International Journal of
Mathematical Modelling & Computations
Vol. 12, No. 04, Fall 2022, 249- 263
DOI:10.30495/ijm2c.2022.1963557.1259

A Computational Approach for Fractal Mobile-Immobile Transport with Caputo-Fabrizio Fractional Derivative

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Abstract. This paper deals with a spectral collocation method for the numerical solution of linear and nonlinear fractal Mobile/Immobile transport (FM/IT) model with Caputo-Fabrizio fractional derivative (C-F-FD). In the time direction, the finite difference procedure is used to construct a semi-discrete problem and afterwards by applying a Chebyshev-spectral method, we obtain the approximate solution. The unconditional stability of the proposed method is proved which provides the theoretical basis of proposed method for solving the considered equation. Finally, some numerical experiments are included to clarify the efficiency and applicability of our proposed concepts in the sense of accuracy and convergence ratio.

Received: 18 July 2022, Revised: 01 September 2022, Accepted: 02 December 2022.

Keywords: Fractal nobile-immobile transport (FM/IT) model; Caputo-Fabrizio fractional derivative (C-F-FD); Spectral approximation; Stability analysis.

AMS Subject Classification: 26A33, 34K37.

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1. Introduction

In the last decades, the use of fractional-order derivatives has become popular due to its nonlocality property which is an intrinsic property of many complex systems. The fractional-order derivatives are widely applied in modeling of physical phenomena such as viscoelasticity, electrochemistry, electromagnetic, nanotechnology, control theory of dynamical systems, financial modeling, random walk, anomalous transport and anomalous diffusion, porous materials, and biological modeling [14, 16]. Several researchers studied fractional calculus that we can mention the following works:

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Atangana and Baleanu [4] suggested a new fractional derivative with nonlocal and nonsingular kernel for solving fractional heat in material with different scales and also those with heterogeneous media. Sharma et al. analyzed [18] nonlinear dynamics of CattaneoChristov heat flux model for third-grade power-law fluid. Tateishi and et al. [19] solved the fractional diffusion equation without external forces and according to the free diffusion boundary conditions. Atangana and Qureshi [6] proposed fractal-fractional derivatives which estimate the chaotic behavior of some attractors from applied mathematics. Yuste and Acedo [21] suggested a set of continuum fractional diffusion equations to investigate the behavior of a reaction front in the $A + B \to C$ reactionsubdiffusion process. The concept of fractional-order derivatives based on of the exponential and Mittag-Leffler laws, are described in [1, 5].

Recently, fractional partial differential equations (FPDEs) have attracted increasing attention in both theory and application. The superior capabilities of FPDEs to accurately model different phenomena have raised significant interest in assaying analytical and numerical methods for obtaining the solutions to such problems. It is usually difficult to obtain closed-form solutions for FPDEs. Therefore, the approximate solutions of these equations have been the subject of many publications. From the numerical point of view, some various approximation methods have been presented for solving FPDEs. The main aim of [11] is to propose an implicit difference approximation scheme (IDAS) for the numerical solution of fractional diffusion equation. Baeumer et al. [7] developed a practical method based on operator splitting for solving of fractional reaction-diffusion equations. Chen et al. [12] applied the Kansa method for solving the time fractional diffusion equations, in which the Multi-Quadrics and thin plate spline serve as the radial basis function. The main aim of [22] is to present an implicit numerical method to solve the nonlinear fractional reactionsubdiffusion equations.

Recently, Caputo and Fabrizio [10] have defined a new fractional derivative without a singular kernel. The new definition is called as the CaputoFabrizio fractional derivative (C-F-FD) by some researchers. The models with the new C-F-FD can describe the fluctuations of different scales and material heterogeneities, which cannot be described by classical local theories or by fractional models with a singular kernel. So far, some researchers have started analytical and numerical studies on the basis of the new C-F-FD; see [2, 3, 8, 13]. However, the studies on the numerical methods for FPDEs with the C-F-FD have been rarely reported.

In this paper, we deal with the linear and nonlinear fractal Mobile/Immobile transport (FM/IT) model with C-F-FD [15]:

Linear FM/IT model with C-F-FD:

$$\lambda_1 \frac{\partial \mathcal{U}(x,t)}{\partial t} + \lambda_2 \, {}_0^{CF}\partial_t^{\alpha} \mathcal{U}(x,t) = \gamma_1 \partial_x^2 \mathcal{U}(x,t) - \gamma_2 \mathcal{U}(x,t) + f(x,t), \tag{1}$$

Nonlinear FM/IT model with C-F-FD:

$$\lambda_1 \frac{\partial \mathcal{U}(x,t)}{\partial t} + \lambda_2 \, {}_0^{CF}\partial_t^{\alpha} \mathcal{U}(x,t) = \gamma \partial_x^2 \mathcal{U}(x,t) + \mathcal{Q}(\mathcal{U}) + f(x,t), \tag{2}$$

where $(x,t) \in \Omega \times (0,T]$, $\Omega = (-1,1)$, $\mathcal{U} = \mathcal{U}(x,t)$ is a sufficiently differentiable function in $\overline{\Omega} \times [0,T]$ and the time-fractional derivative ${}_0^{CF}\partial_t^{\alpha}\mathcal{U}(x,t)$ is the C-F-FD defined by

$$ {}_{0}^{CF}\partial_t^{\alpha}\mathcal{U}(x,t) := \int_0^t \frac{\partial \mathcal{U}(x,s)}{\partial s}\vartheta_{\alpha}(t-s)ds,\ 0<\alpha<1, $$

in which $\vartheta_{\alpha}(t) := \frac{\exp\left(-\frac{\alpha}{1-\alpha}(t)\right)}{1-\alpha}$.
The term $\mathcal{Q}(\mathcal{U})$ in (2) satisfies the following conditions:

- There exists a positive constant c such that $|\mathcal{Q}(\mathcal{U})| \leqslant c|\mathcal{U}|$,
- There exists a positive constant c such that $|\mathcal{Q}^{'}(\mathcal{U})| \leqslant c$.

For Eqs. (1) and (2), the initial condition:

$$ \mathcal{U}(x,t)|_{t=0} = h(x),\ x \in \overline{\Omega}, \tag{3} $$

and the Dirichlet boundary conditions:

$$ \mathcal{U}(x,t)|_{x\in\partial\Omega} = 0,\ t>0, \tag{4} $$

are considered.

In this paper, we present a spectral method to compute the approximate solution for linear and nonlinear FM/IT models with C-F-FD. The rest of this paper is organized as follows. In Section 2, we present a computational approach to construct numerical solution for fractal Mobile/Immobile transport model with C-F-FD. We prove the convergence and the stability of the method in this section. Some test problems are presented and the results are shown in Section 3 and we discuss the numerical performance of our method. Finally, in Section 4 some concluding remarks are presented.

