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Apply New Optimized MRA & Invariant Solutions on the Generalized-FKPP Equation

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So far, the numerous methods for solving and analyzing differential equations are proposed. Meanwhile; the combined methods are beneficial; one of them is the Optimized MRA method (OMRA). This method is based on the Father wavelets (dependent on the invariant solutions obtained by the Lie symmetry method) and correspondent MRA. In this paper, we apply the OMRA on the generalized version of FKPP equation (GFKPP) with function coefficient

$$fu_{tt}(x,t) + u_t(x,t) = u_{xx}(x,t) + u(x,t) - u^2(x,t),$$

where f is a smooth function of either x or t. We will see that by the suitable Father wavelets, this method proposes attractive approximate solutions.

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1. Introduction

At the end of the nineteenth century, the theory of Lie symmetry groups of differential equations was developed by Sophus Lie in order to study the solutions

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of ordinary differential equations (ODEs). He showed that the order of an ODE could be reduced by one if it is invariant under one-parameter Lie group of point transformations. Lie devoted the remainder of his scientific career to developing these continuous groups that have now an impact on many areas of mathematically based sciences. Lie and Emmy Noether mainly established the applications of Lie groups to differential equation systems and then lie Cartan advocated their works [3].

Indeed, a Lie point symmetry of a differential system is a local group of transformations that maps every solution of the system to another solution of the same system. In other words, it maps the solution set of the system to itself. Primary and elementary examples of Lie groups are translations, rotations, and scalings (for many other applications of Lie symmetries see [11], [12]).

The Fisher's equation (also called the FisherKolmogorov equation and the FisherKPP equation, named after R. A. Fisher and A. N. Kolmogorov) is the PDE: $u_t = u_{xx} + u - u^2$. This equation describes the spatial spread of an advantageous allele and explores its traveling wave solutions and is a model of diffusion in biomathematics [17].

Such equations occur, e.g., in ecology, physiology, combustion, crystallization, plasma physics, and in general phase transition problems, this equation is a well-known and widely applied nonlinear reaction-diffusion equation and is traditionally applied to model the spread of genes in population genetics [16].

The generalized version of FKPP equation with function coefficient as follows

$$fu_{tt} + u_t = u_{xx} + u - u^2,$$

where f is a smooth function of either x or t. So far, this version of FKPP solved by numerical methods and did not found any explicit solution. By the Lie symmetry method, the generalized FKPP equation will be converted to ODEs, and all symmetries, and generalized vector fields will be determined (for more details and computations, see [17], [16]).

The wavelet theory proposes useful, and efficient tools for analyzing problems in functional and harmonic analysis. The Hungarian mathematician Alfrd Haar introduced the first wavelet in 1909 [7]. The wavelets have numerous applications in many fields of science and technology: seismology, image processing, signal processing, coding theory, biosciences, financial mathematics, fractals, and so on [1]. The application of wavelets for solving differential equations limited to ODEs or PDEs with the numerical solutions in the particular conditions. The famous wavelets such as Haar, Daubechies, Coiflet, Symlet, CDF, Mexican hat and Gaussian are extendible to two or more variables by some methods (for example, tensor product). Here, the wavelets with two or more variables (in connection with PDEs) are constructed by the different methods. In this paper, we apply the OMRA to the generalized version of FKPP (GFKPP) and obtain approximate solutions [15].

The paper is organized as follows. In Section 2, we briefly review some needed results to construct differential invariants, the Father wavelets, and MRA. In section 3, the OMRA method is mentioned. In sections 4, the OMRA method applies to the GFKPP equation. Finally, the conclusions & future works are presented.

2. Preliminaries

In this section, we recall some concepts, definitions, formulas, and theorems about Lie symmetries, Fushchych method, and wavelets. For more details and proofs, please see related resources.

