International Journal of
Mathematical Modelling & Computations
Vol. 12, No. 03, Summer 2022, 201- 212
DOI:10.30495/ijm2c.2022.1934934.1223

Comparison of Estimators of the PDF and the CDF of the Three-Parameter Inverse Weibull Distribution

F. Maleki Jebely[a], K. Zare[a,*], S. Shokri[b] and P. Karami[c]

[a]*Department of Statistics, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran,*
[b]*Department of Statistics, Lahijan Branch, Islamic Azad University, Lahijan, Iran,*
[c]*Department of Mathematics, Nourabad Mamasani Branch, Islamic Azad University, Nourabad Mamasani, Iran.*

Abstract. The purpose of the present study is to consider the estimation of the PDF and CDF of the three-parameter inverse Weibull (IWD) distribution. To do so, we propose the following well-known methods: moment (MM) estimation, maximum likelihood (ML) estimation, and a developed method entitled the location and scale parameters free maximum likelihood (LSPF) derived from Nagatsuka et al. (2013). Having estimated the parameters, we would consider estimating the PDF and the CDF of the IWD distribution with these three methods. Then, analytical expressions are derived for the mean integrated squared error (MISE) to compare the estimators. According to the results of simulation and two real data for estimation of the PDF and CDF, when the shape parameter is greater than 1, the LSPF method performs better than the others, and when the shape parameter is equal to or smaller than 1, the ML method is better than the others.

Received: 06 July 2021, Revised: 07 February 2022, Accepted: 13 March 2022.

Keywords: Cumulative distribution function; Maximum likelihood estimation; Moment estimation; Probability density function; Weibull distribution.

AMS Subject Classification: 62F10, 62F12.

Index to information contained in this paper

1. Introduction

The Weibull distribution, first presented by Weibull [21], is the most widely used in reliability and lifetime studies. The cumulative distribution function (CDF) and the probability density function (PDF) of the three-parameter Weibull distribution are given, respectively, as follows:

$$H(x;\alpha,\beta,\gamma) = 1 - exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right] \quad (1)$$

and

$$h(x;\alpha,\beta,\gamma) = \frac{\alpha}{\beta}\left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right] \quad (2)$$

for $\alpha > 0$, $\beta > 0$ and $\gamma < x$ (see [9]).

[*]Corresponding author. Email: karim.zare@iau.ac.ir

http://ijm2c.ctb.iau.ir

If X has the Weibull distribution with the CDF and PDF given by (1) and (2), then $1/X$ is said to have the three-parameter inverse Weibull distribution with the CDF and PDF given, respectively,

$$F(x;\alpha,\beta,\gamma)=exp\left[-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right] \tag{3}$$

and

$$f(x;\alpha,\beta,\gamma)=\frac{\alpha}{\beta}\left(\frac{\beta}{x-\gamma}\right)^{\alpha+1} exp\left[-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right] \tag{4}$$

for $\alpha>0,\beta>0$ and $\gamma<x$.

The inverse Exponential distribution, the inverse Rayleigh distribution, the inverse Gamma distribution obtained respectively when in IWD, β, respectively, is equal 1, 2 and 0.5.

The IWD model was developed by Erto [7]. Since then, Cohen and Whitten [5]; Lawless [11]; Muthry et al. [12], and Nelson [16] have conducted studies on the use of the IWD in varied fields of science and technology.

Application of the IWD in degradation of mechanical components such as piston, crankshafts of diesel engines, and the breakdown of insulating fluid has been studied by Keller and Kamath [10]. Also, other usages of the IWD have been studied by Shuaib khan and King [18] and Shuaib Khan et al. [19]. As reported, the IWD can model various failure characteristics such as infant mortality, useful life, and wear-out periods. Due to several applications, the efficient estimation of the PDF and the CDF of the three-parameter IWD is the goal of the present study. In this vein, three estimation methods, including ML estimation, MM estimation, and LSPF method suggested by Nagatsuka et al. [14], are used. It should be mentioned that other research papers have been carried out in this field. For instance, the estimation of the PDF and CDF of the three-parameter IWD has been studied by Alizadeh et al. [2], when all parameters are considered known parameters except for its shape parameter. In the simulation phase conducted in these studies, the mean square error was used to compare estimators. However, in the present paper, all parameters are unknown, and by referring to Silverman [20] and Shirahata and Chu [17], MISE is utilized to make comparisons between the estimators. According to the these mentioned researches,

$$MISE\big[\tilde{f}(.)\big]=E\big\{ISE\big[\tilde{f}(.)\big]\big\}$$

where

$$ISE\big[\tilde{f}(.)\big]=\int_R\big[\tilde{f}(x)-f(x)\big]^2\,dx$$

and

$$MISE[\tilde{F}(.)]=E\{ISE[\tilde{F}(.)]\}$$

where

$$ISE[\tilde{F}(.)]=\int_R[\tilde{F}(x)-F(x)]^2\,dx$$

where $\tilde{f}(x)$ and $\tilde{F}(x)$ are estimators for the PDF and CDF.