2. FM/IT model with C-F-FD

2.1 *Linear FM/IT model with C-F-FD*

2.1.1 Discretization of C-F-FD and semi-discrete scheme

In this subsection, we deal with the linear FM/IT model with C-F-FD. For discretization of time variable, let $t_k := k\delta t,\ k = 0,1,...,N$ be an equidistant partition of $[0,T]$, where $\delta t = \frac{T}{N}$. We analogize the C-F-FD term by using the finite difference scheme:

$$ \begin{aligned} &{}_{0}^{CF}\partial_t^{\alpha}\mathcal{U}^{k+1}(x) \\ &= \begin{cases} \bar{c}_{\alpha,\delta t}[\mathcal{D}_{\alpha,k+1}^{k+1}(\mathcal{U}^{k+1}(x)-\mathcal{U}^{k}(x)) + \sum_{j=1}^{k}\mathcal{D}_{\alpha,j}^{k+1}(\mathcal{U}^{j}(x)-\mathcal{U}^{j-1}(x))], & k\geqslant 1 \\ \bar{c}_{\alpha,\delta t}\mathcal{D}_{\alpha,1}^{1}(\mathcal{U}^{1}(x)-\mathcal{U}^{0}(x)), & k=0, \end{cases} + r_{\mathcal{U}}^{k+1}(x), \end{aligned} \tag{5} $$

where

$$ \bar{c}_{\alpha,\delta t} = (\alpha\delta t)^{-1}, $$

and

$$ \mathcal{D}_{\alpha,j}^{k+1} = \exp(-\frac{\alpha\delta t}{1-\alpha}(k+1-j)) - \exp(-\frac{\alpha\delta t}{1-\alpha}(k-j+2)),\ (j=1,2,\cdots,k+1). $$

Theorem 2.1 *([15]) For any $0 < \alpha < 1$, the coefficients of $\mathcal{D}_{\alpha,j}^{k+1}$, $j = 1, 2, \cdots, k+1$ satisfy the following properties*

- $\mathcal{D}_{\alpha,j}^{k+1} > 0, \ \forall j \leqslant k+1;$
- $\mathcal{D}_{\alpha,j}^{k+1} \leqslant \mathcal{D}_{\alpha,j+1}^{k+1}, \ \forall j \leqslant k;$
- $\mathcal{D}_{\alpha,k+1}^{k+1} = \mathcal{D}_{\alpha,1}^{1}, \ \mathcal{D}_{\alpha,k}^{k+1} = \mathcal{D}_{\alpha,1}^{2};$
- $\sum_{j=1}^{k-1}(\mathcal{D}_{\alpha,j+1}^{k+1} - \mathcal{D}_{\alpha,j}^{k+1}) + \mathcal{D}_{\alpha,1}^{k+1} = \mathcal{D}_{\alpha,k}^{k+1} = \mathcal{D}_{\alpha,1}^{2}.$

Theorem 2.2 *([15]) For any $0 < \alpha < 1$, it holds*

$$r_{1,\mathcal{U}}^{k+1}(x) = -\frac{1}{1-\alpha}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(s - t_{j-\frac{1}{2}})\frac{\partial^2\mathcal{U}(x,s)}{\partial s^2}|_{s=\varsigma_j}\exp(-\frac{\alpha}{1-\alpha}(t_{k+1}-s))ds,$$

$$|r_{1,\mathcal{U}}^{k+1}(x)| \leqslant \frac{c}{\alpha}\exp(\frac{2\alpha}{1-\alpha})\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t^2, \ \ -1 \leqslant k \leqslant N-1, \ \forall x \in \Omega,$$

where $\varsigma_j \in (t_{j-1}, t_j)$ and c are independent of δt.

Also, the first order temporal derivative can be approximated as follows

$$\frac{\partial\mathcal{U}^{k+1}(x)}{\partial t} = \frac{\mathcal{U}^{k+1}(x) - \mathcal{U}^{k}(x)}{\delta t} + r_{2,\mathcal{U}}^{k+1}(x), \tag{6}$$

where the truncation error $r_{2,\mathcal{U}}^{k+1}(x)$ satisfies $|r_{2,\mathcal{U}}^{k+1}(x)| \leqslant c\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t$, in which c is independent δt.
Substituting (5) and (6) into (1), we get

$$\mathcal{B}_{1,\alpha,\delta t}\mathcal{U}^{k+1}(x) - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1\partial_x^2\mathcal{U}^{k+1}(x) = \mathcal{P}_t^{\alpha}\mathcal{U}^{k}(x) + F^{k+1}(x) + \mathcal{R}_{\mathcal{U}}^{k+1}(x), \ x \in \overline{\Omega},$$

where

$$\mathcal{P}_t^{\alpha}\mathcal{U}^{k}(x) = \begin{cases} (\lambda_1\alpha + \lambda_2\mathcal{D}_{\alpha,1}^{1})\mathcal{U}^0(x), & k = 0,\\ \mathcal{B}_{2,\alpha,\delta t}\mathcal{U}^{k}(x) + \lambda_2\sum_{j=1}^{k-1}(\mathcal{D}_{\alpha,j+1}^{k+1} - \mathcal{D}_{\alpha,j}^{k+1})\mathcal{U}^{j}(x) + \lambda_2\mathcal{D}_{\alpha,1}^{k+1}\mathcal{U}^0(x), & k \geqslant 1,\end{cases}$$

$$F^{k+1} = \bar{c}_{\alpha,\delta t}^{-1}f(x, t_{k+1}), \ k = 0, 1, \cdots, N-2, \ \mathcal{U}^0(x) = h(x),$$

and

$$\mathcal{B}_{1,\alpha,\delta t} = \lambda_1\alpha + \lambda_2\mathcal{D}_{\alpha,1}^{1} + \gamma_2\bar{c}_{\alpha,\delta t}^{-1}, \ \mathcal{B}_{2,\alpha,\delta t} = \lambda_1\alpha + \lambda_2(\mathcal{D}_{\alpha,1}^{1} - \mathcal{D}_{\alpha,1}^{2}).$$

Furthermore, the truncation error $\mathcal{R}_{\mathcal{U}}^{k+1}(x)$ satisfy

$$|\mathcal{R}_{\mathcal{U}}^{k+1}(x)| \leqslant \frac{c}{\alpha}\exp(\frac{2\alpha}{1-\alpha})\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t^2, \ \ -1 \leqslant k \leqslant N-1, \ \forall x \in \Omega.$$