2.1 *The Lie symmetry method*

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations (see [3], [11] and [10]). Suppose that

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{1}$$

the general case of a nonlinear system of partial differential equations of order nth in p independent and q dependent variables that involving $x = (x^1, \cdots, x^p)$, $u = (u^1, \cdots, u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system (1):

$$(\tilde{x}^i, \tilde{u}^j) = (x^i, u^i) + s(\xi^i, \eta^j) + O(s^2), \quad i = 1 \cdots, p, \ j = 1 \cdots, q,$$

where s is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator $\mathbf{v} = \sum_{i=1}^{p} \xi^i \partial_{x^i} + \sum_{j=1}^{q} \eta^j \partial_{u^j}$ is correspondent to this group of transformations. A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. The invariance of the system (1) under the infinitesimal transformations leads to the invariance conditions ([11]):

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta_\nu(x, u^{(n)})\big] = 0, \quad \Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l,$$

where $\mathrm{Pr}^{(n)}$ is called the n^{th} order prolongation of the infinitesimal generator and given by

$$\mathrm{Pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J} \phi_J^\alpha(x, u^{(n)}) \partial_{u_J^\alpha},$$

where $J = (j_1, \cdots, j_k)$, $1 \leqslant j_k \leqslant p$, $1 \leqslant k \leqslant n$ and the sum is over all J's of order $0 < \#J \leqslant n$. If $\#J = k$, the coefficient ϕ_J^α of $\partial_{u_J^\alpha}$ will only depend on k-th and lower order derivatives of u, and $\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha$, where $u_i^\alpha := \partial u^\alpha / \partial x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / \partial x^i$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket. The first advantage of symmetry group methods is to construct new solutions from known solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it into a linear system. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the PDE to be connected directly to some order differential equations. To do this, particular linear combinations of infinitesimals are considered, and their corresponding invariants are determined.

For every generator vector field, the characteristics system as follows

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}.$$

by solving this system, we obtain differential invariants corresponding to every vector field. The PDE is expressed in the coordinates (x, t, u), so for reducing this equation, we should search for its form in the specific coordinates. Those coordinates will be constructed by searching for independent invariants (y, v) corresponding to the infinitesimal generator. Hence by using the chain rule, the expression of the differential equation in the new coordinate allows us to the reduced equation. For more information and examples, see [11].

2.2 *The Fushchych method*

Let us now consider the Fushchych Method: A fundamental and broadly comprehensive notion has been introduced by Fushchych [13].

Definition 2.1 Let us say that X is a *conditional* symmetry of the equation $\Delta_\nu(x, u^{(n)}) = 0$ in the sense of Fushchych, if there is a supplementary equation $E = 0$ such that X is an exact symmetry of the system

$$\Delta_\nu(x, u^{(n)}) = 0, \quad E = 0.$$

The simplest and more frequent case is obtained by choosing as supplementary equation the *side condition* or *invariant surface condition.*

$$X_Q u = \xi_i \frac{\partial u}{\partial x_i} - \phi = 0.$$

where X_Q is the symmetry that written in *evolutionary form* [2].

This corresponds to the usual (properly called) conditional symmetry (CS) (also called Q-conditional symmetry), and the above condition, indicates that we are looking precisely for solutions which are invariant under X. To avoid unessential complications with notations, we will consider from now on only the case of a single PDE: $\Delta_\nu = 0$ for a single unknown function $u(x)$. The extension to more general cases is entirely straightforward.

Definition 2.2 A vector field X is called a *partial* symmetry (of order σ), if it is a symmetry of the system

$$\Delta_\nu = 0, \qquad \mathrm{Pr}^{(1)}\mathbf{X}\big[\Delta_\nu(x, u^{(n)})\big] = 0, \cdots, \mathrm{Pr}^{(\sigma-1)}\mathbf{X}\big[\Delta_\nu(x, u^{(1)})\big] = 0,$$

Definition 2.3 A vector field X is a *weak* CS (of order σ), if it is a symmetry of the system

$$\begin{gathered} \Delta_\nu = 0, \qquad X_Q(u) = 0, \\ \mathrm{Pr}^{(1)}\mathbf{X}\big[\Delta_\nu(x, u^{(n)})\big] = 0, \qquad \cdots \qquad \mathrm{Pr}^{(\sigma-1)}\mathbf{X}\big[\Delta_\nu(x, u^{(1)})\big] = 0, \end{gathered}$$

moreover this corresponds to the existence of a system of σ reduced equations, which gives X-invariant solutions of $\Delta_\nu = 0$.

