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An Introduction to the Application of Tensorial Manifold Learning Methods in the Digital Image Processing and Computer Vision

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Abstract. Tensors as vector fields structures and manifolds as great geometrical-topological structures have many applications in the fields of big data analysis. Types of norms, metrics, and scalable structures have been defined from various aspects. Nowadays, the hybrid methods between tensorial algorithms and manifold learning (MaL) methods have been attracted some attention. In image and signal processing, from image recovery to face recognition, these methods have appeared very excellent. According to our experiments by **MATLAB R2021a**, the hybrid algorithms are powerful other than algorithms based on the efficient popular parameters.

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1. Introduction

Matrix and tensor completion methods have many applications in various fields of big data analysis, prediction based on collected data, image processing, and computer vision. Incomplete, distorted, and noisy data has always been a major challenge in the field of big data analysis, especially image processing [5]. This problem appears in digital images as a variety of noise and image distortion. Matrix and

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tensor completion methods have the ability to compensate to a significant degree (up to 90 percent distortion) [12]. On the other hand, manifold learning methods based on a manifold theory with the ability to reduce the dimension and eliminate noise and outliers data, significantly increase computational efficiency [4]. Today, the use of hybrid methods has become very common. By combining these large mathematical structures in geometry (manifolds) and algebra (tensors), advanced and precise methods can be achieved that, while having high efficiency with significant detection and recovery rates, also have high computational efficiency [13].

In this paper, we introduce tensorial manifold learning methods and applications to image processing and computer vision. our experiments show the high power, efficiency, and computational cost of these hybrid methods in various fields from image recovery to face recognition.

The rest of the paper is organized as follows: In section 2, some preliminaries and basic concepts from tensor and manifold theory are proposed, the hybrid method between tensor and manifolds are introduced in section 3, and finally, results and conclusions are considered in section 4.

2. Basic concepts

In this section, we briefly state some preliminaries from tensor, tensor calculus, tensor completion, and manifold theory.. For more details and information, please refer to [5], and [12].

Definition 2.1 A tensor is a multidimensional array, The dimensionality of it is described as its order. An Nth-order tensor is an N-way array, also known as N-dimensional or N-mode tensor, denoted by X. We use the term order to refer to the dimensionality of a tensor (e.g., Nth-order tensor), and the term mode to describe operations on a specific dimension (e.g., mode-n product) [1]. We denote the set of all n-dimensional tensors of order m by $T_{m,n}$. For a tensor A, if all of $a_{i_1,\ldots,i_n}$ are invariant under any permutation of indices, then A is called a symmetric tensor. We show the set of all real n-dimensional symmetric tensors of order m with $S_{m,n}$.

Tensors are simply mathematical objects that can be used to describe physical properties. In fact, tensors are merely a generalization of scalars, vectors, and matrices; a scalar is a zero rank tensor, a vector is a first-rank tensor and a matrix is the second rank tensor [12].

Definition 2.2 The inner product of two tensors X and Y of the same size is defined as $< X, Y >$. Unless otherwise specified, we treat it as dot product defined as follows [8]:

$$< X, Y >:= \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} X_{i_1,i_2,\cdots,i_N} y_{i_1,i_2,\cdots,i_N} \tag{1}$$

Definition 2.3 Generalized from matrix Frobenius norm, the F-norm of a tensor X is defined as [5]:

$$||X||_F := \sqrt{< X, X >} = \sqrt{\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} X^2_{i_1,i_2,\cdots,i_N}} \tag{2}$$

Definition 2.4 The well-known optimization problem for matrix completion as

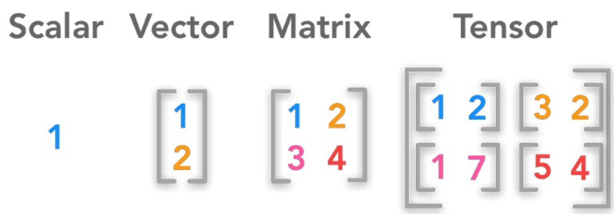

Figure 1. The representation of tensors as a generalization of other linear algebraic structures.

follows:

$$Min_X : \frac{1}{2}||X - M||_{\Omega}^2$$
$$(s.t.) \quad rank(X) \leqslant r,$$

where $X, M \in \mathbb{R}^{p \times q}$, and the elements of M in the set Ω are geven while the remaining are missing. We aim to use a low rank matrix X to approximate the missing elements [2].

