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Pseudo-Triangular Entropy of Uncertain Variables: An Entropy-Based Approach to Uncertain Portfolio Optimization

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Abstract. In this paper we introduce concepts of pseudo-triangular entropy as a supplement measure of uncertainty in the uncertain portfolio optimization. We first prove that logarithm entropy and triangular entropy for uncertain variables sometimes may fail to measure the uncertainty of an uncertain variable. Then, we propose a definition of pseudo-triangular entropy as a supplement measure to characterize the uncertainty of uncertain variables and we derive its mathematical properties. We also give a formula to calculate the pseudo-triangular entropy of uncertain variables via inverse uncertainty distribution. Moreover, we use the pseudo-triangular entropy to characterize portfolio risk and establish some uncertain portfolio optimization models based on different types of entropy. A genetic algorithm (GA) is implemented in MATLAB software to solve the corresponding problem. Numerical results show that pseudo-triangular entropy as a quantifier of portfolio risk outperforms logarithm entropy and triangular entropy in the uncertain portfolio optimization.

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1. Introduction

The financial crisis of 2007-2008 showed that unpredictable events can challenge conventional ideas about portfolio construction. However, we do not know in advance when such events will occur and how traumatic they can be, we can modify current risk management frameworks to better manage these rare and dangerous events.

The prime reason for the occurrence of such anomalies lies in the conventional approach to applying mean-variance model which was introduced by Markowitz [13]. The basic assumption in the mean-variance model and many other models is that future returns will be independent and normally distributed. Sheikh and Qiao [17] declared that in many cases it can be empirically observed that returns are not normally distributed and under non- normality, variance becomes inefficient as the quantifier of portfolio risk.

Entropy of random variables was first proposed by Shannon [16] in logarithm form. The study carried out by Philippatos and Wilson [15] was the pioneering work to associate entropy with a measure of risk in portfolio optimization. They argued that entropy is more general and better suited in portfolio optimization than variance.

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Furthermore, Simonelli [18] showed that entropy as a measure of risk is better than variance in wealth allocation.

In the mentioned literatures of investigating entropy, it is assumed that the security returns are random variable with probability distribution. The fundamental assumption for using probability theory in the portfolio optimization is that the probability distribution of security returns is similar to the past one and close enough to frequencies. However, it is difficult to ensure this assumption.

In financial businesses, sometimes we have historical data scarcity. Thus, we ask domain experts to evaluate the belief degree that each event will happen. Using fuzzy set theory is a way to handle portfolio optimization problems with returns given by experts' evaluations. Liu [9] showed that fuzzy set theory is not self-consistent in mathematics and may lead to wrong results in practice. The main mistake of fuzzy set theory is based on the wrong assumption that the belief degree of a union of events is the maximum of the belief degrees of the individual events no matter if they are independent or not. Furthermore, Liu [11] presented a counterexample to show that modeling belief degree which uses subjective probability may lead to counterintuitive results.

Liu [6] first put forward the entropy of uncertain variables in logarithm form. After that, several scholars have been investigating entropy under uncertainty theory. Chen et al [3] investigated the cross-entropy to measure the divergence degree of uncertain variables and proposed the minimum cross-entropy principle. Chen and Dai [2] proposed the maximum entropy principle for uncertain variables. Moreover, Dai and Chen [5] presented a formula to calculate the entropy of uncertain variables. As a supplement of logarithm entropy, several types of entropy for uncertain variables have been investigated ([4], [19] and [20]).

In this paper, after providing some preliminaries about uncertainty theory in Section 2, a definition of pseudo-triangular entropy as a supplement measure of uncertainty and its mathematical properties such as translation invariance and positive linearity are presented in Section 3. Then in Section 4, applications of pseudo-triangular entropy in the uncertain portfolio optimization together with a numerical example are given. Finally, conclusions and suggestions are presented in Section 5.

2. Model formulation

Uncertainty theory is a branch of mathematics and was founded by Liu in 2007. Having sample scarcity, we should ask domain experts to evaluate the degree of belief in the occurrence of an event. Modeling belief degree which uses subjective probability or fuzzy set theory may lead to counterintuitive results. Therefore, we use uncertainty theory to model belief degree. This section comes with reviewing some necessary definitions and theorems in uncertainty theory.

Definition 2.1 [8] Let Γ be a nonempty set (sometimes called universal set). A collection $\mathcal{L}$ consisting of subsets of Γ is called an algebra over Γ if the following three conditions hold: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \dots, \Lambda_n \in \mathcal{L}$ then

$$\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}.$$

The collection $\mathcal{L}$ is called a $\sigma -$ algebra over Γ if the condition (c) is replaced with closure under countable union, i.e., when $\Lambda_1, \Lambda_2, \dots \in \mathcal{L}$, we have

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}.$$

Definition 2.2 [8] Let Γ be a nonempty set, and let $\mathcal{L}$ be a $\sigma-$algebra over Γ. Then $(\Gamma,\mathcal{L})$ is called a measurable space, and any element in $\mathcal{L}$ is called a measurable set.

Definition 2.3 [8] The smallest $\sigma-$algebra $\boldsymbol{\mathcal{B}}$ containing all open intervals is called the Borel algebra over the set of real numbers, and any element in $\boldsymbol{\mathcal{B}}$ is called a Borel set.

Definition 2.4 [8] A function ξ from a measurable space $(\Gamma,\mathcal{L})$ to the set of real numbers is said to be measurable if

$$\xi^{-1}(\mathrm{B})=\{\Upsilon\in\Gamma|\xi(\Upsilon)\in\mathrm{B}\}\in\mathcal{L}$$

for any Borel set B of real numbers.

Let $(\Gamma,\mathcal{L})$ be a measurable space. Each element Λ in $\mathcal{L}$ is called a measurable set. We rename measurable set as event in uncertainty theory. We also define an uncertain measure $\mathcal{M}$ on the $\sigma-$algebra $\mathcal{L}$. That is, a number $\mathcal{M}\{\Lambda\}$ will be assigned to each event Λ to indicate the belief degree with which we believe Λ will happen. The uncertain measure $\mathcal{M}$ must have certain mathematical properties. In order to rationally deal with belief degree, Liu [8] suggested the following three axioms:

Axiom 1 (Normality) $\mathcal{M}\{\Gamma\}=1$ for the universal set Γ.

Axiom 2 (Duality) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$ for any event Λ.