Theorem 2.4 *Any vector field X is either an exact, or a standard CS, or a weak CS. Similarly, any X is either an exact or a partial symmetry.*

Proof For more details and examples refer to [17]. ■

2.3 *Wavelets*

In this section, we briefly explain some needed concepts and formulas from wavelet theory. Here, we introduce the wavelets as functions belong to $L^2(\mathbb{R}^2)$ (the space of squared integrable functions with integral norm). The basic needed concepts are Mother wavelet, Father wavelet, and wavelet family.

Definition 2.5 The function ψ belongs to $L^2(\mathbb{R}^2)$ that satisfies in the following *admissible condition*

$$C_\psi = \int_{\mathbb{R}^2} \frac{|F(\psi)(\omega)|^2 d\omega}{|\omega|^2} > 0.$$

is called a *wavelet*, where $F(\psi)(\omega)$ is the *Fourier transformation* of wavelet ψ and given by

$$F(\psi)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \exp(-ix.\omega)\psi(x) d\omega.$$

where C_ψ is called the wavelet coefficient of ψ. Here, $\omega = (\omega_1, \omega_2)$ and $x = (x_1, x_2)$ belong to $\mathbb{R}^2$. For further information and examples, see [11].

Definition 2.6 The wavelet ψ is called a *Mother* wavelet, if it satisfies in the following properties,

$$\int_{\mathbb{R}^2} \psi(x) dx = 0,$$

$$\int_{\mathbb{R}^2} |\psi(x)|^2 dx < \infty,$$

$$\lim_{|\omega| \to \infty} F(\psi(\omega)) = 0.$$

Note that, the first condition is equivalent with *admissible condition*. For more details, see [7].

In fact, the mother wavelets have the *admissible condition*, *n-zero moments* and *exponential decay* properties. For generating wavelet family, two parameters: the translation parameter $b = (b_1, b_2)$ and scaling parameter $a > 0$ are applied on the Mother wavelet and the corresponding wavelet family as below

$$\psi_{a,b}(x) = \psi(\frac{x-b}{a}) = \psi(\frac{x_1 - b_1}{a}, \frac{x_2 - b_2}{a}).$$

Definition 2.7 The wavelet ϕ is called a *Father* wavelet, if it has the following properties,

$$\int_{\mathbb{R}^2} \phi(x) dx = 1, \tag{2}$$

$$\int_{\mathbb{R}^2} |\phi(x)|^2 dx = 1, \tag{3}$$

$$\langle \phi(x), \phi(x-n) \rangle = \delta(n) \tag{4}$$

where $\langle , \rangle$ is scalar product derived from the integral norm of $L^2(\mathbb{R}^2)$ and $\delta(n)$ is Kronecker delta.

For more details, see [7]. In fact, the Father wavelets are *scaling functions* for decomposing $L^2(\mathbb{R}^2)$ (in MRA).

Multiresolution Analysis:

Definition 2.8 A *multiresolution analysis* (MRA) of $L^2(\mathbb{R}^2)$ is defined as sequence of non-empty closed subspace $V_j \subset L^2(\mathbb{R}^2), (j \in \mathbb{Z})$ such that

$$\{0\} \subset ... \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ... \subset L^2(\mathbb{R}^2).$$

have the following properties:

1 . $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^2)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
2 . $f(x) \in V_j$ iff, $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$,
3 . $f(x) \in V_j$ iff, $f(x_1 - 2^s j, x_2 - 2^s k) \in V_j, \quad \forall (j,k) \in \mathbb{Z}^2$,
4 . There exist a function $\phi(x) \in V_0$ with nonvanishing integral, such that the set $\{\phi_{0,k}(x) = \phi(x-k), k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

where $x = (x_1, x_2) \in \mathbb{R}^2$.

Actually, with a given Father wavelet $\phi(x)$, the related wavelet family is

$$\phi^s_{j,k}(x) = 2^{s/2}\phi(2^s x_1 - j, 2^s x_2 - k). \tag{5}$$

thus, $\phi^0_{j,k}(x) = \phi_{j,k}(x) = \phi(x_1 - j, x_2 - k)$, and the function $\phi_{j,k}(x)$ is called the scaling function associated with MRA.