The contents of the present paper are organized as follows: the estimation methods of the parameters of the three-parameter IWD are derived in Sections 2. The simulations of the estimators and the PDF and CDF are presented in Section 3. The estimators are compared by two real data sets in Section 4. Finally, some discussions on the possible use of the results are provided in Section 5.

2. Some estimation methods of the parameters in the three-parameter inverse Weibull distribution

2.1 *Maximum likelihood estimation method*

Let $X_1, X_2, \ldots., X_n$ be a random sample distributed as (3) with the vector of parameters (α,β,γ) and $x_1, x_2, \ldots, x_n$ be the observed values of this random sample. The log-likelihood function for the vector of parameters can be written as

$$l(\alpha,\beta,\gamma) = nlog\alpha - nlog\beta + (\alpha+1)\sum_{i=1}^{n} log\left(\frac{\beta}{x_i-\gamma}\right) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i-\gamma}\right)^{\alpha}$$

where $x_i > \gamma,\ i = 1,2,\ldots,n$. The maximum likelihood estimates of α, β and γ are obtained by solving the non-linear equations $l'(\alpha) = l'(\beta) = l'(\gamma) = 0$, where $l'(\alpha)$, $l'(\beta)$ and, $l'(\gamma)$ are the score functions for α, β and, γ, respectively, and $x_{(1)} > \gamma$.

2.2 *Moment Estimation Method*

It is necessary to emphasize the importance of the moments in any statistical analysis, especially in applications. Some of the most fundamental features and characteristics can be studied through moments. It is known that if the random variable X has the three-parameter IWD with the PDF given (3), the k-th moment about γ is

$$E(X-\gamma)^k = \beta^k\,\Gamma\left(1-\frac{k}{\alpha}\right).$$

Now, we can write the first three theoretical moments of X as

$$E(X) = \beta\Gamma\left(1-\frac{1}{\alpha}\right) - \gamma,$$

$$E(X^2) = 2\gamma E(X) - \gamma^2 + \beta^2\Gamma\left(1-\frac{2}{\alpha}\right),$$

$$E(X^3) = \gamma^3 - 3\gamma^2 E(X) + 3\gamma E(X^2) + \beta^3\Gamma\left(1-\frac{3}{\alpha}\right).$$

Now by equating the first three theoretical moments [i.e. $E(X)$, $E(X^2)$ and $E(X^3)$] with the sample moments ($\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\overline{X^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ and $\overline{X^3} = \frac{1}{n}\sum_{i=1}^{n} X_i^3$), respectively, the moment estimations of the three-parameter IWD are obtained.

2.3 *The LSPF method*

MM and ML estimation methods are used to estimate the parameters in many distributions. As Chen and Amin [4] mentioned, they do not always give satisfactory estimations of the parameters for certain three-parameter distributions. For example, Gamma, Lognormal and Weibull distributions along with three parameters include paths in the parameter space in which shifted origin tends to the smallest observation, and the likelihood estimation can face problems. To investigate the problems of ML estimation, studying further resources in this field is highly suggested.

Griffths [8] suggested a method for estimating the three-parameter Lognormal distribution parameters. Lawless [11] reported all the given details of the descriptions of the various methods on estimating the parameters for the three-parameter Weibull distribution. Regardless of this point that which method is used, the existence and the uniqueness of the estimators are very crucial. Besides, we expect our estimators to have consistent property (See [13-15]). These issues made the scholars investigate other distributions such as the three-parameter IWD, familiar with a new estimation method called the LSPF method. The method, as mentioned earlier, included suitable properties for the three-parameter IWD.

Based on Nagatsuka et al. [14], in the current study, the new method for estimating the parameters of the three-parameter IWD is explored and then compared with ML and MM estimations. In the following, this new method is presented and discussed in detail.

2.3.1 *Estimation of the shape parameter*

If $X_1, X_2, \dots, X_n$ be a random sample of size n ($n > 2$) from the three-parameter IWD, can defined the new order statistics:

$$W_i = \frac{X_{(i)} - X_{(1)}}{X_{(n)} - X_{(1)}}, \quad i = 1,2,3,\dots,n. \tag{5}$$

It is obvious that the distribution of W_i's, $i = 2, \dots, n-1$ would not depend on β and γ and it merely would depend on α. It is clear that, if $w_2, w_3, \dots, w_{n-1}$ are the observed values of $W_2, W_3, \dots, W_{n-1}$ and $w_1 = 0$, $w_n = 1$, then the likelihood function of α is

$$l(\alpha; w_2, \dots, w_{n-1})$$
$$= n!\,\alpha^n \int_0^\infty \int_0^\infty (v)^{n-2} \left\{\prod_{i=1}^n (u + vw_i)\right\}^{-(\alpha+1)} exp\left[-\left\{\sum_{i=1}^n (u + vw_i)^{-\alpha}\right\}\right] du\,dv$$

where $0 = w_1 \le w_2 \le \dots \le w_{n-1} \le w_n = 1$.(See [14]). It can be easily shown that the likelihood function can be varied with respect to α and also $l'(\alpha; w_2, \dots, w_{n-1}) = 0$ always has a unique solution, namely $\hat{\alpha}_w$, which is consistent for $\alpha > 0$.