Axiom 3 (Subadditivity) For every countable sequence of events $\Lambda_1,\Lambda_2,\ldots$, we have

$$\mathcal{M}\{\bigcup_{i=1}^{\infty}\Lambda_i\}\leq\sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}$$

Definition 2.5 [8] The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, duality and subadditivity axioms.

Definition 2.6 [8] Let Γ be a nonempty set, let $\mathcal{L}$ be a $\sigma-$algebra over Γ, and let $\mathcal{M}$ be an uncertain measure. Then the triplet $(\Gamma,\mathcal{L},\mathcal{M})$ is called an uncertainty space.

The product uncertain measure $\mathcal{M}$ on the product $\sigma-$algebra $\mathcal{L}$ is defined by the following axiom (Liu [6]).

Axiom4 (Product) Let $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ be uncertainty spaces for $K=1,2,\ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $K=1,2,\ldots$, respectively.

Definition 2.7 [8] An uncertain variable is a function ξ from an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to the set of real numbers such that $\{\xi\in B\}$ is an event for any Borel set B of real numbers.

Definition 2.8 [8] The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x)=\mathcal{M}\{\xi\leq x\}$$

for any real number x.

Definition 2.9 [10] An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0<\Phi(x)<1$, and $\lim_{x\to-\infty}\Phi(x)=0$, $\lim_{x\to+\infty}\Phi(x)=1$.

Definition 2.10 [12] A real-valued function $\Phi(x)$ on $\mathbb{R}$ is an uncertainty distribution if and only if it is a monotone increasing function satisfying

$$0 \le \Phi(x) \le 1, \quad \Phi(x) \not\equiv 0, \quad \Phi(x) \not\equiv 1.$$
$$\Phi(x_0) = 1 \quad \text{if} \quad \Phi(x) = 1 \text{ for any } x > x_0.$$

Definition 2.11 [10] The uncertain variables $\xi_1, \xi_2, \dots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \dots, B_n$ of real numbers.

Definition 2.12 [10] An uncertain variable ξ is called normal denoted by $N(m, \sigma)$ if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + exp\left(\frac{\pi(m-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R},$$

where m and σ $(\sigma > 0)$ are real numbers.

Definition 2.13 [1] An uncertain variable ξ is called skew-normal denoted by $SN(m, p, \sigma)$ if it has a skew-normal uncertainty distribution

$$\Phi(x) = \left(1 + exp\left(\frac{\pi(m-x)}{\sqrt{3}\sigma}\right)\right)^{-p}, \quad x \in \mathbb{R},$$

where m, σ $(\sigma > 0)$ and p $(p > 0)$ are real numbers.

Parameter m specifies distribution location, parameter $\sigma\,(\sigma > 0)$ specifies distribution spread and parameter p $(p > 0)$ specifies distribution shape. There are three conditions for p as follows:

1. If $p = 1$, then an uncertain variable ξ is normal uncertainty distribution.
2. If $p > 1$, then an uncertain variable ξ is positive skew-normal uncertainty distribution.
3. If $0 < p < 1$, then an uncertain variable ξ is negative skew-normal uncertainty distribution.

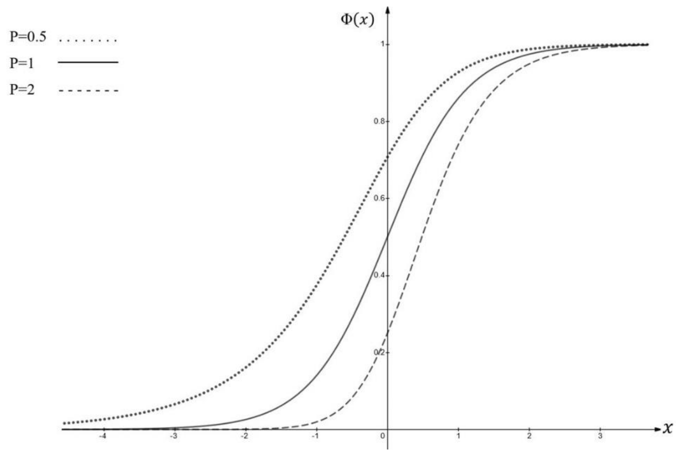

Figure 1. Skew-normal uncertainty distribution.

In order to estimate the parameter vector $\Theta = (m, p, \sigma)$, we employ the least squares principle proposed by Liu [7] to minimize the sum of squares of the distance between the expert's experimental data and the uncertainty distribution. Assume that the uncertainty distribution to be determined has a function form $\Phi(x|\Theta)$ with an unknown parameter vector $\Theta = (m, p, \sigma)$. The optimal solution vector $\widehat{\Theta} = (\widehat{m}, \hat{p}, \hat{\sigma})$ of $Min \sum_{i=1}^{n} (\Phi(x|\Theta) - \alpha_i)^2$ is called the least squares estimate of the parameter vector $\Theta = (m, p, \sigma)$. In the following example we show how the optimal solution vector $\widehat{\Theta} = (\widehat{m}, \hat{p}, \hat{\sigma})$ is obtained.

Example 2.1 Suppose that we invite a stock market expert to evaluate the belief degree about a specific stock return for the next month. We assume a consultation process is as follows:

Question 1: What is your evaluation about the maximum return for the next month?
Answer 1: I am 100% sure that the stock return is less than +5%. This means the belief degree of "the stock return is less than +5%" is 1. (An expert's experimental data (5, 1) is acquired)

Question 2: What is your evaluation about the minimum return for the next month?
Answer 2: I am 0% sure that the stock return is less than -5%. This means the belief degree of "the stock return is less than -5%" is 0. (An expert's experimental data (-5, 0) is acquired)

Question 3: To what degree do you think that the stock return for the next month is less than -1%?
Answer 3: I am 10% sure that the stock return is less than -1%. This means the belief degree of "the stock return is less than -1%" is 0.1. (An expert's experimental data (-1, 0.1) is acquired)

Question 4: To what degree do you think that the stock return for the next month is less than 0%?
Answer 4: I am 28% sure that the stock return is less than 0%. This means the belief degree of "the stock return is less than 0%" is 0.28. (An expert's experimental data (0, 0.28) is acquired)

Question 5: To what degree do you think that the stock return for the next month is less than 0.14%?
Answer 5: I am 30% sure that the stock return is less than 0.14%. This means the belief degree of "the stock return is less than 0.14%" is 0.30. (An expert's experimental data (0.14, 0.30) is acquired)

Question 6: To what degree do you think that the stock return for the next month is less than 1%?
Answer 6: I am 33% sure that the stock return is less than 1%. This means the belief degree of "the stock return is less than 1%" is 0.33. (An expert's experimental data (1, 0.33) is acquired)