Theorem 2.9 *If $M = \{V_j\}$ is a MRA for $L^2(\mathbb{R}^2)$, with $g = \phi(x-k)$ as the scaling function, then, we can make wavelets ϕ and ψ such that for every $j, k \in \mathbb{Z}$,*

$$\{\{\phi_{j,k}\}, \{\psi_{j,k}\}\}_{j,k\in\mathbb{Z}}.$$

is an orthonormal basis of $L^2(\mathbb{R}^2)$. The wavelets ϕ and ψ are the Father and mother wavelets (respectively).

Proof For proof and more details, see [9]. ■

With given Father wavelet ϕ, and by obtaining corresponding MRA and producing corresponding mother wavelet ψ, We can define the following subspaces:

Definition 2.10 In a MRA with the scaling function $g = \phi(x-k)$, the Father and mother wavelets, ϕ and ψ (respectively), the approximation subspaces V_j and wavelet subspaces W_j (respectively) are defined as follows

$$V_s := \text{span}\{\phi^s_{j,k} \mid \phi^s_{j,k}(x) = 2^{s/2}\phi(2^s x_1 - j, 2^s x_2 - k)\}, \tag{6}$$

$$W_s := \text{span}\{\psi^s_{j,k} \mid \psi^s_{j,k}(x) = 2^{s/2}\psi(2^s x_1 - j, 2^s x_2 - k)\}. \tag{7}$$

Remark 2.11 Indeed, W_j is the orthogonal complement of V_j in V_{j+1}, i.e.,

$$V_{s+1} = V_s \oplus W_s.$$

By following this process, we found that $\oplus_s W_s = L^2(\mathbb{R}^2)$. On the other hand, since

$\phi(x) \in V_0 \subset V_1$, there exists a sequence $\{a_{j,k}, j, k \in \mathbb{Z}\}$, such that

$$\phi(x) = \sqrt{2} \sum_{j,k} a_{j,k} \phi(2x_1 - j, 2x_2 - k).$$

These equations are called *dilation equations*, *two-scale differential equations*, or *refinement equations*. The Mother wavelet ψ satisfies in the similar equations,

$$\psi(x) = \sqrt{2} \sum_{j,k} w_{j,k} \phi(2x_1 - j, 2x_2 - k), \tag{8}$$

where the coefficients $w_{j,k}$ are given by

$$w_{j,k} = (-1)^{j+k} \bar{a}_{1-j,1-k}, \tag{9}$$

These equations are called the *wavelet equations*.

Definition 2.12 The coefficients $a_{j,k}$ and $w_{j,k}$ are called the *approximation* and *wavelet* coefficients (respectively). In fact, the approximation coefficient $a_{j,k}$ are calculated as

$$a_{j,k} = \langle \phi, \phi_{1;j,k} \rangle = \sqrt{2} \int\int \phi(x,t) \phi(2x - j, 2t - k) dx dt. \tag{10}$$

Where the scaling parameter s is 1.

Theorem 2.13 *If $M = \{V_j\}$ is a MRA for $L^2(\mathbb{R}^2)$, with $g = \phi(x-k)$ as the scaling function, and $W_j's$ as the corresponding wavelet subspaces, then $L^2(\mathbb{R}^2) = \oplus_j W_j$ and every $f \in L^2(\mathbb{R}^2)$ can be uniquely expressed as a sum $\sum w_{j,k} \psi_{j,k}$. Equivalently, the set of all Mother wavelets, $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$.*

Proof For proof and more details, see [4]. ■

3. Optimized-MRA method

The optimized-MRA method (OMRA) have four following steps:

1) Apply equivalence algorithms (for example, the Lie symmetry method) on differential equations, and obtain differential invariants.
2) Propose suitable Father wavelet based on the differential invariants.
3) Apply MRA with Father wavelet as the scaling function, then obtain the approximation and wavelet subspaces, related coefficients and correspondent mother wavelet based on the Father wavelet & invariant solutions.
4) The final solution is in the form of the linear combination of Father wavelet (with the structure based on differential invariants), the Mother wavelet (with a structure based on Father wavelet and invariant solutions) and wavelet coefficients.

