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On Hermite-Hadamard Type Inequalities for Co-ordinated Hyperbolic ρ-Convex Functions

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Abstract. In this study, we first introduce the co-ordinated hyperbolic ρ-convex functions. Then we establish some Hermite-Hadamard type inequalities for co-ordinated hyperbolic ρ-convex functions. The inequalities obtained in this study provide generalizations of some results given in earlier works.

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1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[8], [15], [23, p.137]). These

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inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The paper is organized as follows. After giving definition of co-ordinated convex functions and related Hermite-Hadamard inequality, we present hyperbolic ρ-convex functions and Hermite-Hadamard type inequalities for this kind of convexity. In Section 2, we introduce the concept of co-ordinated hyperbolic ρ-convex functions. We also prove a lemma which will be frequently used next sections. Some Hermite-Hadamard type inequalities for co-ordinated hyperbolic ρ-convex functions are obtained and some special cases of the results are also given in Section 3. Finally, some conclusions and further directions of research are discussed in Section 4.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.1 A function $f : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x, u), (y, v) \in \Delta$ and $t, s \in [0, 1]$, if it satisfies the following inequality:

$$\begin{aligned} &f(tx + (1-t)\, y, su + (1-s)\, v) \\ &\leqslant ts\, f(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v). \end{aligned} \tag{2}$$

The mapping f is a co-ordinated concave on Δ if the inequality (2) holds in reversed direction for all $t, s \in [0, 1]$ and $(x, u), (y, v) \in \Delta$.

In [10], Dragomir proved the following inequalities which are Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.1 Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leqslant \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\leqslant \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ &\qquad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \end{aligned} \tag{3}$$

$$\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

The above inequalities are sharp. The inequalities in (3) hold in reverse direction if the mapping f is a co-ordinated concave mapping.

Over the years, the numerous studies have focused on establishing generalization of the inequality (1) and (3). For some of them, please see ([1]-[7], [16]-[22], [24]-[30]).

1.1 *Hyperbolic rho-convex functions*

First, we give the definition of hyperbolic ρ-convex functions and some related inequalities. Then we define the co-ordinated hyperbolic ρ-convex functions.

Definition 1.2 [9] A function $f : I \to \mathbb{R}$ is said to be hyperbolic ρ-convex, if for any arbitrary closed subinterval $[a,b]$ of I such that we have

$$f(x) \leqslant \frac{\sinh[\rho(b-x)]}{\sinh[\rho(b-a)]} f(a) + \frac{\sinh[\rho(x-a)]}{\sinh[\rho(b-a)]} f(b) \tag{4}$$

for all $x \in [a,b]$. If we take $x = (1-t)a + tb$, $t \in [0,1]$ in (4), then the condition (4) becomes

$$f((1-t)a+tb) \leqslant \frac{\sinh[\rho(1-t)(b-a)]}{\sinh[\rho(b-a)]} f(a) + \frac{\sinh[\rho t(b-a)]}{\sinh[\rho(b-a)]} f(b). \tag{5}$$

If the inequality (4) holds with "$\geqslant$", then the function will be called hyperbolic ρ-concave on I.

The following Hermite-Hadamard inequality for hyperbolic ρ-convex function is proved by Dragomir in [9].

Theorem 1.2 Suppose that $f : I \to \mathbb{R}$ is hyperbolic ρ-convex on I. Then for any $a, b \in I$, we have

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sinh\left[\frac{\rho(b-a)}{2}\right] \leqslant \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{\rho} \tanh\left[\frac{\rho(b-a)}{2}\right]. \tag{6}$$

Moreover in [9], Dragomir proved the following another version of Hermite Hadamard type inequalities for hyperbolic ρ-convex functions.

Theorem 1.3 Assume that $f : I \to \mathbb{R}$ is hyperbolic ρ-convex on I. Then for any $a, b \in I$, we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x) \sec h\left[\rho\left(x - \frac{a+b}{2}\right)\right] dx \leqslant \frac{f(a)+f(b)}{2} \sec h\left[\frac{\rho(b-a)}{2}\right] \tag{7}$$

For the other inequalities for hyperbolic ρ-convex functions, please refer to ([11]-[14]).