Replacing $\mathcal{U}^{k+1}(x)$ by the approximate solution $u^{k+1}(x)$, we can obtain the following semi-discrete problem for (1) and (3)-(4), which is given by:

Scheme I: Given $u^0 = h(x)$ and find u^{k+1} $(k = 0, 1, 2, \cdots, N-1)$, such that:

$$\begin{cases} \mathcal{B}_{1,\alpha,\delta t} u^{k+1}(x) - \bar{c}_{\alpha,\delta t}^{-1} \gamma_1 \partial_x^2 u^{k+1}(x) = \mathcal{P}_t^{\alpha} u^k(x) + F^{k+1}(x), x \in \overline{\Omega}, \\ u^{k+1}|_{x\in\partial\Omega} = 0, -1 \leqslant k \leqslant N-1, \end{cases} \tag{7}$$

2.1.2 Spectral approximation to semi-discrete problem (7)

Consider the Hilbert space of μ-measurable $L^2((-1,1), d\mu(x))$, where

$$d\mu(x) = w(x)dx = (1-x^2)^{-\frac{1}{2}} dx.$$

Furthermore, the Hilbert space $L^2((-1,1), d\mu(x))$ is equipped with inner product

$$\langle u, v\rangle_{0,\omega} = \int_{-1}^{1} u(x)v(x)(1-x^2)^{-\frac{1}{2}} d(x).$$

Theorem 2.3 *([17]) Let $\mathbb{P}_M$ denote the set of polynomials of degree$\leqslant M$. If $\boldsymbol{B}_M$ be a sequence of orthogonal polynomials on $(-1,1)$ of degree$\leqslant M$, i.e.,*

$$\boldsymbol{B}_M = \{u \in \mathbb{P}_M | \langle u, v\rangle_{0,\omega} = 0, \ \forall v \in \mathbb{P}_{M-1}\},$$

then there exists a reproducing kernel $K_M : (-1,1) \times (-1,1) \to \mathbb{R}$ such that

$$u(x) = \langle u, K_M(x, .)\rangle_{0,\omega}, \ \forall u \in \mathbb{P}_M, \forall x \in (-1,1),$$

and

$$0 = \langle (x+1)u, K_M(-1, .)\rangle_{0,\omega} = \langle (1-x)u, K_M(1, .)\rangle_{0,\omega}, \ \forall u \in \mathbb{P}_{M-1}.$$

Let $\{T_M\}_{M\geqslant 0}$ be the Chebyshev polynomials in $L^2((-1,1), d\mu(x))$ with $degree(P_M) = M$, we consider

$$q_M(x) = \frac{T_{M+2}(x) + c_M T_{M+1}(x) + d_M T_M(x)}{(1-x)(x+1)} \in \mathbb{P}_M,$$

where

$$c_M = -\frac{[T_{M+2}(1)T_M(-1) + T_{M+2}(-1)T_M(1)]}{[T_M(-1)T_{M+1}(1) - T_M(1)T_{M+1}(-1)]},$$
$$d_M = -\frac{[T_{M+2}(T_{M+1}(-1) + T_{M+2}(-1)T_{M+1}(1)]}{[T_M(1)T_{M+1}(-1) - T_M(-1)T_{M+1}(1)]}.$$

Hence, $\{q_M\}_{M\geqslant 0}$ is a sequence of orthogonal polynomials in $L^2((-1,1), d\widetilde{\mu}(x))$ equipped with inner product

$$\langle u, v\rangle_{2,\widetilde{\omega}} = \int_{-1}^{1} u(x)v(x)d\widetilde{\mu}(x), \ d\widetilde{\mu}(x) = \widetilde{\omega}(x)dx = (1-x)(x+1)(1-x^2)^{-\frac{1}{2}} dx.$$

It is shown in [17] that

$$K_{M-2}(x,y) = \sum_{m=0}^{M-2} \frac{q_m(x)q_m(y)}{\|q_m\|^2_{2,\widetilde{\omega}}}$$
$$= \frac{k_M(q_{M-1}(x)q_{M-2}(y) - q_{M-2}(x)q_{M-1}(y))}{k_{M+1}\|q_{M-2}\|^2_{0,\widetilde{\omega}}(x-y)},\ x \neq y,$$

where $K_{M-2}(.,y) \in \mathbb{P}_{M-2}$ and $-k_{M+1} < 0$ is the leading coefficient of x^{M+1} in $(x+1)(1-x)q_{M-1}(x)$.

We also have

$$K_{M-2}(x,x) = \sum_{m=0}^{M-2} \frac{q_m^2(x)}{\|q_m\|^2_{0,\widetilde{\omega}}} = \frac{k_M(q'_{M-1}(x)q_{M-2}(x) - q'_{M-2}(x)q_{M-1}(x))}{k_{M+1}\|q_{M-2}\|^2_{0,\widetilde{\omega}}}.$$

Suppose that $\{z_j\}_{j=1}^{M-1}$ denote the $M-1$ simple zero points of q_{M-1} on $(-1,1)$, then we have

$$K_{M-2}(z_i,z_j) = \sum_{m=0}^{M-2} \frac{q_m(z_i)q_m(z_j)}{\|q_m\|^2_{0,\widetilde{\omega}}} = \begin{cases} 0, & i \neq j, \\ \widetilde{\omega}_i^{-1} = \frac{k_M q'_{M-1}(z_i)q_{M-2}(z_i)}{k_{M+1}\|q_{M-2}\|_{0,\widetilde{\omega}}}, & i = j. \end{cases}$$

Let $\{z_j\}_{j=0}^{M}$ denote the $M+1$ simple zero points of $(x+1)(1-x)q_{M-1}$ on $[-1,1]$, it is well known [17] that there exists a unique set of quadrature weights $\{\omega_j\}_{j=0}^{M}$ such that we have

$$\int_{-1}^{1} u(x)\frac{1}{\sqrt{1-x^2}}dx = \sum_{j=0}^{M} \omega_j u(z_j),\ \forall u \in \mathbb{P}_{2M-1}, z_j = -\cos\frac{\pi j}{M}, j = 0,\cdots,M,$$

where

$$\omega_j = \frac{\pi}{\sigma_j M}, j = 0,1,\cdots,M,$$

in which

$$\sigma_j = \begin{cases} 2, j = 0, M, \\ 1, 1 \leqslant j \leqslant M-1, \end{cases}$$