In the following, some OMRA formulas are proposed. First, with given suitable Father wavelet based on differential invariants (DI), we obtain approximation coefficients by writing dilation equations as follows

$$\phi(x,t) = \sum_j \sum_k a_{j,k} \phi_{1;j,k}(x,t) = \sqrt{2} \sum_j \sum_k a_{j,k} \phi(2x - j, 2t - k). \tag{11}$$

where $a_{j,k}$ are the approximation coefficients. Then, we obtain the wavelet coefficients by (9).

Now, If Father wavelet ϕ completely dependent on DE & its DIs, then corresponding Mother wavelet are obtained from relation (8). So, the approximate solution as follows

$$u(x,t) = \sum_{j,k} a_{j,k}\phi_{j,k}(x,t) + \sum_{j,k} w_{j,k}\psi_{j,k}(x,t), \tag{12}$$

After calculating the approximation and wavelet coefficients, we consider the solution as (12), then put it in DE, and by solving resulting ODE for $\psi_{j,k}(x,t)$, the Mother wavelet ψ will be obtained as follows

$$\psi(x,t) = \sum\sum w_{j,k}\psi_{1;j,k}(x,t) = \sqrt{2}\sum_j\sum_k w_{j,k}\psi(2x-j, 2t-k),$$

On the other hand, since $\psi_{j,k}$ made an orthonormal basis for the solution space M, after determining ψ, we can consider the solution of DE as follows

$$u(x,t) = \sum_{j,k} w_{j,k}\psi_{j,k}(x,t), \tag{13}$$

Therefore, by having the corresponding Mother wavelet, we will approximate solution based on the Mother wavelet.

Remark 3.1 In OMRA, the approximation and wavelet subspaces are defined like MRA with scaling parameter $s = 1$. Therefore, the solution space $M^{(n)}$ is decomposed as

$$M^{(n)} = \{\oplus_{j,k}V_{j,k}\} \oplus_{j,k} \{\oplus_{j,k}W_{j,k}\}.$$

For more details, see [7, 13].

4. Apply OMRA on the GFKPP

Here, we implement OMRA on the GFKPP equation and obtain solutions. Finally, the OMRA results will be proposed. First, apply the Lie symmetry method on the GFKPP equation $fu_{tt} + u_t = u_{xx} + u - u^2$, and obtain symmetry groups, vector fields and differential invariants, for more detailed calculations & results of the Lie symmetry method implementation on the GFKPP equation, see [17]. The Lie symmetry method results for the GFKPP equation proposed in the following table (for more details and computations, see [17], [16]):

In Table 1, the symmetry groups are translation and scaling. Here, we offer two below Father wavelets

$$\phi_1(x,t) = \frac{4}{3\pi}\exp(-\frac{x^2+t^2}{0.75}),$$

$$\phi_2(x,t) = \exp(-\frac{x^2+15t^2}{5})\cos(x)\cos(t/3).$$

Table 1. The Lie symmetry method: exact symmetries, differential invariants and invariant solutions.

f	Vector fields	dim(**g**)	Invariants	Invariant solutions
0	$\partial_t,\ \partial_x$ $x\partial_x + 2t\partial_t + 2u\partial_u$	3	$(x,u),(t,u),$ $(\frac{x^2}{t},\frac{u}{t})$	$\int \frac{\sqrt{3}\,du}{\sqrt{2u^3-3u^2+3C_1}} = \lvert t+C_2\rvert,$ $u=(1+F(x)\exp(-t))^{-1}$
c	∂_x,∂_t	2	$(t,u),(x,u)$	$cu''(t)+u'(t)+u(t)^2=u(t)$
$f(x)$	$\partial_t, F(x,t)\partial_u$	∞	(x,u)	$\int \frac{\sqrt{3}\,du}{\sqrt{2u^3-3u^2+3C_1}} = \lvert t+C_2\rvert$
$f(t)$	$\partial_x, x\partial_u, \partial_u,$ $tx+\frac{x^3}{6}\partial_u, t+\frac{x^2}{2}\partial_u$	5	$(t,u),(x,t)$	$f(t)u''(t)+u'(t)=u(t)-u(t)^2$