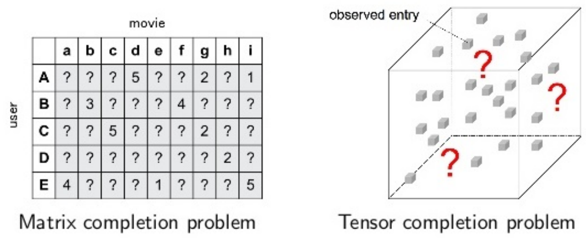

Figure 2. The comparison scheme of matrix vs tensor completions.

Definition 2.5 The tensors is the generalization of the matrix concept. Given a low-rank tensor T with missing entries, the goal of completing it can be formulated as the following optimization problem [3]:

$$Min_X : \frac{1}{2}||X - Y||_F^2$$
$$(s.t.) \quad ||X|| \leqslant c$$
$$Y_{\Omega} = T_{\Omega}$$

where X, Y, T are n-mode tensors with identical size in each mode.

Figure 2 shows the The comparision between matrix and tensor completion problems.

Definition 2.6 A manifold is a Hausdorff topological space which looks locally like a Cartesian space, commonly a finite-dimensional Cartesian space $\mathbb{R}^n$, a topological space in which case one speaks of a manifold of dimension n or n-fold, but possibly an infinite-dimensional topological vector space, in which case one has an infinite-dimensional manifold [6]. The topological manifold of M is called smooth (differentiable) if M has continuous differentials. In fact, the topological manifold C^0 is continuous and the topological manifold whose derivatives of any order are continuous is called C^{∞} or smooth.

For example, A circle is a one-dimensional manifold embedded in two dimensions where each arc of the circle is locally resembles a line segment, Let's now move onto 2D-manifolds. The simplest one is a sphere. You can imagine each infinitesimal patch of the sphere locally resembles a 2D-Euclidean plane. Similarly, any 2D-surface (including a plane) that doesn't self-intersect is also a 2D manifold [6]. Figure 3 shows some examples.

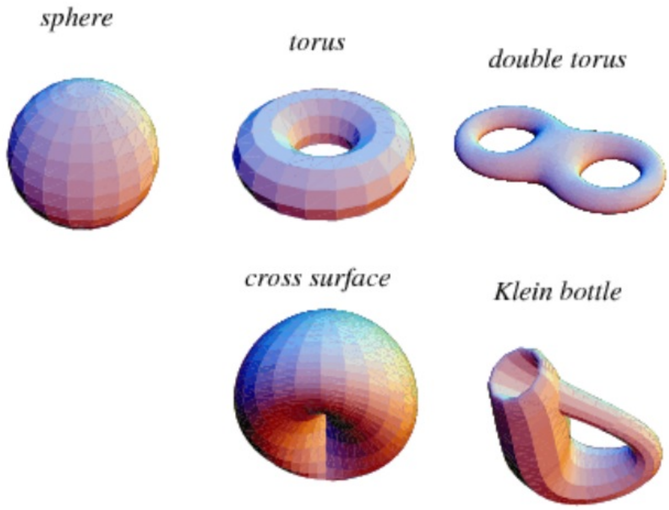

Figure 3. Non-intersecting closed surfaces in $\mathbb{R}^3$ are examples of 2D-manifolds such as a sphere, torus, double torus, cross surfaces and Klein bottle.

For these examples, you can imagine that each point on these manifolds locally resembles a 2D-plane.

3. Main results

Real-world data, such as speech signals, digital photographs, or MRI scans, usually have high dimensionality. In order to handle such real-world data adequately, its dimensionality needs to be reduced. Dimensionality reduction is the transformation of high-dimensional data into a meaningful representation of reduced dimensionality [4].

Dimensionality reduction methods are widely used in the machine learning community for high-dimensional data analysis. Ideally, the reduced representation should have a dimensionality that corresponds to the intrinsic dimensionality of the data. The intrinsic dimensionality of data is the minimum number of parameters needed to account for the observed properties of the data [13]. As a result, dimensionality reduction facilitates among others, classification, visualization, and compression of high-dimensional data. Traditionally, dimensionality reduction was performed using linear techniques such as Principal Components Analysis (PCA), factor analysis, classical scaling, and t-SNE [4]. However, these linear techniques cannot adequately handle complex nonlinear data. In fact, PCA is a statistical procedure that uses an orthogonal transformation to convert a set of observations of possibly correlated variables into a set of values of linearly uncorrelated variables called principal components. Other methods (mentioned above) are generalized versions of the PCA based on the case study. Thus, manifold Learning methods have emerged. manifold learning is an approach to non-linear dimensionality reduction. Algorithms for this task are based on the idea that the dimensionality of many data sets is only artificially high, For more details, see Figure 4 [13].