2.3.2 *Estimation of the location and scale parameters*

After estimating α, the outlined method is used to obtain the estimation of β and γ. Based on ML estimation, through substituting α for $\hat{\alpha}_w$ in (3), the log-likelihood function would be as follows:

$$l(\hat{\alpha}_w, \beta, \gamma) = l_1(\beta, \gamma)$$
$$= nlog\hat{\alpha}_w - nlog\beta + (\hat{\alpha}_w + 1)\sum_{i=1}^n log\left(\frac{\beta}{x_i - \gamma}\right) - \sum_{i=1}^n \left(\frac{\beta}{x_i - \gamma}\right)^{\hat{\alpha}_w}.$$

Now, by maximizing l_1 with respect to γ, the maximum likelihood estimations of γ is shown by $\hat{\gamma}_{init}$,

$$\hat{\gamma}_{init} = X_{(1)} \tag{6}$$

as the initial estimator for γ. Then, through substituting γ for $\hat{\gamma}_{init}$ in l_1, the log-likelihood function will be as

$$l_1(\beta, \hat{\gamma}_{init}) = l_2(\beta)$$
$$= nlog\hat{\alpha}_w - nlog\beta + (\hat{\alpha}_w + 1)\sum_{i=1}^n log\left(\frac{\beta}{x_i - \hat{\gamma}_{init}}\right) - \sum_{i=1}^n \left(\frac{\beta}{x_i - \hat{\gamma}_{init}}\right)^{\hat{\alpha}_w}.$$

This is an equation with one variable and the maximum likelihood estimates of β uniquely is given by

$$\hat{\beta}_{init} = \left[\frac{n}{\sum_{i=1}^{n}(x_i - \hat{\gamma}_{init})^{-\hat{\alpha}_w}}\right]^{\frac{1}{\hat{\alpha}_w}}. \quad (7)$$

Theorem 2.1 Under considered conditions for the three-parameter IWD,

i) $X_{(1)}$ is bias for γ and the bias value is equal to $\int_0^1 \frac{\beta}{\sqrt[\alpha]{-\log(1-z)}} z^{n-1} dz$, where

$$z = 1 - F(x; \alpha, \beta, \gamma) = 1 - exp\left[-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right].$$

ii) $X_{(1)}$ and then the corrected bias estimator $\hat{\gamma}_w = X_{(1)} - \int_0^1 \frac{\hat{\beta}_{init}}{\sqrt[\hat{\alpha}'_\omega]{-\log(1-z)}} z^{n-1} dz$ is consistent for γ .

Proof

i)

$$E\left(X_{(1)} - \gamma\right) = \frac{n\alpha}{\beta}\int_{\gamma}^{\infty}(x-\gamma) exp\left[-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right]\left(\frac{\beta}{x-\gamma}\right)^{\alpha+1}\left\{1 - exp\left[-\left(\frac{\beta}{x-\gamma}\right)^{\alpha}\right]\right\}^{n-1} dx$$

by changing variable $z = 1 - F(x; \alpha, \beta, \gamma)$, we have

$$E\left(X_{(1)} - \gamma\right) = \int_0^1 \frac{\beta}{\sqrt[\alpha]{-\log(1-z)}} z^{n-1} dz. \quad (8)$$

Through substituting $\hat{\alpha}_w$ for α and $\hat{\beta}_{init}$ for β, the bias-corrected estimator for γ, i.e. $\hat{\gamma}_w$, is obtained.

ii) According to (3), the function F^{-1} is defined as

$$F^{-1}(y) = \sup\{x: F(x) \le y\} = \frac{\beta}{\sqrt[\alpha]{-\log(y)}} + \gamma.$$

If we take $p_1 = \frac{1}{n+1}$ and $q_1 = 1 - p_1$ then

$$F^{-1}(p_1) = \frac{\beta}{\sqrt[\alpha]{\log(n+1)}} + \gamma. \quad (9)$$

According to Arnold et al. (1992), $lim_{n\to\infty} E\left(X_{(1)}\right) = \gamma$ and $lim_{n\to\infty} Var\left(X_{(1)}\right) = 0$ and the second part of the theorem is achieved. ■

Similarly, the bias-corrected estimator for β is shown with $\hat{\beta}_w$ and is then obtained as

$$\hat{\beta}_w = \left[\frac{n}{\sum_{i=1}^{n}(x_i - \hat{\gamma}_w)^{-\hat{\alpha}_w}}\right]^{\frac{1}{\hat{\alpha}_w}}. \quad (10)$$

As it was observed, an iterative algorithm is required to compute the LSPF method of β and γ. Also, based on the achieved results in the present paper and the properties of maximum likelihood estimators, $\hat{\beta}_w$ and $\hat{\gamma}_w$ always exist and are consistent for β and γ, respectively.