now, we will apply OMRA with proposed Father wavelets on the GFKPP equation. First, consider the Father wavelet $\phi_1(x,t)$. Then

$$\phi_{1;j,k}(x,t)=\sqrt{2}\phi(2x-j,2t-k)=\frac{4\sqrt{2}}{3\pi}\exp\left(-\frac{(2x-j)^2+(2t-k)^2}{0.75}\right).$$

thus

$$\phi(x,t)=\sqrt{2}\sum_j\sum_k a_{j,k}\phi(2x-j,2t-k).$$

from (10), we can calculate the approximation coefficients as

$$a_{j,k}=\sqrt{2}\iint \frac{16\sqrt{2}}{9\pi^2}\exp\left(-\frac{5x^2+5t^2-4xj-4tk+j^2+k^2}{0.75}\right)dxdt.$$

so, the approximation coefficients are as follows

$$a_{j,k}=\frac{32}{9\pi^2}\exp\left(-\frac{j^2+k^2}{0.75}\right)\iint \exp\left(-\frac{5x^2+5t^2-4(xj+tk)}{0.75}\right)dxdt.$$

where $j,k=0,1,2$ (Generally, for nth-order DE, we assume $j,k=0,1,2,\cdots,n$). So, we should calculate 9 approximation coefficients $a_{j,k}$, the matrix $A=[a_{j,k}]_{j,k}$ are obtained as

$$A=\begin{bmatrix}4.134 & 3.17 & 1.42\\ 3.17 & 2.42 & 1.09\\ 1.42 & 1.09 & 0.49\end{bmatrix}$$

from (9), the matrix of wavelet coefficients $W=[w_{j,k}]_{j,k}$ as follows

$$W=\begin{bmatrix}2.42 & -3.17 & 2.42\\ -3.17 & 4.134 & -3.17\\ 2.42 & -3.17 & 2.42\end{bmatrix}$$

now, we consider solution as (12), and put it in DE, the resulting PDE for $\psi_{j,k}(x,t)$ is

$$f.\{a_{0,0}\phi^{tt}-w_{0,0}\psi^{tt}\}+f.\{a_{0,0}\phi^{t}-w_{0,0}\psi^{t}\}=f.\{a_{0,0}\phi^{xx}-w_{0,0}\psi^{xx}\}+F(u).$$

First, we assume $F(u) = 0$ and $f(x) = f(t) = 1$, So

$$\psi^{tt} + \psi^{t} - \psi^{xx} = (-4.5 + 11.67x^2 + 10.93t - 68.2t^2)\exp(-5.33(x^2 + t^2)).$$

by solving this PDE, the Mother wavelet $\psi(x,t)$ will be obtained as follows

$$\psi(x,t) = \psi_h + \psi_p.$$

where, ψ_h, ψ_p are homogeneous and particular solutions of the GFKPP equation (for more details and information about analytical methods for solving differential equations, see [5]). Here, we can consider ψ_h as a invariant solution of GFKPP, for example, by the invariant solutions of translation, we have

$$\psi_h = \frac{1}{1 + F\exp(-t)}.$$

where F is a function of x (for more details and calculations see [11]). On the other hand, the particular solution as follows

$$\psi_p = -1.025\exp(-5.33(x^2 + t^2)).$$

therefore, the approximate solution based on the Father wavelet ϕ_1 and the corresponding mother wavelet ψ is

$$u_1 = \sum_{i,j} w_{i,j}\{\frac{1}{1 + F(x)\exp(-t)} - 1.025\exp(-5.33(x^2 + t^2))\}.$$

Second, consider the Father wavelet $\phi_2(x,t)$. Then

$$\begin{aligned}\phi_{1;j,k}(x,t) &= \sqrt{2}\phi(2x - j, 2t - k)\\ &= \sqrt{2}\exp(-\frac{(2x-j)^2 + 15(2t-k)^2}{5})\cos(2x - j)\cos((2t-k)/3).\end{aligned}$$

thus

$$\phi(x,t) = \sqrt{2}\sum_j\sum_k a_{j,k}\phi(2x - j, 2t - k).$$

where

$$\begin{aligned}a_{j,k} = \sqrt{2}\iint \{&\exp(-\frac{5x^2 + 285t^2 - 4xj - 60tk + j^2 + 15k^2}{5})\\ &\cos(x)\cos(t/3)\cos(2x - j)\cos(\frac{2t-k}{3})\}dxdt.\end{aligned}$$