Question 7: To what degree do you think that the stock return for the next month is less than 1.5%?
Answer 7: I am 47% sure that the stock return is less than 1.5%. This means the belief degree of "the stock return is less than 1.5%" is 0.47. (An expert's experimental data (1.5, 0.47) is acquired)

Question 8: To what degree do you think that the stock return for the next month is less than 1.7%?
Answer 8: I am 50% sure that the stock return is less than 1.7%. This means the belief degree of "the stock return is less than 1.7%" is 0.50. (An expert's experimental data (1.7,

0.5) is acquired)

Question 9: To what degree do you think that the stock return for the next month is less than 2%?
Answer 9: I am 59% sure that the stock return is less than 2%. This means the belief degree of "the stock return is less than 2%" is 0.59. (An expert's experimental data (2, 0.59) is acquired)

Question 10: To what degree do you think that the stock return for the next month is less than 3.4%?
Answer 10: I am 66% sure that the stock return is less than 3.4%. This means the belief degree of "the stock return is less than 3.4%" is 0.66. (An expert's experimental data (3.4, 0.66) is acquired)

Question 11: Is there another evaluation for the stock return? If yes, what is it?
Answer 11: yes. I am 87% sure that the stock return is less than 4.6%. This means the belief degree of "the stock return is less than 4.6%" is 0.87 (An expert's experimental data (4.6, 0.87) is acquired)

By using the questionnaire survey, eleven expert's experimental data of the specific stock return for the next month are acquired as follows:

$$(\boldsymbol{x}, \boldsymbol{\alpha}) = (-5, 0), (-1, 0.1), (0, 0.28), (0.14, 0.3), (1, 0.33),$$
$$(1.5, 0.47), (1.7, 0.5), (2, 0.59), (3.4, 0.66), (4.6, 0.87), (5, 1).$$

By implementing "nls" function in R software the optimal solution vector $\widehat{\Theta} = (\hat{m}, \hat{p}, \hat{\sigma})$ is obtained (2.1803, 0.8009, 0.3966).

Theorem 2.1 [10] A function $\Phi^{-1}(r) : (0,1) \to \mathbb{R}$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to r.

Example 2.2 [10] Let $\xi \sim N(m, \sigma)$, then the inverse uncertainty distribution of normal uncertain variable ξ is

$$\Phi^{-1}(r) = m + \frac{\sqrt{3}\sigma}{\pi} \ln\left(\frac{r}{1-r}\right), \qquad 0 < r < 1.$$

Theorem 2.2 [1] Let $\xi \sim SN(m, p, \sigma)$, then the inverse uncertainty distribution of skew-normal uncertain variable ξ is

$$\Phi^{-1}(r) = m + \frac{\sqrt{3}\sigma}{\pi} \ln\left(\frac{r^{\frac{1}{p}}}{1 - r^{\frac{1}{p}}}\right) \quad ; \quad 0 < r < 1. \tag{1}$$

Theorem 2.3 [10] Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(r) = f\left(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r)\right).$$

Theorem 2.4 [14] Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. If the expected value of ξ exists, then

$$E[\xi] = \int_0^1 \Phi^{-1}(r)\, dr.$$

where $\Phi^{-1}(r)$ is the inverse uncertainty function of ξ with respect to r.

Theorem 2.5 [1] Let $\xi_i \sim SN(m_i, p_i, \sigma_i)$; $\forall i = 1,2,\dots,n$, then the expected value of skew-normal uncertain variable ξ_i is

$$E[\xi_i] = m_i - \frac{\sqrt{3}\sigma_i}{\pi p_i} - \frac{\sqrt{3}\sigma_i}{\pi}\int_0^1 ln\left(1 - r^{\frac{1}{p_i}}\right) dr. \tag{2}$$

3. Pseudo-triangular entropy

In this section, we introduce concepts of pseudo-triangular entropy for uncertain variable. Moreover, we derive some mathematical properties of pseudo-triangular entropy and a formula to calculate it via inverse uncertainty distribution. We first recall concepts of logarithm entropy and triangular entropy.

Definition 3.1 [6] Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then, the logarithm entropy of ξ is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} L(\Phi(x))\, \mathrm{d}x$$

where $L(s) = -(s)\ln(s) - (1-s)\ln(1-s)$.

Theorem 3.1 [5] Let ξ be an uncertain variable with uncertainty distribution Φ. Then, the logarithm entropy of ξ is

$$H[\xi] = \int_0^1 \Phi^{-1}(r)\ln(\frac{r}{1-r})\, \mathrm{d}r.$$

Definition 3.2 [19] Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then, the triangular entropy of ξ is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} T(\Phi(x))\, \mathrm{d}x$$

where $T(s) = \begin{cases} s, & \text{if } 0 \le s \le \frac{1}{2} \\ 1-s, & \text{if } \frac{1}{2} < s \le 1 . \end{cases}$

Theorem 3.2 [19] Let ξ be an uncertain variable with uncertainty distribution Φ. Then, the triangular entropy of ξ is

$$\mathrm{H}[\xi] = \int_{\frac{1}{2}}^1 \Phi^{-1}(r)\, \mathrm{d}r - \int_0^{\frac{1}{2}} \Phi^{-1}(r)\, \mathrm{d}r.$$

Definition 3.3 Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then, the pseudo-triangular entropy of ξ is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} C(\Phi(x))\, \mathrm{d}x$$

where $C(s) = \begin{cases} (s)^2, & \text{if } 0 \le s \le \frac{1}{2} \\ (1-s)^2, & \text{if } \frac{1}{2} < s \le 1. \end{cases}$

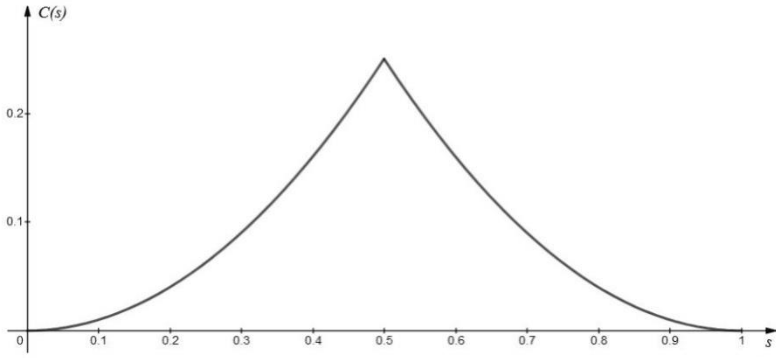

Figure 2. The function $C(s)$.