3. The simulation study to compare the estimators of the CDF and the PDF

Let $(X_1, \ldots, X_n)$ be a random sample from the three–parameter IWD. The ML, MM and, LSPF estimations of α, β and γ, which are shown with $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$, are replaced in (3) and (4). Then, the estimators of the PDF and CDF would be achieved which are illustrated respectively:

$$\tilde{f}(x) = \frac{\tilde{\alpha}}{\tilde{\beta}}\left(\frac{\tilde{\beta}}{x-\tilde{\gamma}}\right)^{\tilde{\alpha}+1} exp\left[-\left(\frac{\tilde{\beta}}{x-\tilde{\gamma}}\right)^{\tilde{\alpha}}\right] \tag{11}$$

and

$$\tilde{F}(x) = exp\left[-\left(\frac{\tilde{\beta}}{x-\tilde{\gamma}}\right)^{\tilde{\alpha}}\right] \tag{12}$$

in these functions, $\tilde{\alpha} > 0$, $\tilde{\beta} > 0$, and $\tilde{\gamma} < x$.

The MISE are computed by replacing $\tilde{f}(x)$ and $\tilde{F}(x)$ in $MISE[\tilde{f}(.)]$ and $MISE[\tilde{F}(.)]$ formulas. It is difficult to find the MISE of these estimators by mathematical methods. According to Carl and Denis (2013), the Weak Law of Large Numbers and Monte Carlo methods can be used in this way.

The Monte Carlo simulation study was carried out to compare the proposed estimators. Three cases for the values of the parameters were considered including (I) $(\alpha,\beta,\gamma) = (5,1,2)$, (II) $(\alpha,\beta,\gamma) = (0.5,1,0)$ and (III) $(\alpha,\beta,\gamma) = (1,2,1)$. The sample sizes were taken for granted as 5, 10, 15, 20, 25, 30, 35, 40 and, 45. All programs were carried out in the R software. To achieve the best results, several other values of the parameters and another sample size were also considered, but the gained results had been quite similar. Also, the following algorithm was used for calculation of MISEs through Mont Carlo simulation:

A. Follow the steps below to generate a random sample of size n from the Weibull distribution:
   1. Fix The preliminary values of n, α, β, and γ.
   2. Generate a random sample of size n from a random variable $U \sim \text{Weibull}(1, \alpha)$.
   3. Make a random sample of size n of the three–parameter IWD through using the below formula:

$$X = \frac{1}{U}\beta + \gamma.$$

B. Estimate the parameters from the generated random samples using the three estimation methods, according to the instructions below:
   1. Estimate the α value in the LSPF method using the "optim" routine in R software. Then, β and γ can be estimated by the related formulas.
   2. Calculate the estimates of parameters in ML and MM methods through the "$nleqslv$" package in R software.

C. Calculate the MISE criterion by following the below steps:
   1. Set the values of estimators in f(x) and F(x), and obtain the PDF and the CDF estimations.
   2. Calculate $\int_R [\tilde{F}(x) - F(x)]^2\,dx$ and $\int_R [\tilde{f}(x) - f(x)]^2\,dx$ through 2000 repetitions.

3. Based on the Weak Law of Large Numbers, calculate the average of the above integrals as an approximation for the MISEs.

Table 1. The expectation of the LSPF method, ML and MM estimators based on 2000 simulations of Case (I).

Parameter	Method	n=5	n=10	n=15	n=20	n=25	n=30	n=35	n=40	n=45
$\alpha = 5$	LSPF	4.06793	4.06626	4.06993	4.07031	4.07395	4.08842	4.09804	4.17045	4.18061
	MLE	4.77734	4.71820	4.77977	4.78093	4.79032	4.76706	4.7882	4.67896	4.70263
	MME	8.68142	8.00534	7.48492	7.64299	7.07904	7.32025	6.79803	5.91635	5.39581
$\beta = 1$	LSPF	0.54542	0.57229	0.59429	0.63699	0.63863	0.64132	0.64642	0.65776	0.66092
	MLE	1.56961	1.56635	1.56192	1.55642	1.55945	1.56205	1.55953	1.56026	1.55839
	MME	1.08478	0.19726	0.23674	0.21393	0.23193	0.20586	0.20656	0.54858	0.43018
$\gamma = 2$	LSPF	2.41904	2.40836	2.40616	2.38246	2.38142	2.35135	2.34694	2.30245	2.29739
	MLE	1.46355	1.46540	1.45407	1.45556	1.45553	1.45467	1.46121	1.45880	1.46257
	MME	2.41904	2.64836	2.66616	2.68246	2.71642	2.70135	2.72694	2.69245	2.69739

Table 2. The expectation of the LSPF method, ML and MM estimators based on 2000 simulations of Case (II).