An approximant u_M^k to u^k can be obtained by calculating a truncated series based on

$$\mathbb{P}_M = span\{\phi_j(x), j = 0,1,\cdots,M\},$$

$$\phi_j(x) = \frac{(x+1)(1-x)q_{M-1}(x)}{((x+1)(1-x)q_{M-1}(x))'|_{x=z_j}(x-z_j)}$$

as

$$u^k(x) \approx u_M^k(x) := \Phi(x)\{\mathbf{v}\}^k,$$

where

$$\Phi(x) = (\phi_0(x), \phi_1(x), \cdots, \phi_M(x)),$$

and

$$\{\mathbf{v}\}^k = (v_0^k, v_1^k, \cdots, v_M^k)^T.$$

Also $\frac{d^m}{dx^m}\Phi(x)$ can be expressed in the following matrix form

$$\frac{d^m}{dx^m}\Phi(x) = \Phi(x)\mathbf{D}^m,\ m \geqslant 1,$$

where

$$\mathbf{D} = [\mathbf{D}_{ij}] = [\phi_j^{'}(z_i)], i, j = 0, 1, \cdots, M,\ \mathbf{D}^m = \underbrace{\mathbf{D}\mathbf{D}\cdots\mathbf{D}}_{m}.$$

The entries of the first-order differentiation matrix $\mathbf{D}$ can be determined by

$$\mathbf{D}_{ij} = \begin{cases} \frac{((x-a)(b-x)q_{M-1}(x))'|_{x=z_i}}{((x-a)(b-x)q_{M-1}(x))'|_{x=z_j}(z_i-z_j)}, & i \neq j, \\ \frac{((x-a)(b-x)q_{M-1}(x))''|_{x=z_i}}{2((x-a)(b-x)q_{M-1}(x))''|_{x=z_i}}, & i = j. \end{cases}$$

$$= \begin{cases} -\frac{2M^2+1}{6}, & i = j = 0, \\ \frac{\sigma_i}{\sigma_j}\frac{(-1)^{i+j}}{z_i - z_j}, & i \neq j,\ 0 \leqslant i, j \leqslant M, \\ -\frac{z_i}{2(1-z_k^2)}, & i = j,\ 1 \leqslant i, j \leqslant M-1, \\ \frac{2M^2+1}{6}, & i = j = M. \end{cases}$$

Then, we approximate $\partial_x^m u_M^k$ by

$$\partial_x^m u_M^k(x) := \Phi(x)\mathbf{D}^m\{\mathbf{v}\}^k,\ m \geqslant 1.$$

Thus, we have:

$$\frac{d^m}{dx^m}u_M^k(z_i) = \sum_{j=0}^{M}(\mathbf{D}^m)_{ij}v_j^k, m \geqslant 1,\ 1 \leqslant i \leqslant M-1.$$

We define the corresponding discrete inner product as

$$\langle u, v\rangle_M = \sum_{j=0}^{M}\omega_j u(z_j)v(z_j),$$

which induces the norm $\|u\|_M = (\langle u, u\rangle_{M,\omega})^{\frac{1}{2}}$ and satisfies

$$\langle u, v\rangle_M = \langle u, v\rangle_{0,\omega},\ \forall u, v : u.v \in \mathbb{P}_{2M-1}.$$

Consider the weight Sobolov space $H^r((-1, 1), d\mu(x))$ as

$$H^r((-1, 1), d\mu(x)) = \{u \in L^2((-1, 1), d\mu(x)) : \|u\|_{r,\omega} = (\sum_{j=0}^{r}\|\partial_x u\|_{0,\omega}^2)^{\frac{1}{2}} < \infty\},$$

moreover, we set $H_0^1((-1,1),d\mu(x)) = \{u \in L^2((-1,1),d\mu(x)) : \partial_x u \in L^2((-1,1),d\mu(x)), u(-1) = u(1) = 0\}$, we also introduce the the bilinear form over $H_0^1((-1,1),d\mu(x))$ as

$$a_\omega\langle u,v\rangle = \langle \partial_x u, \omega^{-1}\partial_x(v\omega)\rangle_{0,\omega} = \int_{-1}^{1} \partial_x u \partial_x(v\omega)dx,\ \forall u,v \in H_0^1((-1,1),d\mu(x)).$$

Now, we will give the representation of numerical solution to semi-discrete problem (7) in the space $\mathbb{P}_M$.
Given $u_M^0 = I_M^c u^0$ and find $u_M^{k+1} \in \mathbb{P}_M$ $(k = 0,1,2,\cdots,N-1)$, such that:

$$\begin{cases} \mathcal{B}_{1,\alpha,\delta t} u_M^{k+1}(z_i) - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1 \partial_x^2 u_M^{k+1}(z_i) = \mathcal{P}_t^\alpha u_M^k(z_i) + F^{k+1}(z_i), 1 \leqslant i \leqslant M-1, \\ u_M^{k+1}(z_i) = 0,\ i = 0, M,\ \ -1 \leqslant k \leqslant N-1, \end{cases} \tag{8}$$

where

$$\mathcal{P}_t^\alpha u_M^k(z_i) = \begin{cases} (\lambda_1\alpha + \lambda_2 \mathcal{D}_{\alpha,1}^1) u_M^0(z_i), & k = 0, \\ \mathcal{B}_{2,\alpha,\delta t} u_M^k(z_i) + \lambda_2 \sum_{j=1}^{k-1}(\mathcal{D}_{\alpha,j+1}^{k+1} - \mathcal{D}_{\alpha,j}^{k+1}) u_M^j(z_i) + \lambda_2 \mathcal{D}_{\alpha,1}^{k+1} u^0(z_i), & k \geqslant 1, \end{cases}$$

and $I_M^c : C[a,b] \to \mathbb{P}_M$ is the interpolation operator associated with $\{z_i,\omega_i\}_{j=0}^M$ such that

$$(I_M^c u)(z_i) = u(z_i),\ i = 0,1,2,\cdots,M.$$

Let us denote $\mathbb{X}_M = \{v_M | v_M \in \mathbb{P}_M,\ v_M(z_0) = v_M(z_M) = 0.\}$, we can reformulate the scheme (8) as the following:
S-A(I): Find the spectral approximation $u_M^{k+1} \in \mathbb{X}_M$ $(k = 0,1,2,\cdots,N-1)$, such that for all $v_M \in \mathbb{X}_M$:

$$\mathcal{B}_{1,\alpha,\delta t}\langle u_M^{k+1}, v_M\rangle_M + \bar{c}_{\alpha,\delta t}^{-1}\gamma_1 a_\omega\langle u_M^{k+1}, v_M\rangle = \langle \mathcal{P}_t^\alpha u_M^k, v_M\rangle_M + \langle I_M^c F^{k+1}, v_M\rangle_M. \tag{9}$$