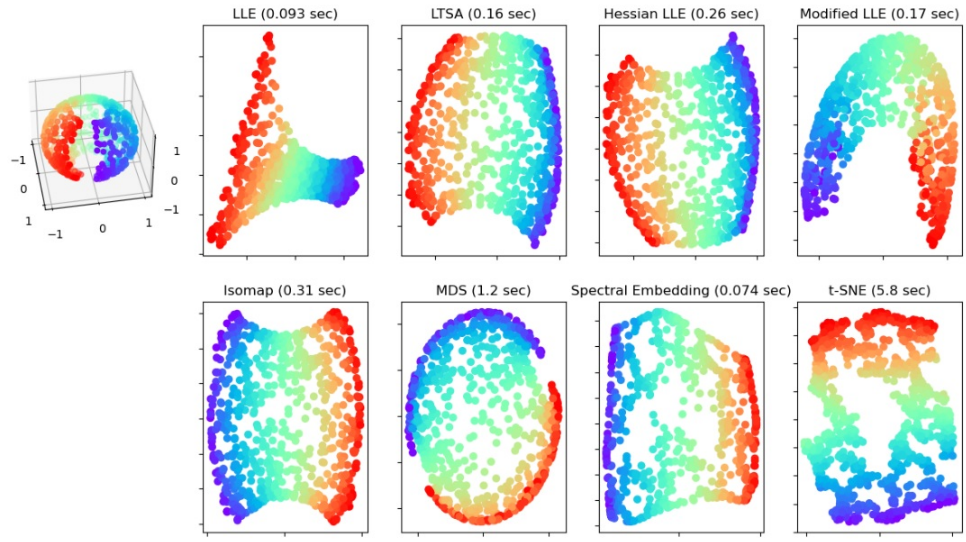

Figure 4. The comparision scheme of some manifold learning methods.

In less than 20 years, with the development of dimensional reduction methods, manifold's theory has been widely used in the field of artificial intelligence and has led to the discovery of a new concept called manifold learning. This is a subfield of machine learning based on the hypothesis that data lies in the vicinity of a low-dimensional manifold. We would like to learn the underlying manifold from data. manifold learning is a subclass of non-linear dimensionality reduction algorithms. These algorithms attempt to discover the low dimensional manifold that the data points have been sampled from manifold learning methods are useful for high dimensional data analysis [4]. Many of the existing methods produce a low-dimensional representation that attempts to describe the intrinsic geometric structure of the original data. Typically, this process is computationally expensive and the produced embedding is limited to the training data. In many real-life scenarios, the ability to produce embedding of unseen samples is essential. In this space, the Euclidean distance indicates the affinity between the original data points with respect to the manifold geometric structure [13].

The main idea is that the dimension of the data set or space is artificially high and with appropriate geometric methods, a low-dimensional manifold can be achieved that contains valuable and important information of the original data space (*Whitney theorem*). This embedded manifold is called the *Whitney theorem*. The main goal of manifold learning methods is to reduce the dimension and increase computational efficiency. The concept of a tensor is also presented in the form of a tensor field, so the combination of tensor methods and manifold learning methods in recent years is very much in the spotlight and promises the emergence of faster and more efficient methods for processing all types of big data, especially high-resolution images. The format of digital images and videos has been changed. In the field of applications of manifold learning methods, we can mention handwriting manifold learning through LLE or Isompe methods (in general, Isompe is one of the most basic methods for manifold learning, which can be considered as MDS and PCA expansion while maintaining geodesic distances between points). Application in image processing in the stages of recovery and recognition in medical images such as brain MRI, face recognition, and high ability to reconstruct human face images is also one of the important applications [4]. Figure 5 shows the place of manifold learning methods between dimensionality reduction methods for big data analysis.

Figure 5. The place of manifold learning methods.