Parameter	Method	n=5	n=10	n=15	n=20	n=25	n=30	n=35	n=40	n=45
$\alpha = 0.5$	LSPF	1.24735	1.24540	1.22477	0.38182	0.38215	0.40374	0.40579	0.41059	0.42708
	MLE	0.41560	0.46120	0.42141	0.16067	0.06808	0.02717	0.06166	0.06331	0.05630
	MME	0.94561	0.91591	0.98542	0.92079	0.19071	0.76181	0.75782	0.17651	0.14323
$\beta = 1$	LSPF	0.71526	0.71856	0.73774	0.74724	0.78963	0.82333	0.83306	0.84172	0.91141
	MLE	0.69991	0.26723	0.37297	0.16153	0.10192	0.02417	0.06198	0.05977	0.04758
	MME	0.96757	0.84421	0.12009	0.09538	0.79255	0.02092	0.03262	0.04157	1.82567
$\gamma = 0$	LSPF	0.02981	-0.0274	0.02155	-0.0199	-0.0132	-0.0094	0.00864	-0.0086	0.00725
	MLE	0.75025	0.19112	-0.0677	-0.0330	-0.0133	-0.0057	-0.0131	-0.0126	-0.0111
	MME	0.02981	-0.0274	0.02155	-0.0423	-0.0142	-0.0015	0.00269	-0.0052	0.00621

Table 3. The expectation of the LSPF method, ML and MM estimators based on 2000 simulations of Case (III).

Parameter	Method	n=5	n=10	n=15	n=20	n=25	n=30	n=35	n=40	n=45
$\alpha = 1$	LSPF	0.06979	0.07158	0.14667	0.15000	0.17220	0.17862	0.18307	0.18692	0.18840
	MLE	0.26102	0.06244	0.17124	0.47811	0.03340	0.14921	0.96004	0.24393	0.00902
	MME	0.64295	0.93617	0.65414	0.72655	0.41146	0.38774	0.50572	0.46861	0.99003
$\beta = 2$	LSPF	0.03461	0.03930	0.03941	0.07270	2.08153	2.04505	2.03772	2.02144	2.01535
	MLE	0.43622	0.04317	2.20285	0.42990	2.08153	0.34231	1.34159	1.11800	0.16232
	MME	0.15493	0.00070	0.14592	0.03701	0.01113	0.11614	0.02621	0.02845	0.03880
$\gamma = 1$	LSPF	0.06033	0.06891	0.12012	0.13953	0.15896	0.37981	0.47652	0.49219	0.49434
	MLE	0.37587	-0.8350	1.72634	0.71402	0.02862	0.64426	0.52420	1.34785	0.35462
	MME	0.02123	0.03517	0.40831	0.10930	0.02862	0.34964	0.04779	0.06087	0.10710

Tables 1-3 display the Monte Carlo simulation results of the estimators: $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ which were run 2000 times for each set of configurations. Also, Figures 1-3 show the simulation results to compare the three estimation methods of the PDF and CDF based on the MISE criterion. The simulation results show that when $\alpha > 1$, the LSPF method for estimating the PDF and the CDF are more appropriate than ML and MM estimators. When $\alpha \leq 1$, the ML estimator is better than the other estimators. Also, when n is large, a small difference is observed between the MISE of the ML and MM estimators and the LSPF method. Indeed, as n would become smaller, the difference would be more considerable. Generally, the MISE for each estimator appears to decrease with increasing sample size.

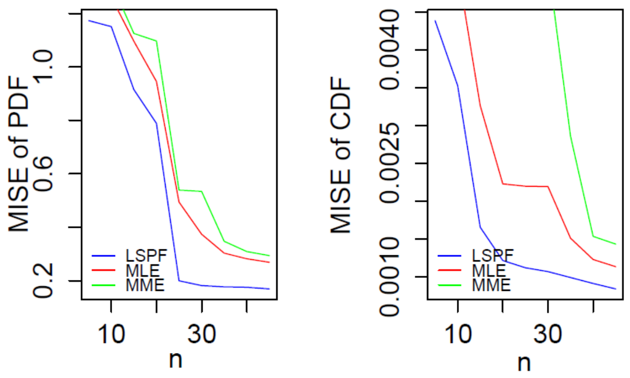

Figure 1. The MISE of the LSPF method, ML, MM estimators based on 2000 simulations of Case (I).

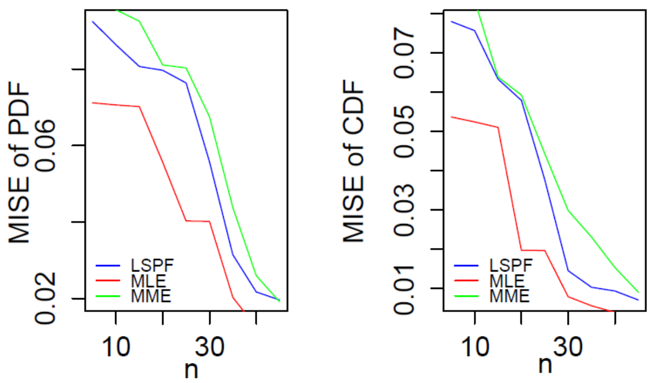

Figure 2. The MISE of the LSPF method, ML, MM estimators based on 2000 simulations of Case (II).