2. Co-ordinated hyperbolic ρ-convex functions

In this section we introduce the concept of co-ordinated hyperbolic ρ-convex functions.

Definition 2.1 A function $f : \Delta =: [a,b] \times [c,d] \to \mathbb{R}$ is said to co-ordinated hyperbolic ρ-convex on Δ, if the inequality

$$\begin{aligned} f(x,y) \leqslant & \frac{\sinh[\rho_1(b-x)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2(d-y)]}{\sinh[\rho_2(d-c)]}f(a,c) \\ & +\frac{\sinh[\rho_1(b-x)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2(y-c)]}{\sinh[\rho_2(d-c)]}f(a,d) \\ & +\frac{\sinh[\rho_1(x-a)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2(d-y)]}{\sinh[\rho_2(d-c)]}f(b,c) \\ & +\frac{\sinh[\rho_1(x-a)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2(y-c)]}{\sinh[\rho_2(d-c)]}f(b,d). \end{aligned} \tag{8}$$

holds.

If the inequality (8) holds with "$\geqslant$", then the function will be called co-ordinated hyperbolic ρ-concave on Δ.

If we take $x = (1-t)a + tb$ and $y = (1-s)c + sd$ for $t, s, \in [0,1]$, then the inequality (8) can be written as

$$\begin{aligned} & f((1-t)a+tb,(1-s)c+sd) \\ \leqslant & \frac{\sinh[\rho_1(1-t)(b-a)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2(1-s)(d-y)]}{\sinh[\rho_2(d-c)]}f(a,c) \\ & +\frac{\sinh[\rho_1(1-t)(b-a)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2 s(d-y)]}{\sinh[\rho_2(d-c)]}f(a,d) \\ & +\frac{\sinh[\rho_1 t(b-a)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2(1-s)(d-y)]}{\sinh[\rho_2(d-c)]}f(b,c) \\ & +\frac{\sinh[\rho_1(b-a)]}{\sinh[\rho_1(b-a)]}\frac{\sinh[\rho_2 s(d-y)]}{\sinh[\rho_2(d-c)]}f(b,d). \end{aligned} \tag{9}$$

Now we give the following useful lemma:

Lemma 2.1 *If $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is hyperbolic co-ordinated ρ-convex function on Δ, then we have the following inequality*

$$\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leqslant \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right] \tag{10}$$

$$\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\frac{\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]}{\cosh\left[\frac{\rho_1(b-a)}{2}\right]}\frac{\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}{\cosh\left[\frac{\rho_2(d-c)}{2}\right]}$$

for all $(x,y) \in \Delta$.

Proof Since f is co-ordinated ρ-convex function on Δ, by the inequality (9) with $t = s = \frac{1}{2}$, we have

$$
\begin{aligned}
& f\left(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2}\right) \\
\leqslant & \frac{\sinh\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sinh\left[\rho_1\left(u_2-u_1\right)\right]} \frac{\sinh\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sinh\left[\rho_1\left(v_2-v_1\right)\right]} f(u_1, v_1) \\
& +\frac{\sinh\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sinh\left[\rho_1\left(u_2-u_1\right)\right]} \frac{\sinh\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sinh\left[\rho_1\left(v_2-v_1\right)\right]} f(u_1, v_2) \\
& +\frac{\sinh\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sinh\left[\rho_1\left(u_2-u_1\right)\right]} \frac{\sinh\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sinh\left[\rho_1\left(v_2-v_1\right)\right]} f(u_2, v_1) \\
& +\frac{\sinh\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sinh\left[\rho_1\left(u_2-u_1\right)\right]} \frac{\sinh\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sinh\left[\rho_1\left(v_2-v_1\right)\right]} f(u_2, v_2) \\
= & \frac{\sinh\left[\frac{\rho_1(u_2-u_1)}{2}\right]}{\sinh\left[\rho_1\left(u_2-u_1\right)\right]} \frac{\sinh\left[\frac{\rho_2(v_2-v_1)}{2}\right]}{\sinh\left[\rho_1\left(v_2-v_1\right)\right]}\left[f(u_1, v_1)+f(u_1, v_2)+f(u_2, v_1)+f(u_2, v_2)\right] \\
= & \frac{1}{\cosh\left[\frac{\rho_1(u_2-u_1)}{2}\right] \cosh\left[\frac{\rho_2(v_2-v_1)}{2}\right]} \frac{f(u_1, v_1)+f(u_1, v_2)+f(u_2, v_1)+f(u_2, v_2)}{4}.
\end{aligned}
$$