The approximate solution u_M^k can be obtained by calculating a truncated series based on $\mathbb{P}_M = span\{\phi_j(x), j = 0,1,\cdots,M\}$ as the following

$$u^k(x) \approx u_M^k(x) := \Phi(x)\{\mathbf{v}\}^k.$$

Therefore, we get

$$\begin{cases} \sum_{j=0}^M \left[\mathcal{B}_{1,\alpha,\delta t}\delta_{ij} - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1(\mathbf{D}^2)_{ij}\right] v_j^{k+1} = \digamma^{k+1}(z_i),\ 1 \leqslant i \leqslant M-1, \\ \Phi(z_0)\{\mathbf{v}\}^{k+1} = \Phi(z_M)\{\mathbf{v}\}^{k+1} = 0. \end{cases} \tag{10}$$

where

$$\digamma^{k+1}(z_i) = \mathcal{P}_t^\alpha u_M^k(z_i) + F^{k+1}(z_i), 1 \leqslant i \leqslant M-1,$$

Let us denote

$$\begin{aligned}
&(\mathbf{B})_{ij} = \mathcal{B}_{1,\alpha,\delta t}\delta_{ij} - \bar{c}_{\alpha,\delta t}^{-1}\gamma_1(\mathbf{D}^2)_{ij},\ 1 \leqslant i \leqslant M-1,\ 0 \leqslant j \leqslant M,\\
&(\mathbf{B})_{0j} = \delta_{0j},\ (\mathbf{B})_{Mj} = \delta_{Mj},\ 0 \leqslant j \leqslant M,\\
&\{\mathbf{c}\}^{k+1} = (0, \mathcal{F}^{k+1}(z_1), \mathcal{F}^{k+1}(z_2), \cdots, \mathcal{F}^{k+1}(z_{M-1}), 0)^T,\\
&\{\mathbf{v}\}^{k+1} = (v_0^{k+1}, v_1^{k+1}, \cdots, v_M^{k+1})^T,
\end{aligned}$$

then, the linear system (10) reduces to

$$\mathbf{B}\{\mathbf{v}\}^{k+1} = \{\mathbf{c}\}^{k+1},\ k = 0, 1, \cdots, N-1.$$

Lemma 2.4 *([9])For any $u \in \mathbb{P}_M$, we have*

$$\|u\|_{0,\omega} \leqslant \|u\|_M \leqslant \sqrt{2}\|u\|_{0,\omega}.$$

Lemma 2.5 *([9]) If $u \in H_0^1((-1,1), d\mu(x))$, then there holds*

$$\|u\|_{0,\omega} \leqslant c\|\partial_x u\|_{0,\omega},$$

where c is positive constant independent of u.

Lemma 2.6 *([9]) For any $u \in H_0^1((-1,1), d\mu(x))$, we have*

$$\begin{aligned}
|a_\omega\langle u, u\rangle| &\leqslant c\|\partial_x u\|_{0,\omega}^2,\\
a_\omega\langle u, u\rangle &\geqslant \frac{1}{4}\|\partial_x u\|_{0,\omega}^2,
\end{aligned}$$

where c is positive constant independent of u.

Lemma 2.7 *([20]) (Discrete Gronwall inequality) Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are nonnegative sequences and c is a nonnegative constant. If*

$$f_i \leqslant c + \sum_{j=0}^{i-1} g_j f_j,\ i \geqslant 0,$$

then

$$f_i \leqslant c \prod_{0 \leqslant j \leqslant i-1} (1 + g_j) \leqslant c e^{\sum_{j=0}^{i-1} g_j},\ i \geqslant 0.$$

Theorem 2.8 *Let $u_M^{k+1} \in \mathbb{X}_M,\ k = 0, 1, \cdots, M-1$ be the solution of scheme (9). Then the scheme (9) is unconditionally stable in the sense that for all $\delta t > 0$.*

Proof We know $\|u_M^{k+1}\|_{0,\omega} \leqslant c\|\partial_x u_M^{k+1}\|_{0,\omega}$. Set

$$\begin{aligned}
C_{1,\alpha,\delta t} &= \min\{\gamma_2\bar{c}_{\alpha,\delta t}^{-1}, \frac{\bar{c}_{\alpha,\delta t}^{-1}\gamma_1}{4}\},\\
C_{2,\alpha,\delta t} &= \max\{\lambda_2, \frac{\lambda_1\alpha\sqrt{2}c}{(\mathcal{D}_{\alpha,1}^1 - \mathcal{D}_{\alpha,1}^2)}\},
\end{aligned}$$

therefore, we can get the following inequality

$$\|u_M^{k+1}\|_M^2 + \|\partial_x u_M^{k+1}\|_{0,\omega}^2 \leqslant \sum_{j=1}^{k} \frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}} (\mathcal{D}_{\alpha,j+1}^{k+1} - \mathcal{D}_{\alpha,j}^{k+1})(\|u_M^j\|_M^2 + \|\partial_x u_M^j\|_{0,\omega}^2)$$
$$+ \lambda_2 C_{1,\alpha,\delta t}^{-1} \mathcal{D}_{\alpha,1}^{k+1} \|u_M^0\ |_M^2 + \frac{1}{3C_{1,\alpha,\delta t}(\lambda_1\alpha + \lambda_2 \mathcal{D}_{\alpha,1}^1)} \|I_M^c F^{k+1}\|_M^2.$$

Noting Lemma 2.7, we have

$$\|u_M^{k+1}\|_M^2 + \|\partial_x u_M^{k+1}\|_{0,\omega}^2 \leqslant \left(\|u_M^0\|_M^2 + \|I_M^c F^{k+1}\|_M^2\right) e^{\frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}(\mathcal{D}_{\alpha,1}^1 - \mathcal{D}_{\alpha,1}^{k+1})}. \quad (11)$$

where $\mathbf{C}_{1,\alpha,\delta t} = \max\{\lambda_2 C_{1,\alpha,\delta t}^{-1} \mathcal{D}_{\alpha,1}^{k+1}, \frac{1}{3C_{1,\alpha,\delta t}(\lambda_1\alpha+\lambda_2\mathcal{D}_{\alpha,1}^1)}\}$ and $\mathbf{C}_{2,\alpha,\delta t} = \frac{C_{2,\alpha,\delta t}}{C_{1,\alpha,\delta t}}$. Using (11), the following inequality is holds

$$\|u_M^{k+1} - \widetilde{u}_M^{k+1}\|_M^2 \leqslant \|u_M^{k+1} - \widetilde{u}_M^{k+1}\|_M^2 + \||\partial_x u_M^{k+1} - \partial_x \widetilde{u}_M^{k+1}\|_M^2$$
$$\leqslant \mathbf{C}_{1,\alpha,\delta t} \|u_M^0 - \widetilde{u}_M^0\|_M^2 e^{\mathbf{C}_{2,\alpha,\delta t}(\mathcal{D}_{\alpha,1}^1 - \mathcal{D}_{\alpha,1}^{k+1})}.$$