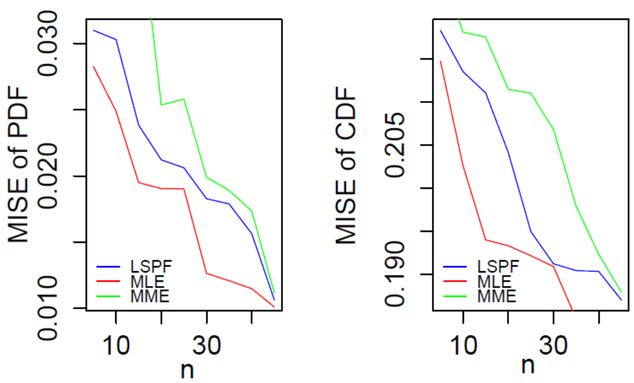

Figure 3. The MISE of the LSPF method, ML, MM estimators based on 2000 simulations of Case (III).

4. Data analysis

Here, we use two real data sets to compare the performances of the LSPF, ML and, MM estimations for the PDF and the CDF.

Example 4.1 The first real data set (Table 4) represents the breaking strengths of the single carbon fibers of different lengths (Alizadeh et al., 2017). Before, they used this data set to fit the three-parameter IWD and finally obtained ML estimation of the parameters (See Table 5). Now, to compare the performances of the three estimation methods, we fit the three-parameter IWD to this data set and obtain the estimation of the parameters (See Table 5). We compare the estimation methods by means of Kolmogorov-Smirnov (K-S) test statistic (the distance between the empirical CDFs and the fitted CDFs). Table 5 also gives values of the K-S statistic for the three different estimation methods for the real data set 1. The K-S statistic shows that the LSPF method provides the best fit.

Table 4. The real data set 1.

2.247	2.640	2.842	2.908	3.099	3.126	3.245	3.328	3.355	3.383
3.572	3.581	3.681	3.726	3.727	3.727	3.727	3.783	3.785	3.786
3.898	3.912	3.964	4.050	4.063	4.082	4.111	4.118	4.141	4.216
4.251	4.262	4.326	4.402	4.458	4.466	4.519	4.542	4.555	4.614
4.632	4.634	4.636	4.678	4.698	4.738	4.832	4.924	5.043	5.099
5.134	5.359	5.473	5.571	5.684	5.721	5.998	6.060		

Table 5. Estimation of the parameters and K-S statistic for the real data set 1.

Parameter	LSPF	MLE	MME
α	3.499999	5.881455	0.371473
β	2.699999	5.097411	0.891971
γ	1. 102633	-1.332681	2.246301
K-S statistic	0.112398	0.133330	0.441729

Figures 4-6 show the Q-Q plots (observed quantiles versus expected quantiles), the density plots (fitted PDF versus empirical PDF), and the distribution plots (fitted CDF versus empirical CDF) for the three different estimation methods. In this data, the LSPF estimation of α is greater than 1, and as we expected, all Figures show that the LSPF method provides the best fit. This result corresponds to the theoretical and simulation results.

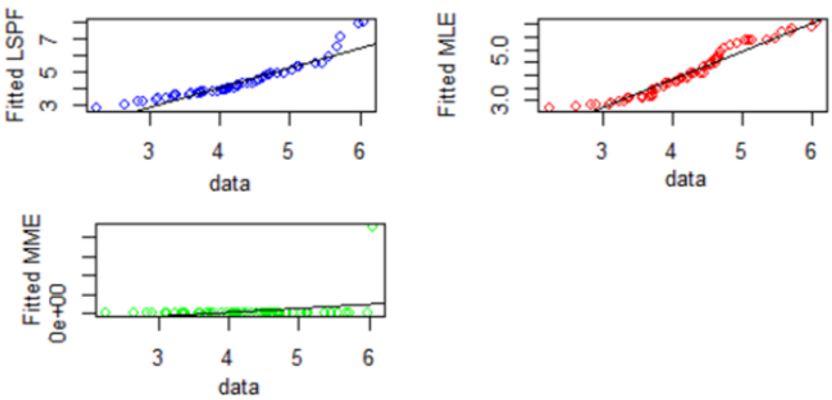

Figure 4. Q-Q plots for the real data set 1 based on the three different estimation methods.

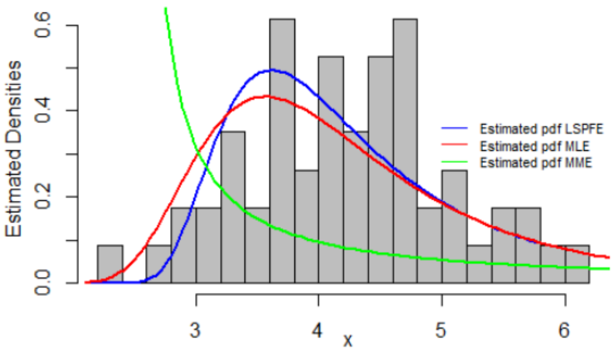

Figure 5. The fitted PDFs and the histogram for the real data set 1 based on for three different estimation methods.