Theorem 3.3 Suppose that ξ is an uncertainty variable with regular uncertainty distribution Φ. Then

$$H[\xi] = -\int_0^1 \Phi^{-1}(r)C'(r)\,\mathrm{d}r$$

where $C(r) = \begin{cases} (r)^2, & \text{if } 0 \le r \le \frac{1}{2} \\ (1-r)^2, & \text{if } \frac{1}{2} < r \le 1. \end{cases}$

Proof It is clear that $C(r)$ is a derivable function whose derivative has the form

$$C'(r) = \begin{cases} 2(r), & \text{if } 0 \le r \le \dfrac{1}{2} \\ -2(1-r), & \text{if } \dfrac{1}{2} < r \le 1. \end{cases}$$

Since,

$$C(\Phi(x)) = \int_0^{\Phi(x)} C'(r)\,\mathrm{d}r = -\int_{\Phi(x)}^1 C'(r)\,\mathrm{d}r$$

we have

$$\mathrm{H}[\xi] = \int_{-\infty}^{+\infty} C(\Phi(x))\,\mathrm{d}x = \int_{-\infty}^{0}\int_0^{\Phi(x)} C'(r)\,\mathrm{d}r\,\mathrm{d}x - \int_0^{+\infty}\int_{\Phi(x)}^1 C'(r)\,\mathrm{d}r\,\mathrm{d}x.$$

It follows from Fubini theorem that

$$\begin{aligned} \mathrm{H}[\xi] &= \int_0^{\Phi(0)}\int_{\Phi^{-1}(r)}^{0} C'(r)\,\mathrm{d}x\,\mathrm{d}r - \int_{\Phi(0)}^{1}\int_0^{\Phi^{-1}(r)} C'(r)\mathrm{d}x\,\mathrm{d}r \\ &= -\int_0^{\Phi(0)} \Phi^{-1}(r)C'(r)\,\mathrm{d}r - \int_{\Phi(0)}^1 \Phi^{-1}(r)C'(r)\,\mathrm{d}r \\ &= -\int_0^1 \Phi^{-1}(r)C'(r)\,\mathrm{d}r. \end{aligned}$$

The theorem is verified. ■

Theorem 3.4 Suppose that ξ is an uncertainty variable with regular uncertainty distribution Φ. Then, the pseudo-triangular entropy of ξ is

$$\mathrm{H}[\xi] = \int_{\frac{1}{2}}^{1} 2(1-r)\Phi^{-1}(r)\,\mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r)\Phi^{-1}(r)\,\mathrm{d}r. \tag{3}$$

Proof According to Theorem 8 we have

$$\begin{aligned} H[\xi] &= -\int_0^1 \Phi^{-1}(r)C'(r)\,\mathrm{d}r \\ &= -\left(\int_0^{\frac{1}{2}} 2(r)\,\Phi^{-1}(r)\,\mathrm{d}r + \left(\int_{\frac{1}{2}}^{1} -2(1-r)\Phi^{-1}(r)\,\mathrm{d}r\right)\right) \\ &= \int_{\frac{1}{2}}^{1} 2(1-r)\Phi^{-1}(r)\,\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\Phi^{-1}(r)\,\mathrm{d}r. \end{aligned}$$

The theorem is proved. ■

Now, we prove entropy for uncertain variables in forms of logarithm function and triangular function sometimes may fail to measure the uncertainty of an uncertain variable.

Suppose that the uncertain variable ξ_* has an uncertainty distribution

$$\Phi_*(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}\ ;\ \ x \in R$$

and inverse uncertainty distribution

$$\Phi_*^{-1}(r) = \tan\left(\pi\left(r - \frac{1}{2}\right)\right)\ ;\ 0 < r < 1.$$

According to Theorem 6, the logarithm entropy of ξ_* is obtained as follows:

$$\begin{aligned} H[\xi_*] &= \int_0^1 \Phi_*^{-1}(r)\ln(\frac{r}{1-r})\,\mathrm{d}r = \int_0^1 \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\ln(\frac{r}{1-r})\,\mathrm{d}r \\ &= \int_0^1 \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\ln(r)\,\mathrm{d}r - \int_0^1 \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\ln(1-r)\,\mathrm{d}r \\ &= \int_0^1 \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\ln(r)\,\mathrm{d}r - \int_0^1 \tan\left(\pi\left(\frac{1}{2}-r\right)\right)\ln(r)\,\mathrm{d}r \\ &= 2\left(\int_0^1 \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\ln(r)\,dr\right) \\ &\geq \int_0^1 tan\left(\pi\left(r-\frac{1}{2}\right)\right)\ln(r)\,dr = \infty. \end{aligned}$$

According to theorem 7, the triangular entropy of ξ_* is obtained as follows:

$$H[\xi_*] = \int_{\frac{1}{2}}^{1} \Phi_*^{-1}(r)\,\mathrm{d}r - \int_0^{\frac{1}{2}} \Phi_*^{-1}(r)\,\mathrm{d}r$$
$$= \int_{\frac{1}{2}}^{1} \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r - \int_0^{\frac{1}{2}} \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r$$
$$\geq \int_{\frac{1}{2}}^{1} \tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r \geq \int_{\frac{1}{2}}^{1} \sin\left(\pi\left(r-\frac{1}{2}\right)\right)\tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r = \infty.$$

Thus, logarithm entropy and triangular entropy for the uncertain variable ξ_* with the uncertainty distribution $\Phi_*(x)$ are infinite and they fail to measure the uncertainty of an uncertain variable ξ_* with the uncertainty distribution $\Phi_*(x)$. Now, we prove pseudo-triangular entropy for the uncertain variable ξ_* with the uncertainty distribution $\Phi_*(x)$ is finite.