This completes the proof of Theorem 2.8. ■

2.2 *Nonlinear FM/IT model with C-F-FD*

2.2.1 Semi-discrete scheme and spectral approximation

In this subsection, we consider the nonlinear FM/IT model with C-F-FD. Using Taylor series expansion, we have

$$\begin{cases} \mathcal{Q}(\mathcal{U}^1) = \mathcal{Q}(\mathcal{U}^0) + \mathcal{Q}_{\mathcal{U}}(\mathcal{U}_\vartheta)\partial_t \mathcal{U}(x, t_\tau)\delta t, \quad k = 0 \\ \mathcal{Q}(\mathcal{U}^{k+1}) = 2\mathcal{Q}(\mathcal{U}^k) - \mathcal{Q}(\mathcal{U}^{k-1}) + O(\delta t^2), \, k \geqslant 1. \end{cases}$$

Therefore, we can get:

$$\mathcal{S}_{1,\alpha,\delta t}\mathcal{U}^{k+1}(x) - \bar{c}_{\alpha,\delta t}^{-1}\gamma\partial_x^2\mathcal{U}^{k+1}(x)$$
$$= \mathcal{P}_t^\alpha \mathcal{U}^k(x) + \begin{cases} \mathcal{Q}(\mathcal{U}^0) + F^1(x), & k = 0, \\ 2\mathcal{Q}(\mathcal{U}^k) - \mathcal{Q}(\mathcal{U}^{k-1}) + F^{k+1}(x), & k \geqslant 1, \end{cases} + \mathcal{R}_{\mathcal{U}}^{k+1}(x),$$

where

$$\mathcal{P}_t^\alpha \mathcal{U}^k(x)$$
$$= \begin{cases} (\mathcal{S}_{2,\alpha,\delta t} + \lambda_2 \mathcal{D}_{\alpha,1}^2)\mathcal{U}^0(x), & k = 0, \\ \mathcal{S}_{2,\alpha,\delta t}\mathcal{U}^k(x) + \lambda_2 \sum_{j=1}^{k-1}(\mathcal{D}_{\alpha,j+1}^{k+1} - \mathcal{D}_{\alpha,j}^{k+1})\mathcal{U}^j(x) + \lambda_2 \mathcal{D}_{\alpha,1}^{k+1}\mathcal{U}^0(x), & k \geqslant 1, \end{cases}$$

$$F^{k+1} = \bar{c}_{\alpha,\delta t}^{-1} f(x, t_{k+1}), \; k = 0, 1, \cdots, N-2, \; \mathcal{U}^0(x) = h(x),$$

and

$$\mathcal{S}_{1,\alpha,\delta t} = \lambda_1\alpha + \lambda_2\mathcal{D}_{\alpha,1}^1, \; \mathcal{S}_{2,\alpha,\delta t} = \lambda_1\alpha + \lambda_2(\mathcal{D}_{\alpha,1}^1 - \mathcal{D}_{\alpha,1}^2).$$

Furthermore, it holds

$$|\mathcal{R}_{\mathcal{U}}^{k+1}(x)| \leqslant \frac{c}{\alpha}\exp(\frac{2\alpha}{1-\alpha})\max_{t\in(0,T]}|\partial_t^2\mathcal{U}(x,t)|\delta t^2, \ -1\leqslant k\leqslant N-1, \ \forall x\in\Omega.$$

Replacing $\mathcal{U}^{k+1}(x)$ by the approximate solution $u^{k+1}(x)$, we can obtain the following semi-discrete problem for (2) and (3)-(4), which is given by:
Scheme II: Given $u^0 = h(x)$ and find u^{k+1} $(k = 0, 1, 2, \cdots, N-1)$, such that

$$\begin{aligned}\mathcal{S}_{1,\alpha,\delta t}u^{k+1}(x) - &\bar{c}_{\alpha,\delta t}^{-1}\gamma\partial_x^2 u^{k+1}(x)\\ &= \mathcal{P}_t^{\alpha}u^k(x) + \begin{cases}\mathcal{Q}(u^0) + F^1(x), & k=0,\\ 2\mathcal{Q}(u^k) - \mathcal{Q}(u^{k-1}) + F^{k+1}(x), & k\geqslant 1,\end{cases}\end{aligned} \tag{12}$$

$$u^{k+1}|_{x\in\partial\Omega} = 0, -1\leqslant k\leqslant N-1, \tag{13}$$

Now, we will give the representation of numerical solution to semi-discrete problem (12)-(13) in the space $\mathbb{X}_M$.
S-A(II): Find the spectral approximation $u_M^{k+1} \in \mathbb{X}_M$ $(k = 0, 1, 2, \cdots, N-1)$, such that for all $v_M \in \mathbb{X}_M$:

$$\begin{aligned}&\mathcal{S}_{1,\alpha,\delta t}\langle u_M^{k+1}, v_M\rangle_M + \bar{c}_{\alpha,\delta t}^{-1}\gamma a_\omega\langle u_M^{k+1}, v_M\rangle\\ &= \langle \mathcal{P}_t^{\alpha}u_M^k, v_M\rangle_M + \begin{cases}\langle I_M^c G^0, v_M\rangle_M, \ k=0,\\ \langle I_M^c G^k, v_M\rangle_M, \ k\geqslant 1,\end{cases}\end{aligned} \tag{14}$$

where

$$G^k = \begin{cases}\mathcal{Q}(u_M^0) + F^1, & k=0,\\ 2\mathcal{Q}(u_M^k) - \mathcal{Q}(u_M^{k-1}) + F^{k+1}, & k\geqslant 1.\end{cases}$$

Similar to Theorem 2.8, we have the following theorem:

Theorem 2.9 *Let $u_M^{k+1} \in \mathbb{X}_M, \ k = 0, 1, \cdots, N-1$ be the solution of scheme (14). Then the scheme (14) is unconditionally stable in the sense that for all $\delta t > 0$.*

3. Illustrative test problems and discussion

We have studied some numerical examples to test the performance of the proposed methods. We illustrate the accuracy and stability of the proposed methods by performing **S-A(I)** and **S-A(II)** for different values of M and N.
1. (Error measurement criterion) As the exact solution is known, the maximum absolute error $\mathrm{e}_\infty^{M,N}$ and the root mean square error $\mathrm{e}_{rms}^{M,N}$ are measured with the following formulas:

$$\mathrm{e}_\infty^{M,N} = \max_{0\leqslant i\leqslant M}|\mathcal{U}^N(z_i) - u_M^N(z_i)|,$$

and

$$\mathrm{e}_{rms}^{M,N} = \sqrt{\frac{1}{M+1}\sum_{i=0}^{M}|\mathcal{U}^N(z_i) - u_M^N(z_i)|^2}.$$