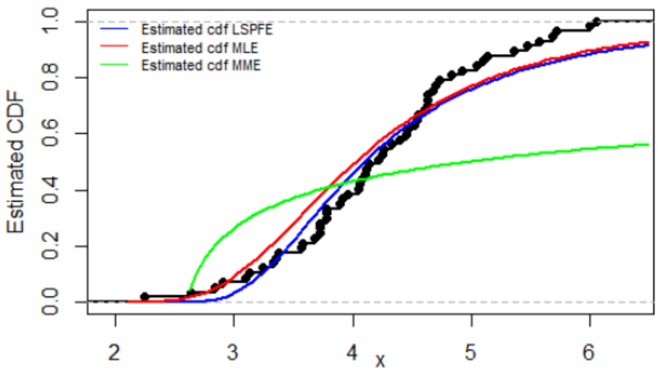

Figure 6. The fitted CDFs versus the empirical CDF for the real data set 1 based on the three different estimation methods.

Example 4.2 The second real data set is in Table 6 (Abd Ellah [1]). Before, he used this data set to fit the three-parameter IWD and finally to obtain ML estimation of the parameters (See Table 7). Now, to compare the performances of the three estimation methods, we fit the three-parameter IWD to this data set and obtain the estimation of the parameters (See Table 7). Table 7 also gives the values of the K-S statistic for the three different estimation methods for the real data set 2.

Table 6. The real data set 2.

0.0130	0.0270	0.0290	0.0307	0.0314	0.0820	0.1210	0.1240	0.1360	0.1540
0.2100	0.2140	0.2400	0.3200	0.3600	0.7600	1.0400	1.2800	5.2600	

Table 7. Estimation of the parameters and K-S statistic for the real data set 2.

parameter	LSPF	MLE	MME
α	0.765841	0.825806	0.396235
β	0.926680	0.577889	2.842535
γ	-0.015241	0.000000	-0.145468
K-S statistic	0.653923	0.561429	0.678768

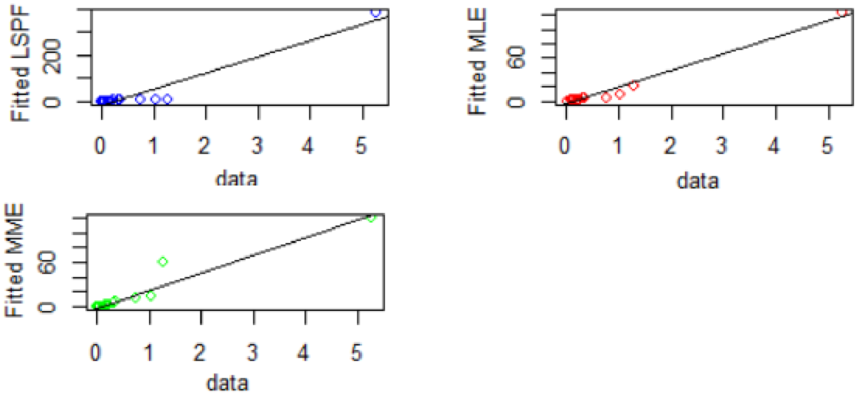

Figure 7. Q-Q plots for the real data set 2 based on the three different estimation methods.

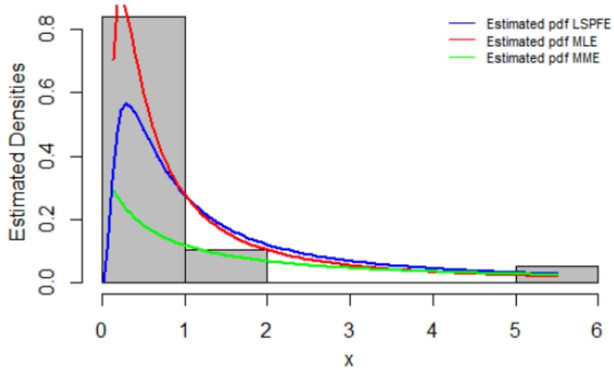

Figure 8. The fitted PDFs and the histogram for the real data set 2 based on the three different estimation methods.

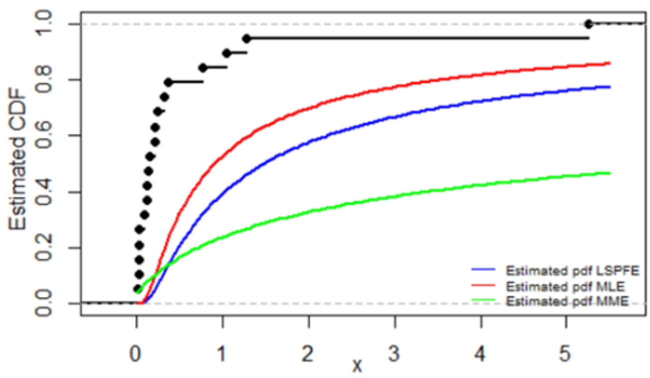

Figure 9. Fitted the CDFs versus the empirical CDF for the real data set 2 based on the three different estimation methods.