According to Theorem 9, we have

$$\mathrm{H}[\xi_*] = \int_{\frac{1}{2}}^{1} 2(1-r)\,\Phi_*^{-1}(r)\,\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\,\Phi_*^{-1}(r)\,\mathrm{d}r$$
$$= \int_{\frac{1}{2}}^{1} 2(1-r)\tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r$$
$$= \int_0^{\frac{1}{2}} 2(r)\tan\left(\pi\left(\frac{1}{2}-r\right)\right)\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r$$
$$= -\int_0^{\frac{1}{2}} 2(r)\tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r$$
$$= 2\left(-\int_0^{\frac{1}{2}} 2(r)\tan\left(\pi\left(r-\frac{1}{2}\right)\right)\mathrm{d}r\right)$$
$$\leq 2\left(-\int_0^{\frac{1}{2}} sin\left(\pi\left(r-\frac{1}{2}\right)\right)dr\right) = 2\left(\frac{1}{\pi}\right) = \frac{2}{\pi} < \infty.$$

Example 3.1 Let $\xi \sim N(m,p,\sigma)$, then the pseudo-triangular entropy has a closed form for $p=1$ which can be derived as follows:

$$\mathrm{H}[\xi] = \int_{\frac{1}{2}}^{1} 2(1-r)\Phi^{-1}(r)\,\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\,\Phi^{-1}(r)\,\mathrm{d}r$$
$$= \int_{\frac{1}{2}}^{1} 2(1-r)\left(m+\frac{\sqrt{3}\sigma}{\pi}\ln\left(\frac{\mathrm{r}}{1-\mathrm{r}}\right)\right)\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\left(m+\frac{\sqrt{3}\sigma}{\pi}\ln\left(\frac{r}{1-r}\right)\right)\mathrm{d}r$$
$$= \left(m+\frac{2\sqrt{3}\ln 2}{\pi}\sigma-\frac{3}{4}m-\frac{\sqrt{3}}{2\pi}\sigma-\frac{\sqrt{3}\ln 2}{\pi}\sigma\right)-\left(\frac{1}{4}\mathrm{m}+\frac{4\sqrt{3}}{8\pi}\sigma-\frac{\sqrt{3}\ln 2}{\pi}\sigma\right)$$
$$= \frac{(2\ln 2-1)\sqrt{3}}{\pi}\sigma.$$

Theorem 3.5 (Translation invariance) Let ξ be an uncertain variable, and c be a real number. Then,

$$H[\xi + c] = H[\xi].$$

Proof Suppose that ξ has an uncertainty distribution Φ, i.e. $\Phi(x) = \mathrm{M}\{\xi \leq x\}$. Then, the uncertain variable $\xi + c$ has an uncertain distribution $\Psi(x) = \mu\{\xi + c \leq x\} = \mu\{\xi \leq x - c\} = \Phi(x - c)$. By the definition of pseudo-triangular entropy we have

$$H[\xi + c] = \int_{-\infty}^{+\infty} C\big(\Psi(x)\big)\, dx = \int_{-\infty}^{+\infty} C\big(\Phi(x - c)\big)\, dx = \int_{-\infty}^{+\infty} C\big(\Phi(x)\big) dx = H[\xi].$$

Theorem 3.6 Let $f: \Re^n \to \Re$ be strictly monotone function, and $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variable with uncertainty distributions $\Phi_1, \Phi_2, \dots, \Phi_n$, respectively. Then, the pseudo-triangular entropy of uncertain variable $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ is

$$\mathrm{H}[\xi] = \left| \int_{\frac{1}{2}}^{1} 2(1-r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r \right|.$$

Proof
Case I Suppose f is an increasing function. Then, ξ has an inverse uncertainty distribution

$$\Phi^{-1}(r) = f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right)$$

by Theorem 3. It follows from Theorem 9 that

$$\mathrm{H}[\xi] = \int_{\frac{1}{2}}^{1} 2(1-r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r.$$

Case II Suppose f is a decreasing function. Then, ξ has an inverse uncertainty distribution

$$\Phi^{-1}(r) = f\left(\Phi_1{}^{-1}(1-r), \dots, \Phi_n{}^{-1}(1-r)\right)$$

by Theorem 3. It follows from Theorem 9 that

$$\begin{aligned}\mathrm{H}[\xi] &= \int_{\frac{1}{2}}^{1} 2(1-r) f\left(\Phi_1{}^{-1}(1-r), \dots, \Phi_n{}^{-1}(1-r)\right) \mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r) f\left(\Phi_1{}^{-1}(1-r), \dots, \Phi_n{}^{-1}(1-r)\right) \mathrm{d}r \\ &= -\int_{\frac{1}{2}}^{0} 2(r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r + \int_{1}^{\frac{1}{2}} 2(1-r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r\end{aligned}$$

$$= \int_0^{\frac{1}{2}} 2(r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r - \int_{\frac{1}{2}}^{1} 2(1-r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r.$$

Thus, we have

$$\mathrm{H}[\xi] = \left| \int_{\frac{1}{2}}^{1} 2(1-r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r - \int_0^{\frac{1}{2}} 2(r) f\left(\Phi_1{}^{-1}(r), \dots, \Phi_n{}^{-1}(r)\right) \mathrm{d}r \right|.$$

The theorem is proved. ■

Theorem 3.7 (positive linearity) Let ξ and η be independent uncertain variables with uncertainty distribution Φ and Ψ respectively. Then for any real numbers a and b we have:

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta].$$

Proof Suppose that ξ and η have uncertainty distributions Φ and Ψ, respectively. The theorem will be proved via three steps.

Step 1. We prove $H[a\xi] = aH[\xi]$.

If $a > 0$, then the uncertain variable $a\xi$ has an inverse uncertain distribution $\Upsilon^{-1}(r) = a\Phi^{-1}(r)$.

By Theorem 9 we have

$$\mathrm{H}[a\xi] = \int_{\frac{1}{2}}^{1} 2a(1-r)\Phi^{-1}(r)\,\mathrm{d}r - \int_0^{\frac{1}{2}} 2a(r)\Phi^{-1}(r)\,\mathrm{d}r$$

$$= a\left(\int_{\frac{1}{2}}^{1} 2(1-r)\Phi^{-1}(r)\,\mathrm{d}r - \int_0^{\frac{1}{2}} 2(r)\,\Phi^{-1}(r)\,\mathrm{d}r\right) = a\mathrm{H}[\xi].$$

If $a = 0$, then $H[a\xi] = 0 = aH[\xi]$.

If $a < 0$, then the uncertain variable $a\xi$ has an inverse uncertain distribution $\Upsilon^{-1}(r) = a\Phi^{-1}(1-r)$.