2. (Convergence ratio) As the exact solution is known, the convergence ratio is given by

$$\textbf{Ratio} = \log_2 \left[\frac{\mathrm{e}_{\infty}^{M,N/2}}{\mathrm{e}_{\infty}^{M,N}} \right].$$

Example 3.1 Consider Eq. (1) on $(-1,1) \times (0,1]$ with the following terms

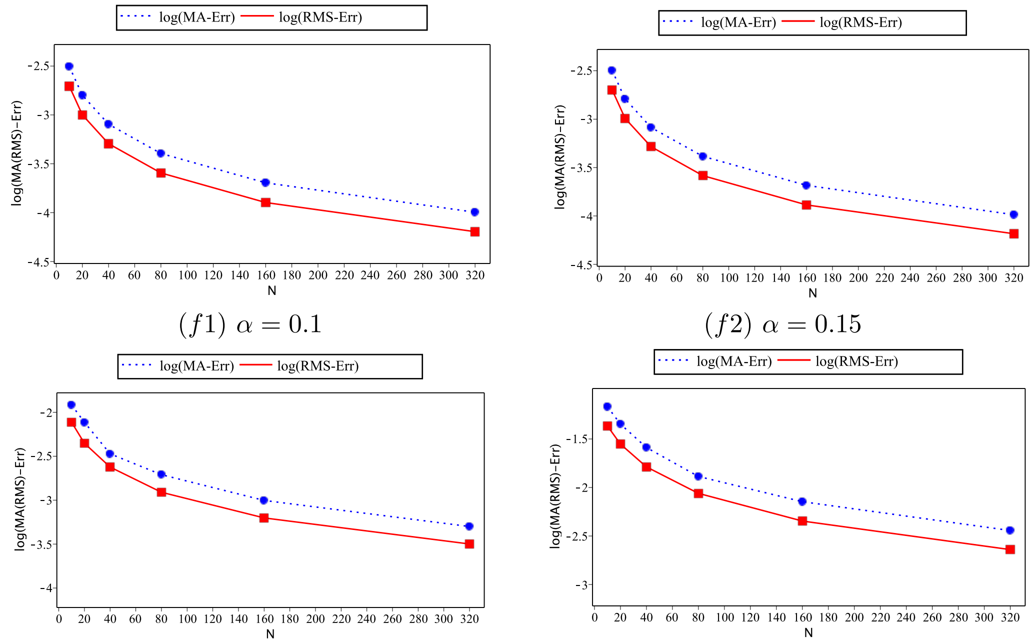
$$\begin{cases} \textit{Parameters}: \ \lambda_1 = 1, \lambda_2 = 1, \gamma_1 = 1, \gamma_2 = 1, \\ \textit{Source term}: \ f(x,t) = 3e^t \sin(2\pi x) - e^{\frac{\alpha t}{-1+\alpha}} \sin(2\pi x) + 4e^t \sin(2\pi x)\pi^2, \\ \textit{Initial condition}: \ \mathcal{U}(x,0) = \sin(2\pi x), \\ \textit{Dirichlet boundary conditions}: \ \mathcal{U}(-1,t) = \mathcal{U}(1,t) = 0. \end{cases}$$

The exact solution of 3.1 is given by $\mathcal{U}(x,t) = e^t \sin(2\pi x)$.

Experimental results of S-A(I): Table 1 presents the experimental results of **S-A(I)** in temporal direction based on Chebyshev polynomials for Example 3.1 with $\alpha = 0.2, 0.4, 0.7, 0.8$. From the obtained results given in Table 1, we observe that, the numerical results agree precisely with the theoretical rate of convergence. More detailed observation of changes of $\log_{10}[\mathrm{e}_{\infty}^{M,N}]$ and $\log_{10}[\mathrm{e}_{rms}^{M,N}]$ against N for $\alpha = 0.1, 0.15, 0.6, 0.81$ are plotted in Figures 1 (f1-f4). To check the spatial accuracy, we present the maximum absolute error $\mathrm{e}_{\infty}^{M,N}$ and the root mean square error $\mathrm{e}_{rms}^{M,N}$ for $\alpha = 0.1, 0.15, 0.6, 0.81$ with respect to the polynomial degree M for $N = 160$ in Figures 2 (f5-f8).

Table 1. **S-A(I):** The maximum absolute error $\mathrm{e}_{\infty}^{M,N}$ and the root mean square error $\mathrm{e}_{rms}^{M,N}$ for different values of α with $M = 17$ (Example 3.1).

	$\alpha = 0.2$			$\alpha = 0.4$		
N	$\mathrm{e}_{\infty}^{M,N}$	$\mathrm{e}_{rms}^{M,N}$	**Ratio**	$\mathrm{e}_{\infty}^{M,N}$	$\mathrm{e}_{rms}^{M,N}$	**Ratio**
10	3.3029e-3	2.0883e-3	-	4.7579e-3	3.0082e-3	-
20	1.6915e-3	1.0694e-3	0.9654	2.5442e-3	1.6086e-3	0.9031
40	8.5607e-4	5.4126e-4	0.9825	1.3160e-3	8.3204e-4	0.9511
80	4.3066e-4	2.7229e-4	0.9912	6.6929e-4	4.2317e-4	0.9755
160	2.1598e-4	1.3656e-4	0.9956	3.375e-4	2.1340e-4	0.9877
320	1.0816e-4	6.8385e-5	0.9977	1.6947e-4	1.0715e-4	0.9939
	$\alpha = 0.7$			$\alpha = 0.8$		
N	$\mathrm{e}_{\infty}^{M,N}$	$\mathrm{e}_{rms}^{M,N}$	**Ratio**	$\mathrm{e}_{\infty}^{M,N}$	$\mathrm{e}_{rms}^{M,N}$	**Ratio**
10	2.5040e-2	1.5831e-2	-	6.2126e-2	3.9280e-2	-
20	1.5109e-2	9.5528e-3	0.7288	4.0299e-2	2.5479e-2	0.6244
40	8.2872e-3	5.2396e-3	0.8664	2.2915e-2	1.4488e-2	0.8144
80	4.3385e-3	2.7431e-3	0.9337	1.2213e-2	7.7215e-3	0.9079
160	2.2195e-3	1.4033e-3	0.9670	6.3035e-3	3.9855e-3	0.9542
320	1.1225e-3	7.0973e-3	0.9835	3.2021e-3	2.0246e-3	0.9771



$(f1)$ $\alpha = 0.1$ $(f2)$ $\alpha = 0.15$

$(f3)$ $\alpha = 0.6$ $(f4)$ $\alpha = 0.81$

Figure 1. **S-A(I):** The changes of $\log_{10}(\mathrm{e}_{\infty}^{M,N})$ and $\log_{10}(\mathrm{e}_{rms}^{M,N})$ against N for different values of α with $M = 17$ (Example 3.1).