In our work, we implement 57 tensor completion methods on the image datasets for some goals such as recovering, reconstruction, and face detection and recognition. After that, we hybrid these methods with 33 manifold learning methods for dimensionality reduction and improving some important parameters like recovery rate, recognition rate, and facial shape recovery ratio. our experiments show that hybrid methods besides computational costs (time spent for processing and workload on processors) have more efficient accuracy. For example, In the facial shape recovery, our hybrid method could completely extract 3D-facial shape with zero error. Figure 6 shows the result of algorithm implementation [11].

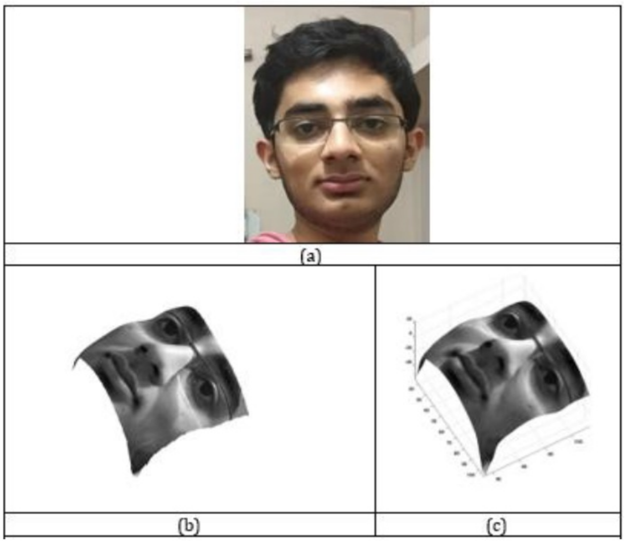

Figure 6. The implementation of facial shape recovery by manifold tensorial methods. (a): original photo, (b): tensor representation output and (c): manifold-tensorial representation output.

In the image recovery, the hybrid method could boost the ability of recovery of noisy images by missing rates from 70 to 95 percent, for more details, see Figure 7, for more information and details, please see [10].

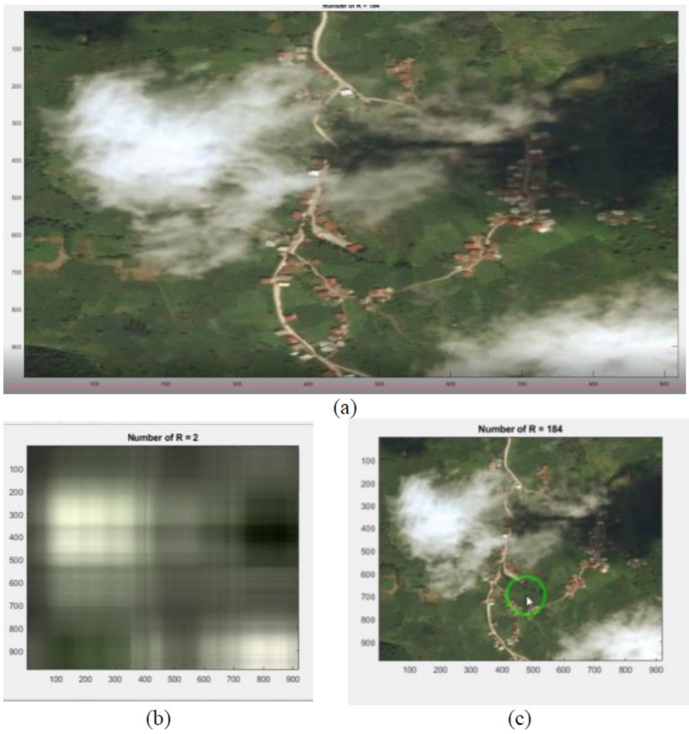

Figure 7. Image recovery: (a): Original image, (b): noisy image up to 90 percent (Type: dissolved), (c): Reconstructed image (after image processing by tensorial manifold algorithm.

In face detection and recognition, a hybrid between tensor completion and PCA (as a manifold learning or dimensionality reduction method) could improve the recognition rate from 70 to 91.27 percent and the error is about 10^{-6}, for more details, see Figure 8, for more information, we refer the reader to [9].

