $\int_r^\infty |\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda = O(r^{-2k}(\log r)^{2\gamma})$ *as* $r \longrightarrow +\infty,$

where

$$Lip(k, \gamma, 2) = \{f \in \mathrm{L}_{2,\alpha}, \|\mathrm{T}_h f(t) - f(t)\|_{2,\alpha} = O\left(\frac{h^k}{(\log\frac{1}{h})^\gamma}\right) \ as\ h \longrightarrow 0\}.$$

The main aim of this paper is to establish a generalization of Theorem 1.3

2. Main Results

In this section we present the main result of this paper. We first need to define the $(\delta, \gamma, 2)$-Bessel Lipschitz class.

Definition 2.1 *Let $0 < \delta < 1$, $\gamma \geqslant 0$ and $r = 0, 1, ..., k$. A function $f \in \mathrm{W}^k_{2,\alpha}$ is said to be in the $(\delta, \gamma, 2)$-Bessel Lipschitz class, denoted by $Lip(\delta, \gamma, 2)$; if*

$$\|\Delta^k_h \mathrm{B}^r f(x)\|_{2,\alpha} = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \longrightarrow 0.$$

Lemma 2.2 *Let $f \in \mathrm{W}^k_{2,\alpha}$. Then*

$$\|\Delta^k_h \mathrm{B}^r f(x)\|^2_{2,\alpha} = \frac{1}{(2^\alpha \Gamma(\alpha+1))^2} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

where $r = 0, 1, ..., k$

Proof From formula (2), we have

$$\widehat{\mathrm{B}^r f}(\lambda) = (-1)^r \lambda^{2r} \widehat{f}(\lambda) \tag{4}$$

By formulas (1) and (4), we conclude that

$$\widehat{\mathrm{T}^i_h \mathrm{B}^r f}(\lambda) = (-1)^r t^{2r} j^i_\alpha(th) \widehat{f}(\lambda), 1 \leqslant i \leqslant k. \tag{5}$$

From formulas (3) and (5) follows that the Bessel transform of $\Delta^k_h \mathrm{B}^r f(x)$ is $(-1)^r \lambda^{2r} (j_\alpha(\lambda h) - 1)^k \widehat{f}(\lambda)$.

By Parseval's identity we have the result. ∎

Theorem 2.3 *Let $f \in \mathrm{W}^k_{2,\alpha}$. Then the followings are equivalents*

(1) $f \in Lip(\delta, \gamma, 2)$,
(2) $\int_s^\infty \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right)$ *as* $s \longrightarrow +\infty$.

Proof 1) $\Longrightarrow$ 2) Assume that $f \in Lip(\delta, \gamma, 2)$. Then we have from Lemma 2.2

$$\|\Delta^k_h \mathrm{B}^r f(x)\|^2_{2,\alpha} = \frac{1}{(2^\alpha \Gamma(\alpha+1))^2} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $\lambda h \geqslant 1$ and Lemma 1.2 implies that

$$1 \leqslant \frac{1}{c^{4k}} |1 - j_\alpha(\lambda h)|^{2k}.$$

Then

$$\int_{1/h}^{2/h}\lambda^{4r}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda \leqslant \frac{1}{c^{4k}}\int_{1/h}^{2/h}\lambda^{4r}|1-j_\alpha(\lambda h)|^{2k}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda$$
$$\leqslant \frac{1}{c^{4k}}\int_{0}^{\infty}\lambda^{4r}|1-j_\alpha(\lambda h)|^{2k}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda$$
$$= O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right).$$

We have

$$\int_{s}^{2s}\lambda^{4r}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda \leqslant C\frac{s^{-2\delta}}{(\log s)^{2\gamma}},$$

where C is a positive constant.
So that

$$\int_{s}^{\infty}\lambda^{4r}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda = \left[\int_{s}^{2s}+\int_{2s}^{4s}+\int_{4s}^{8s}+....\right]|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda$$
$$\leqslant C\frac{s^{-2\delta}}{(\log s)^{2\gamma}}+C\frac{(2s)^{-2\delta}}{(\log 2s)^{2\gamma}}+C\frac{(4s)^{-2\delta}}{(\log 4s)^{2\gamma}}+......$$
$$\leqslant C\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\left(1+2^{-2\delta}+(2^{-2\delta})^2+(2^{-2\delta})^3+.....\right)$$
$$\leqslant CC_\delta\frac{s^{-2\delta}}{(\log s)^{2\gamma}},$$

where $C_\delta=(1-2^{-2\delta})^{-1}$ since $2^{-2\delta}<1$.
This proves that

$$\int_{s}^{\infty}\lambda^{4r}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right)\ \ as\ s\longrightarrow+\infty$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_{s}^{\infty}\lambda^{4r}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right)\ \ as\ s\longrightarrow+\infty$$

We write

$$\int_{0}^{\infty}\lambda^{4r}|1-j_\alpha(\lambda h)|^{2k}|\widehat{f}(\lambda)|^2\lambda^{2\alpha+1}d\lambda = \mathrm{I}_1+\mathrm{I}_2$$

where

$$\mathrm{I}_1 = \int_0^{1/h} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

and

$$\mathrm{I}_2 = \int_{1/h}^{\infty} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

Estimate the summands I_1 and I_2 from above. It follows from the formula $|j_\alpha(\lambda h)| \leqslant 1$ that

$$\begin{aligned}
\mathrm{I}_2 &= \int_{1/h}^{\infty} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\
&\leqslant 4^k \int_{1/h}^{\infty} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\
&= O\left(\frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right)
\end{aligned}$$

To estimate I_1, we use the inequality (2) of Lemma 1.1
Set

$$\psi(x) = \int_x^{\infty} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

Using integration by parts, we obtain

$$\begin{aligned}
\mathrm{I}_1 &\leqslant -C_1 h^{4k} \int_0^{1/h} t^{4k} \psi'(t) dt \\
&\leqslant C_1 \psi(\frac{1}{h}) + 4C_1 k h^{4k} \int_0^{1/h} t^{4k-1} \psi(t) dt \\
&\leqslant C_2 h^{4k} \int_0^{1/h} t^{4k-1} t^{-2\delta} (\log t)^{-2\gamma} dt \\
&\leqslant C_3 h^{-2\delta} (\log \frac{1}{h})^{-2\gamma},
\end{aligned}$$

where C_1, C_2 and C_3 are positive constants and this ends the proof. ■

COROLLARY 2.4 *Let* $f \in \mathrm{W}^k_{2,\alpha}$, *and let*

$$f \in Lip(\delta, \gamma, 2)$$

Then

$$\int_{s}^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta-4r}}{(\log s)^{2\gamma}}\right) \ as \ s \longrightarrow +\infty$$

3. Conclusion

In this work we have succeded to generalise the theorem 5 in [4] for the Bessel transform in the Sobolev space $W^k_{2,\alpha}$ constructed by the Bessel operator B. We proved that $f \in Lip(\delta, \gamma, 2)$ if and only if $\int_s^{+\infty} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right) \ as \ s \longrightarrow +\infty$.

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