By Theorem 9 we have

$$\mathrm{H}[a\xi] = \int_{\frac{1}{2}}^{1} 2a(1-r)\Phi^{-1}(1-r)\,\mathrm{d}r - \int_0^{\frac{1}{2}} 2a(r)\,\Phi^{-1}(1-r)\,\mathrm{d}r$$

$$= -\int_{\frac{1}{2}}^{0} 2a(r)\Phi^{-1}(r)\,\mathrm{d}r + \int_1^{\frac{1}{2}} 2a(1-r)\,\Phi^{-1}(r)\,\mathrm{d}r$$

$$= \int_0^{\frac{1}{2}} 2a(r)\Phi^{-1}(r)\,\mathrm{d}r - \int_{\frac{1}{2}}^{1} 2a(1-r)\,\Phi^{-1}(r)\,\mathrm{d}r$$

$$= -a\left(\int_{\frac{1}{2}}^{1} 2(1-r)\Phi^{-1}(r)\,\mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r)\,\Phi^{-1}(r)\,\mathrm{d}r\right) = -a\mathrm{H}[\xi].$$

Thus, we have $H[a\xi] = |a|H[\xi]$.

Step 2. We prove $H[\xi+\eta] = H[\xi] + H[\eta]$. It follows from Theorem 3 that $\xi+\eta$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(r) = \Phi^{-1}(r) + \Psi^{-1}(r).$$

It follows from Theorem 9 that

$$H[\xi+\eta] = \int_{\frac{1}{2}}^{1} 2(1-r)\left(\Phi^{-1}(r) + \Psi^{-1}(r)\right)\mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r)\left(\Phi^{-1}(r) + \Psi^{-1}(r)\right)\mathrm{d}r$$

$$= \int_{\frac{1}{2}}^{1} 2(1-r)\Phi^{-1}(r)\,\mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r)\,\Phi^{-1}(r)\,\mathrm{d}r + \int_{\frac{1}{2}}^{1} 2(1-r)\Psi^{-1}(r)\,\mathrm{d}r - \int_{0}^{\frac{1}{2}} 2(r)\Psi^{-1}(r)\,\mathrm{d}r$$

$$= H[\xi] + H[\eta].$$

Step 3. For any real numbers a and b, we have

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta].$$

The theorem is proved. ■

4. Application to uncertain portfolio optimization

Suppose that there are n securities whose returns are uncertain variables $\xi_1, \xi_2, \dots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \dots, \Phi_n$, respectively. Let x_i denote the investment proportion of allocation in security $i, i = 1,2, \dots, n$. The uncertain portfolio total return denoted by $\boldsymbol{x}^T\boldsymbol{\xi} = \sum_{i=1}^{n} x_i\xi_i$ is an uncertain variable. Let ψ be the uncertainty distribution of the uncertain portfolio total return $\boldsymbol{x}^T\boldsymbol{\xi} = \sum_{i=1}^{n} x_i\xi_i$. Since $x_1, x_2, \dots, x_n$ are non-negative, the uncertainty distribution of the uncertain portfolio total return can be derived as follows:

$$\Psi^{-1}(r) = \sum_{i=1}^{n} x_i\,\Phi^{-1}(r). \tag{4}$$

To make sure that the uncertain portfolio risk is under control, we minimize entropy as the objective function. Moreover, we set expected value greater than some preset value c.

The uncertain portfolio optimization model is formulated as below:

$$\begin{aligned} & min\;\; H[\boldsymbol{x}^T\boldsymbol{\xi}] \\ & \text{s.t.}\;\; E[\boldsymbol{x}^T\boldsymbol{\xi}] \ge C \\ & \qquad \sum_{i=1}^{n} x_i = 1 \\ & \qquad 0 \le x_i \le 1, \quad \forall i = 1,2, \dots, n. \end{aligned} \tag{5}$$

Suppose that $\xi_1, \xi_2, \dots, \xi_n$ are independent skew-normal uncertain variables denoted by $\xi_i \sim SN(m_i, p_i, \sigma_i)$; $\sigma_i > 0$, $p_i > 0, i = 1,2,\dots,n$. By applying Eq. (1) and Eq. (4) the inverse uncertainty distribution $\Psi^{-1}(r)$ of the uncertain portfolio total return $\sum_{i=1}^{n} x_i\xi_i$ is derived as follows:

$$\Psi^{-1}(r) = \sum_{i=1}^{n} x_i\,\Phi^{-1}(r) = \sum_{i=1}^{n} x_i\left(m_i + \frac{\sqrt{3}\sigma_i}{\pi}\ln\left(\frac{r^{\frac{1}{p_i}}}{1-r^{\frac{1}{p_i}}}\right)\right)$$

where $x_i \geq 0,\ i = 1,2,\dots,n$ and $r \in (0,1)$.

By using Eq. (1), Eq. (2) and Eq. (4) the expected value of the uncertain portfolio total return $x^T\xi = \sum_{i=1}^{n} x_i\xi_i$ is derived as follows:

$$E\left[\sum_{i=1}^{n} x_i\xi_i\right] = \int_0^1\left(\sum_{i=1}^{n} x_i\left(m_i + \frac{\sqrt{3}\sigma_i}{\pi}\ln\left(\frac{r^{\frac{1}{p_i}}}{1-r^{\frac{1}{p_i}}}\right)\right)\right)dr$$
$$= \sum_{i=1}^{n} x_i\left(m_i - \frac{\sqrt{3}\sigma_i}{\pi p_i} - \frac{\sqrt{3}\sigma_i}{\pi}\int_0^1 \ln\left(1-r^{\frac{1}{p_i}}\right)dr\right).$$

Now, by applying Eq. (3) and Eq. (4) the pseudo-triangular entropy of the uncertain portfolio total return $\boldsymbol{x}^T\boldsymbol{\xi} = \sum_{i=1}^{n} x_i\xi_i$ is obtained as follows:

$$\mathrm{H}\left[\sum_{i=1}^{n} x_i\xi_i\right] = \int_{\frac{1}{2}}^1\left(\sum_{i=1}^{n} x_i(2-2r)\,\Phi_i^{-1}(r)\right)dr - \int_0^{\frac{1}{2}}\left(\sum_{i=1}^{n} x_i(2r)\,\Phi_i^{-1}(r)\right)dr$$
$$= \sum_{i=1}^{n} x_i\left(\int_{\frac{1}{2}}^1 (2-2r)\left(m_i + \frac{\sqrt{3}\sigma_i}{\pi}\ln\left(\frac{r^{\frac{1}{p_i}}}{1-r^{\frac{1}{p_i}}}\right)\right)dr - \int_0^{\frac{1}{2}}(2r)\left(m_i + \frac{\sqrt{3}\sigma_i}{\pi}\ln\left(\frac{r^{\frac{1}{p_i}}}{1-r^{\frac{1}{p_i}}}\right)\right)dr\right).$$