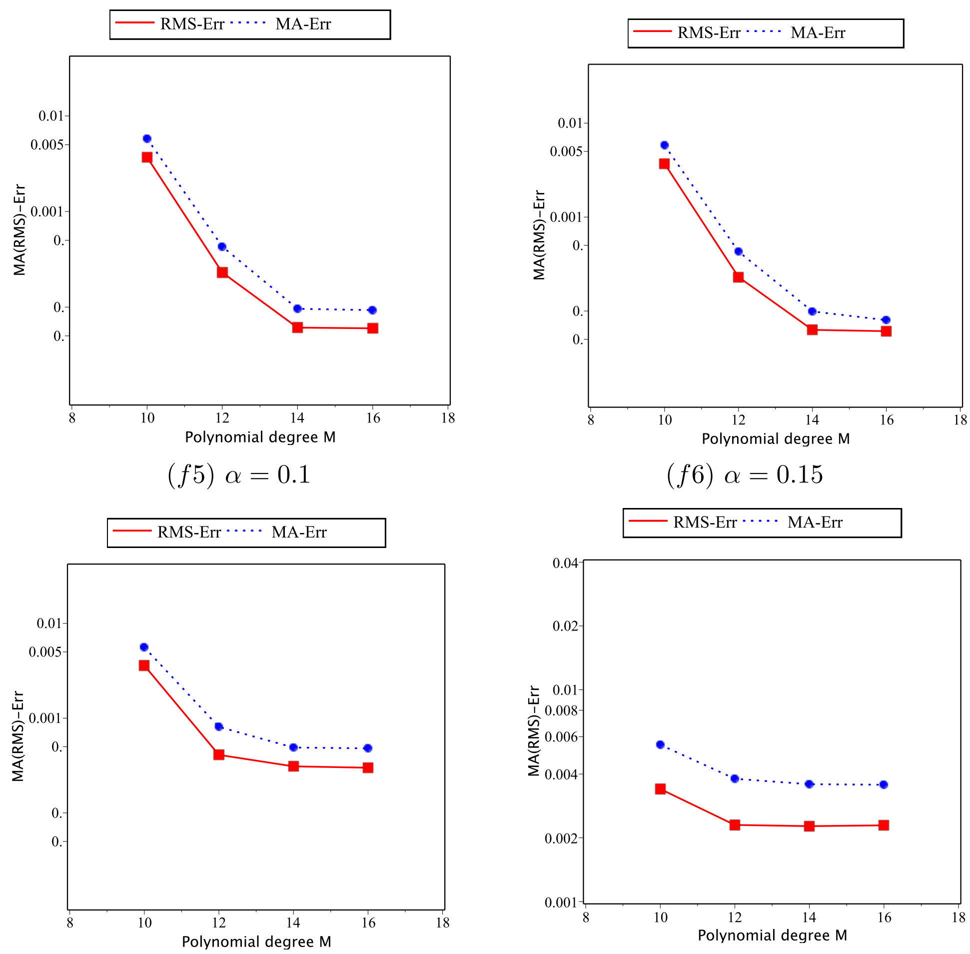
Example 3.2 Consider Eq. (1) on $(-1, 1) \times (0, 1]$ with the following terms

$$\begin{cases} Parameters: \ \lambda_1 = 1, \lambda_2 = 1, \gamma = 1, \\ Nonlinear\ term: \mathcal{Q}(\mathcal{U}) = -\sin(\mathcal{U}), \\ Source\ term: \ f(x,t) = 2e^t \sin(\pi x) - e^{\frac{\alpha t}{-1+alpha}} \sin(\pi x) + e^t \sin(\pi x)\pi^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sin(e^t \sin(\pi x)), \\ Initial\ condition: \ \mathcal{U}(x,0) = \sin(\pi x), \\ Dirichlet\ boundary\ conditions: \ \mathcal{U}(-1,t) = \mathcal{U}(1,t) = 0. \end{cases}$$

The exact solution of 3.2 is given by $\mathcal{U}(x,t) = e^t \sin(\pi x)$.
Experimental results of S-A(II): Table 2 presents the experimental results of **S-A(II)** in temporal direction based on Chebyshev polynomials for Example 3.2 with $\alpha = 0.1, 0.15, 0.6$.

4. Conclusions

In this paper, a numerical method is developed to solve FM/IT model with C-F-FD. Furthermore, the unconditional stability of the numerical method is discussed which provides the theoretical basis of the proposed method. The proposed method is computationally effective due to its simple implementation but with reasonable accuracy. It can be easily viewed from obtained numerical solutions and error norms that this is an excellent method to achieve a numerical solution of the time-fractional Mobile/Immobile transport model.



$(f5)\ \alpha = 0.1$ $(f6)\ \alpha = 0.15$

$(f7)\ \alpha = 0.6$ $(f8)\ \alpha = 0.81$

Figure 2. **S-A(I):** The changes of $e_{\infty}^{M,N}$ and $e_{rms}^{M,N}$ against M for $\alpha = 0.1, 0.15, 0.6, 0.81$ with $N = 320$ (Example 3.1).

Table 2. **S-A(II):** The maximum absolute error $e_{\infty}^{M,N}$ and the root mean square error $e_{rms}^{M,N}$ for different values of α with $M = 16$ (Example 3.2).

α	N	80	160	320
0.1	$e_{\infty}^{M,N}$	1.4117e-3	7.2585e-3	3.6786e-3
	$e_{rms}^{M,N}$	8.6583e-3	4.4498e-3	2.2547e-4
	Ratio	-	0.9597	0.9805
0.15	$e_{\infty}^{M,N}$	1.4445e-3	7.4247e-4	3.7635e-4
	$e_{rms}^{M,N}$	8.8597e-4	4.5517e-4	2.3067e-4
	Ratio	-	0.9602	0.9803
0.6	$e_{\infty}^{M,N}$	6.8372e-3	3.4971e-3	1.7683e-3
	$e_{rms}^{M,N}$	4.1907e-3	2.1433e-3	1.0837e-3
	Ratio	-	0.9672	0.9838

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