3.1 *Tips for practical use*

There are the following considerations for applying manifold learning methods:

1) Make sure the same scale is used overall features. Because manifold learning methods are based on a nearest-neighbor search, the algorithm may perform poorly otherwise.
2) The reconstruction error computed by each routine can be used to choose the optimal output dimension. For a d-dimensional manifold embedded in a D-dimensional parameter space, the reconstruction error will decrease as n-components is increased until n-components $= d$.
3) Note that noisy data can short-circuit the manifold, in essence acting as a bridge between parts of the manifold that would otherwise be well-separated. manifold learning on noisy and/or incomplete data is an active area of research.
4) Certain input configurations can lead to singular weight matrices, for example when more than two points in the dataset are identical, or when the data is split into disjointed groups. The easiest way to address this is to use a singular matrix, though it may be very slow depending on the number of input points. Alternatively, one can attempt to understand the source of the singularity: if it is due to disjoint sets, increasing n neighbors may help. If it is due to identical points in the dataset, removing these points

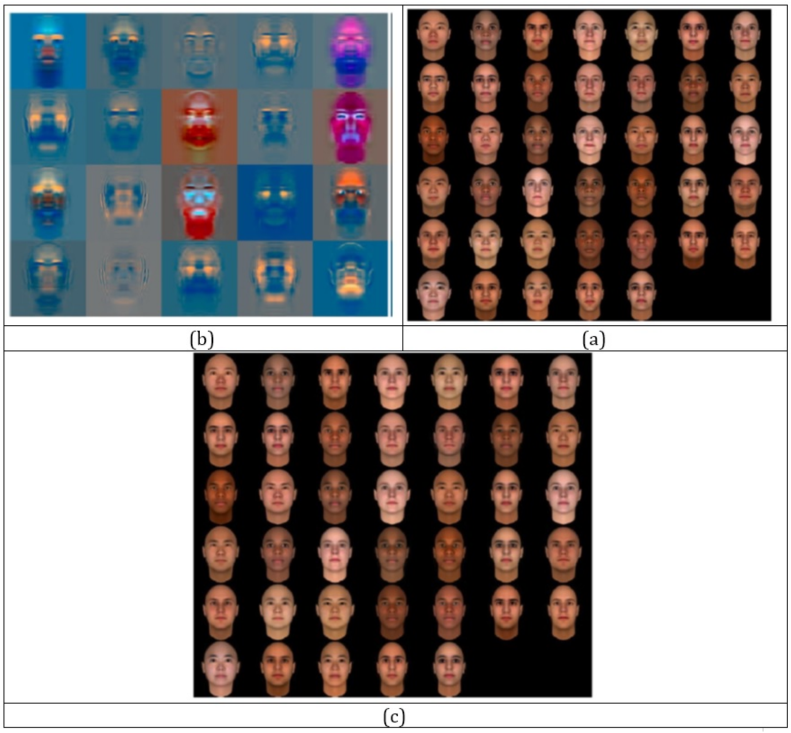

Figure 8. Face detection: (a): Original images, (b): Tensorized images, (c): Recognitioned images (after image processing by tensorial manifold algorithm.

may help.

For defeating these problems, we use tensor and matrix completion methods. The hybrid method is based on the tensor completion and manifold tools, and more powerful and efficient than other methods.

4. Conclusions

In this paper, we have introduced new computational methods that lead to advanced hybrid algorithms for the registration, reconstruction, recovery, and recognition of objects and human images (like faces). Practically, we implement 57 tensor completion methods on the image datasets for some goals such as recovering, reconstruction, and face detection and recognition. After that, we hybrid these methods with 33 manifold learning methods for dimensionality reduction and improving some important parameters like recovery rate, recognition rate, and facial shape recovery ratio. According to studies conducted in our experiments, in addition to computational savings due to reduced dimensions, these methods have high detection and recognition rates up to 99.9 percent and recovery rates up to 99 percent. Hence, these methods have suitable computational costs and are more efficient than other methods. The combination of conventional linear dimensional reduction methods such as *PCA* and *LDA* with tensors and the development of new algorithms such as *MPCA* and *MLDA* is a testament to this claim. In any type of problem, depending on the case study conditions such as type of images or data (structured, semi-structured, or unstructured), by choosing the appropriate tensor analysis method, multiplication and metric, the type of optimization method depending on equal or unequal constraints of the problem, or convexity or concavity, the best method, and algorithm for achieved a better result, should be selected. Finally, the hybrid between tensors and manifolds methods result in efficient and hopeful methods for big data analysis, especially digital images.

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