Thus, Model (5) based on pseudo-triangular entropy is equivalent to the following model:

$$\min \sum_{i=1}^{n} x_i\left(\int_{\frac{1}{2}}^1 (2-2r)\left(m_i + \frac{\sqrt{3}\sigma_i}{\pi}\ln\left(\frac{r^{\frac{1}{p_i}}}{1-r^{\frac{1}{p_i}}}\right)\right)dr - \int_0^{\frac{1}{2}}(2r)\left(m_i + \frac{\sqrt{3}\sigma_i}{\pi}\ln\left(\frac{r^{\frac{1}{p_i}}}{1-r^{\frac{1}{p_i}}}\right)\right)dr\right)$$
$$s.t.\ \sum_{i=1}^{n} x_i\left(m_i - \frac{\sqrt{3}\sigma_i}{\pi p_i} - \frac{\sqrt{3}\sigma_i}{\pi}\int_0^1 \ln\left(1-r^{\frac{1}{p_i}}\right)dr\right) \geq C$$
$$\sum_{i=1}^{n} x_i = 1$$
$$0 \leq x_i \leq 1\ ;\ \forall i = 1,2,\dots,n.$$

Now, in order to further investigate the outperformance of pseudo-triangular entropy as a quantifier of portfolio risk together with the effect of different parameters of skew-normal uncertainty distribution on investment allocation, let us consider the following example.

Example 4.1 suppose there are four investment portfolios containing twelve securities. According to expert's evaluation, the future returns are independent skew-normal uncertain variables which are depicted in Table 1. Moreover, the parameter C in Model (5) is designated to 2 by investor.

Table 1. Uncertain return of securities.

Portfolio 1		**Portfolio 2**	
Security i	Uncertain return $\xi_i \sim SN(m_i, p_i, \sigma_i)$	Security i	Uncertain return $\xi_i \sim SN(m_i, p_i, \sigma_i)$
1	$\xi_i \sim SN(0.5, 0.4, 0.5)$	1	$\xi_i \sim SN(3, 3, 0.4)$
2	$\xi_i \sim SN(1, 0.5, 0.6)$	2	$\xi_i \sim SN(3, 2, 0.7)$
3	$\xi_i \sim SN(1.5, 1, 0.7)$	3	$\xi_i \sim SN(2, 2, 0.5)$
4	$\xi_i \sim SN(2, 1.5, 0.8)$	4	$\xi_i \sim SN(2, 1, 0.6)$
5	$\xi_i \sim SN(2.5, 2, 0.9)$	5	$\xi_i \sim SN(1, 0.5, 0.5)$
6	$\xi_i \sim SN(3, 3, 1)$	6	$\xi_i \sim SN(1, 0.4, 0.5)$
7	$\xi_i \sim SN(0.5, 3, 0.5)$	7	$\xi_i \sim SN(3, 0.4, 1)$
8	$\xi_i \sim SN(1, 2, 0.6)$	8	$\xi_i \sim SN(3, 0.7, 1)$
9	$\xi_i \sim SN(1.5, 1.5, 0.7)$	9	$\xi_i \sim SN(2, 0.5, 1)$
10	$\xi_i \sim SN(2, 1, 0.8)$	10	$\xi_i \sim SN(0.5, 3, 0.4)$
11	$\xi_i \sim SN(2.5, 0.5, 0.9)$	11	$\xi_i \sim SN(3, 1, 0.5)$
12	$\xi_i \sim SN(3, 0.4, 1)$	12	$\xi_i \sim SN(1, 1, 0.4)$
Portfolio 3		**Portfolio 4**	
Security i	Uncertain return $\xi_i \sim SN(m_i, p_i, \sigma_i)$	Security i	Uncertain return $\xi_i \sim SN(m_i, p_i, \sigma_i)$
1	$\xi_i \sim SN(4, 3, 2)$	1	$\xi_i \sim SN(4, 4, 0.5)$
2	$\xi_i \sim SN(4, 3, 1)$	2	$\xi_i \sim SN(4, 0.5, 1)$
3	$\xi_i \sim SN(0.5, 3, 0.3)$	3	$\xi_i \sim SN(3, 4, 0.4)$
4	$\xi_i \sim SN(1.5, 2, 0.4)$	4	$\xi_i \sim SN(4, 0.4, 0.2)$
5	$\xi_i \sim SN(2, 1, 0.5)$	5	$\xi_i \sim SN(1, 1, 0.5)$
6	$\xi_i \sim SN(3, 1, 0.4)$	6	$\xi_i \sim SN(1, 3, 1)$
7	$\xi_i \sim SN(1, 1, 0.3)$	7	$\xi_i \sim SN(0.5, 1, 0.3)$
8	$\xi_i \sim SN(1, 2, 0.3)$	8	$\xi_i \sim SN(0.7, 4, 0.2)$
9	$\xi_i \sim SN(4, 2, 0.5)$	9	$\xi_i \sim SN(0.6, 1, 0.2)$
10	$\xi_i \sim SN(4, 2, 0.4)$	10	$\xi_i \sim SN(0.3, 2, 0.4)$
11	$\xi_i \sim SN(4, 1, 0.3)$	11	$\xi_i \sim SN(0.5, 0.4, 0.5)$
12	$\xi_i \sim SN(4, 3, 0.5)$	12	$\xi_i \sim SN(3, 0.4, 0.4)$

To show the outperformance of pseudo-triangular entropy as a quantifier of portfolio risk, we consider Model (5) for logarithm entropy, triangular entropy and pseudo-triangular entropy. The optimal solutions are obtained by implementing GA in MATLAB software. Objective value and investment allocation in securities are illustrated in Table 2.

Table 2. Objective value and investment allocation in securities.