In this case, since $(j, k = 0, 1, 2)$, we should calculate 9 approximation coefficients $a_{j,k}$. After calculation (from (10)), the matrix $A = [a_{j,k}]_{j,k}$ are obtained as follows

$$A = \begin{bmatrix} 0.25618 & 0.024 & 0.00002\\ 0.0195 & 0.0189 & 0.00015\\ -0.9346 & -0.00002 & 0.000006 \end{bmatrix}$$

from (9), the matrix $W = [w_{j,k}]_{j,k}$ of wavelet coefficients are obtained as

$$W = \begin{bmatrix} 0.0189 & -0.0195 & 0.00131 \\ -0.024 & 0.25618 & -0.024 \\ 0.0189 & 0.0195 & 0.0189 \end{bmatrix}$$

again, we consider solution as (12) and put it in DE, the resulting PDE for $\psi_{j,k}(x,t)$ is

$$\begin{aligned} &\psi^{tt} + \psi^{t} - \psi^{xx} \\ &= \Big\{ -105.5\sin(2x)\cos(\frac{2t}{3}) + 49.54x^2\sin(2x)\cos(\frac{2t}{3}) - 123.86x\cos(2x)\cos(\frac{2t}{3}) \\ &\quad + 3.09t\sin(2x)\cos(\frac{2t}{3}) + 12.9\sin(2x)\sin(\frac{2t}{3}) - 3.62t^2\sin(2x)\cos(\frac{2t}{3}) \\ &\quad - 3.017t\sin(2x)\sin(\frac{2t}{3})\Big\}\exp(-\frac{4x^2+60t^2}{5}). \end{aligned}$$

by considering the right hand side as $F(x,t)$, we get

$$\psi^{tt} + \psi^{t} - \psi^{xx} = F(x,t).$$

by solving this PDE, the Mother wavelet $\psi(x,t)$ will obtained as follows

$$\psi(x,t) = \psi_h + \psi_p.$$

where, ψ_h, ψ_p are homogeneous and particular solutions of the FKPP equation (see [5]). Here, we can consider ψ_h as a invariant solution of the FKPP equation, for instance, by considering the corresponding SCS at $f(x) = f(t) = c$, we have

$$\psi_h = F(x)\exp(-\frac{t}{c}) + G(x).$$

where, F, G are functions of x. In other hand, according to the form of $F(x,t)$, the particular solution is

$$\psi_p = -19.16\exp(-\frac{4x^2+60t^2}{5})\cos(2x)\cos(2t/3).$$

therefore, the approximate solution based on the Father wavelet ϕ_2 and the corresponding mother wavelet ψ as follows

$$u_2 = \sum_{i,j} w_{i,j}\{F(x)\exp(-\frac{t}{c}) + G(x) - 19.16\exp(-\frac{4x^2+60t^2}{5})\cos(2x)\cos(2t/3)\}.$$

Note that, for the case of $F(u) \neq 0$ and known $f(x), f(t)$, the same calculations can be done and the similar final solution will be obtained.

5. Conclusions & future works

In this paper, we applied the novel method based on the Father wavelets (is called the optimized MRA (OMRA) method) on the GFKPP equation. We proposed suitable Father wavelets based on the Lie symmetry method results such as symmetry groups, differential invariants, and invariant solutions. Indeed, we used the results obtained by the equivalence methods like the Lie symmetry and Fushchych methods for constructing the Father wavelets and applied OMRA based on them on the GFKPP equation. Finally, we proposed results as approximate solutions. This research shows the power and performance of OMRA for analyzing and solving different PDEs. In the future works, by implementing OMRA on the other PDEs, we will propose the proper Father wavelets for every symmetry group and hope that can generalize this method for solving PDEs at every order & every number of independent variables with every initial condition.

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