Figures 7-9 show the Q-Q plots, the density plots and the distribution plots for the three different estimation methods. In this data, the LSPF estimation of α is smaller than 1. The K-S statistic and Figures 7-9 show that the ML estimation method provides the best fit.

This result corresponds to the theoretical and simulation results.

5. Conclusion

Here, we discussed some methods for estimating the three-parameter IWD when all the three parameters are unknown. To achieve our goals, we first presented the usual methods (i.e., MM and ML estimations) to estimate the parameters. Then, we proposed the LSPF method for the three-parameter IWD as a recommended method in the three-parameter distributions. We showed that the estimates based on the LSPF method are unique and consistent. Furthermore, we used a bias correction for the shifted origin to improve our estimates.

The simulation results and the analysis of real data sets show that the performance of the estimators depends on α. In other words, for $\alpha > 1$, the LSPF estimator of the PDF and the CDF is more appropriate than ML and MM estimators. When $\alpha \leq 1$, the ML estimator is better than the other estimators.

References

[1] A. H. Abd Ellah, Bayesian and non-Bayesian estimation of the inverse Weibull model based on generalized order statistics, Intelligent Information Management, **4** (2012) 23-31.

[2] M. Alizadeh, S. Rezaei, S. Nadarajah and F. Bagheri, On the estimation of the inverse Weibull distribution, Journal of Data Science, **15** (2017) 167-182.

[3] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, A First Course in Order Statistics, John Wiley & Sons, New York, (1992).

[4] R. C. H. Chen and N. A. K. Amin, Estimating the parameters in continuous univariate distributions with a shifted origin, Journal of the Royal Statistical Society, **45** (1983) 394-403.

[5] A. C. Cohen and B. J. Whitten, Parameter Estimation in Reliability and Life Span Models, Marcel Dekker, New York, (1988).

[6] G. Carl and T. Denis, Stochastic Simulation and Monte Carlo Methods, Springer, New York, (2013).

[7] P. Erto, Properties and identification of the inverse Weibull: Unknown or just forgotten, Quality and Reliability Engineering International, **9** (1986) 383-385.

[8] D. A. Griffths, Interval estimation for the three-parameter lognormal distribution via the likelihood function, Journal of the Royal Statistical Society, **29** (1980) 58-68.

[9] N. L. Johnson, S. Kotz, N. Balakrishnan, Continuous Univariate Distributions, Second Edition, John Wiley & Sons, New York, (1995).

[10] Keller, A. Z. and Kamath, A. R. R. (1982). Alternative reliability models for mechanical systems. Proceeding of the Third International Conference on Reliability and Maintainability, 411-415.

[11] J. F. Lawless, Statistical Models and Methods for Lifetime Data, Second Edition, John Wiley & Sons, New York, (2003).

[12] D. N. P. Muthry, M. Xie and R. Jiang, Weibull Models, John Wiley & Sons, New York, (2004).

[13] H. Nagatsuka and N. Balakrishnan, A consistent method of estimation for the parameters of the three-parameter inverse Gaussian distribution, Journal of Statistical Computation and Simulation, **83** (2012) 1915-1931.

[14] H. Nagatsuka, N. Balakrishnan and T. Kamakura, A consistent method of estimation for the three-parameter Weibull distribution, Computational Statistics and Data Analysis, **58** (2013) 210-226.

[15] H. Nagatsuka, N. Balakrishnan and T. Kamakura, A consistent method of estimation for the three-parameter gamma distribution, Communications in Statistics: Theory and Methods, **43** (2014) 3905-3926.

[16] W. Nelson, Applied Life Data Analysis, John Wiley & Sons, New York, (1982).

[17] S. Shirahata and I. S. Chu, Integrated squared error of kernel-type estimator of distribution function, Annals of the Institute of Statistical Mathematics, **44 (3)** (1992) 579-591.

[18] M. Shuaib Khan and R. King, New generalized inverse Weibull distribution for lifetime modeling, Communications for Statistical Applications and Methods, **23 (2)** (2016) 147-161.

[19] M. Shuaib Khan, G. R. Pasha and A. H. Pasha, Theoretical analysis of inverse Weibull distribution, WSEAS Transactions on Mathematics, **7 (2)** (2008) 30-38.

[20] B. W. Silverman, Density Estimation for Statistics and Data Analysis, Chapman and Hall, London (1986).

[21] W. Weibull, A Statistical Theory of Strength of Materials, Generalstabens Litografiska Anstalts Förlag, Stockholm, (1939).