Portfolio 1		
Entropy	Objective value	Investment allocation
Logarithm	1.0720	(0, 0.049, 0.053, 0.051, 0.045, 0.038, 0.559, 0.032, 0.039, 0.049, 0.054, 0.031)
Triangular	0.4111	(0.02, 0, 0.026, 0.077, 0.012, 0.011, 0.643, 0.071, 0.096, 0.033, 0.005,0.005)
Pseudo-triangular	0.1273	(0, 0.057, 0.045, 0.056, 0.003, 0.035, 0.614, 0.037, 0.027, 0.052, 0.034, 0.039)
Portfolio 2		
Entropy	Objective value	Investment allocation
Logarithm	0.7824	(0.366, 0.007, 0.009, 0.036, 0.004, 0.181, 0.004, 0.004, 0.001, 0.319, 0.007, 0.062)
Triangular	0.3654	(0.503, 0, 0, 0, 0.149, 0.187, 0, 0, 0.002, 0.098, 0, 0.059)
Pseudo-triangular	0.0988	(0.484, 0.003, 0.001, 0.001, 0.001, 0.146, 0.185, 0, 0.001, 0.12, 0.002, 0.056)
Portfolio 3		
Entropy	Objective value	Investment allocation
Logarithm	0.5290	(0.002, 0.001, 0.307, 0.268, 0.005, 0.025, 0.132, 0.147, 0.033, 0.01, 0.06, 0.009)
Triangular	0.2422	(0.002, 0.046, 0.27, 0.073, 0.015, 0.021, 0.22, 0.186, 0.039, 0.042, 0.039, 0.046)
Pseudo-triangular	0.0680	(0.003, 0.046, 0.275, 0.073, 0.018, 0.021, 0.223, 0.189, 0.038, 0.038, 0.035, 0.04)
Portfolio 4		
Entropy	Objective value	Investment allocation
Logarithm	0.4112	(0.019, 0.001, 0.056, 0.019, 0.01, 0.008, 0.091, 0.381, 0.325, 0.078, 0.004, 0.007)
Triangular	0.3288	(0.043, 0, 0, 0, 0.025, 0, 0.115, 0.393, 0.097, 0.112, 0.214, 0)
Pseudo-triangular	0.0570	(0.149, 0.006, 0.022, 0.037, 0.036, 0.003, 0.095, 0.349, 0.2, 0.087, 0.009, 0.006)

According to Table 2, the objective value for pseudo-triangular entropy has the lowest value in four portfolios amongst different types of entropy. Therefore, a portfolio based on pseudo-triangular entropy is less risky than logarithm entropy and triangular entropy. Furthermore, parameters σ and p take priority over parameter m in capital allocation. In other words, more capital is allocated to securities with greater parameter p and smaller parameter σ.

5. Conclusion

In this paper, concepts of pseudo-triangular entropy as a supplement measure of uncertainty in the uncertain portfolio optimization were proposed and its mathematical properties were studied. We proved that entropy for uncertain variables in forms of logarithm function and triangular function sometimes may fail to measure the uncertainty of an uncertain variable. We also presented a numerical example to show the performance of pseudo-triangular entropy as the quantifier of risk in the uncertain portfolio optimization problem. To solve the corresponding problem, a genetic algorithm (GA) was implemented in MATLAB software and optimal solutions were obtained. The example illustrated that a portfolio based on pseudo-triangular entropy is less risky than a portfolio based on logarithm entropy and triangular entropy. Furthermore, allocated capital to securities with positive skew-normal uncertainty distribution with smaller parameter σ and greater parameter p is more than other securities.

Meanwhile, some issues remain to be discussed. The outperformance of pseudo-triangular entropy in comparison to logarithm entropy and triangular entropy was investigated, which further researches are required on possible outperformance of pseudo-triangular entropy over other forms of entropy. Moreover, effects of different values of parameter C in proposed model on capital allocation require to be further explored.

References

[1] S. H. Abtahi, G. Yari, F. Hosseinzadeh Lotfi and R. Farnoosh, Multi-objective portfolio selection based on skew-normal uncertainty distribution and asymmetric entropy, International Journal of Fuzzy Logic and Intelligent Systems, **21 (1)** (2021) 38–48, doi:10.5391/IJFIS.2021.21.1.38.

[2] X. Chen and W. Dai, Maximum entropy principle for uncertain variable, International Journal of Fuzzy Systems, **13 (3)** (2011) 232–236.

[3] X. Chen, S. Kar and D. Ralescu, Cross-entropy measure of uncertain variables, Information Sciences, **201** (2012) 53–60.

[4] W. Dai, Quadratic entropy of uncertain variables, Soft Computing, **22 (17)** (2018) 5699–5706.

[5] W. Dai and X. Chen, Entropy of function of uncertain variables, Mathematical and Computer Modelling, **55 (3-4)** (2012) 754–760.

[6] B. Liu, Some research problems in uncertainty theory, journal of Uncertain Systems, **3 (1)** (2009) 3–10.

[7] B. Liu, Toward uncertain finance theory, Journal of Uncertainty Analysis and Applications, **1 (1)** (2013) 1, doi:10.1186/2195-5468-1-1.

[8] B. Liu, Uncertainty Theory, 2nd edition, Springer, Berlin, (2007).

[9] B. Liu, Uncertainty Theory, 5th edition, Springer, Berlin, (2018).

[10] B. Liu, Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty, Springer, Berlin, (2010).

[11] B. Liu, Why is there a need for uncertainty theory, Journal of Uncertain Systems, **6 (1)** (2012) 3–10.

[12] Y. Liu and W. Lio, A Revision of Sufficient and Necessary Condition of Uncertainty Distribution, Journal of Intelligent & Fuzzy Systems, **38 (4)** (2020) 4845–4854, doi:10.3233/JIFS-191535.

[13] H. Markowitz, Portfolio selection, The Journal of Finance, **7 (1)** (1952) 77–91, doi:10. 1111/j.1540-6261.1952.tb01525.x.

[14] Z. Peng, and K. Iwamura, A sufficient and necessary condition of uncertainty distribution, Journal of Inter disciplinary Mathematics, **13 (3)** (2010) 277–285.

[15] G. C. Philippatos and C. J. Wilson, Entropy market risk and the selection of efficient portfolios, Applied Economics, **4 (3)** (1972) 209–220, doi:10.1080/00036847200000017.

[16] C. E. Shannon, A mathematical theory of communication, Bell System Technical Journal, **21 (3)** (1948) 379–423, doi:10.1002/j.1538-7305.1948.tb01338.x.

[17] A. Z. Sheik and H. Qiao, Non-normality of market returns: A framework for asset allocation decision making, The Journal of Alternative Investments, **12 (3)** (2010) 8–35, doi:10.3905/JAI.2010.12.3.008.

[18] M. R. Simonelli, Indeterminacy in portfolio selection, European Journal of Operational Research, **163 (1)** (2005)170–176, doi:10.1016/j.ejor.2004.01.006.

[19] W. Tang and W. N. Gao, Triangular entropy of uncertain variables, INFORMATION-An International Interdisciplinary Journal, **16 (2(A))** (2013) 1279–1282.

[20] K. Yao, J. Gao and W. Dai, Sine entropy for uncertain variable, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **21 (5)** (2013) 743–753.