If we take $u_1 = x$, $u_2 = a + b - x$, $v_1 = y$ and $v_2 = c + d - y$, we have

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (11) \\
\leqslant & \frac{f(x, y)+f(x, c+d-y)+f(a+b-x, y)+f(a+b-x, c+d-y)}{4 \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}.
\end{aligned}
$$

Since $\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right], \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] \geqslant 0$ for $(x, y) \in \Delta$ with $0 < b-a < \frac{\pi}{\rho_1}$ and $0 < d-c < \frac{\pi}{\rho_2}$, if we multiply (11) by

$$
4 \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right],
$$

then we obtain the first inequality in (10).

On the other hand, as f is co-ordinated ρ-convex function on Δ, by the inequality

(8), we get the following inequalities

$$\begin{aligned} f(x,y) \leqslant & \frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c) \\ & +\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d) \\ & +\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c) \\ & +\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d), \end{aligned} \tag{12}$$

$$\begin{aligned} f(x,c+d-y)) \leqslant & \frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c) \\ & +\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d) \\ & +\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c) \\ & +\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d), \end{aligned} \tag{13}$$

$$\begin{aligned} f(a+b-x,y) \leqslant & \frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c) \\ & +\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d) \\ & +\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c) \\ & +\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d), \end{aligned} \tag{14}$$

and

$$\begin{aligned} f(a+b-x,c+d-y) \leqslant & \frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c) \\ & +\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d) \\ & +\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c) \end{aligned} \tag{15}$$

$$+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d)$$

for $(x, y) \in \Delta$. By summing the inequalities (12)-(15), we obtain

$$f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)$$

$$\leqslant \frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d)$$

$$+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d)$$

$$+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d)$$

$$+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d)$$

$$+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d)$$

$$+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d)$$

$$+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,c)+\frac{\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(a,d)$$

$$+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,c)+\frac{\sinh\left[\rho_1\left(b-x\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}f(b,d)$$

$$=\frac{\sinh\left[\rho_1\left(b-x\right)\right]+\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}\frac{\sinh\left[\rho_2\left(d-y\right)\right]+\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}$$
$$\times\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right].$$

It can be easily seen that

$$\frac{\sinh\left[\rho_1\left(b-x\right)\right]+\sinh\left[\rho_1\left(x-a\right)\right]}{\sinh\left[\rho_1\left(b-a\right)\right]}=\frac{\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]}{\cosh\left[\frac{\rho_1(b-a)}{2}\right]}$$

and

$$\frac{\sinh\left[\rho_2\left(d-y\right)\right]+\sinh\left[\rho_2\left(y-c\right)\right]}{\sinh\left[\rho_2\left(d-c\right)\right]}=\frac{\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}{\cosh\left[\frac{\rho_2(d-c)}{2}\right]}.$$

This completes the proof of second inequality in (10). ■

3. Hermite-Hadamard type inequalities

In this section, we establish some Hermite-Hadamard type inequalities for co-ordinated hyperbolic ρ-convex functions.

Theorem 3.1 If $f:\Delta\to\mathbb{R}$ is a co-ordinated hyperbolic ρ-convex function on Δ, then we have the following Hermite-Hadamard type inequalities

$$\frac{4}{\rho_1\rho_2}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\sinh\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leqslant\int_a^b\int_c^d f(x,y)dydx \tag{16}$$
$$\leqslant\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\tanh\left[\frac{\rho_2(d-c)}{2}\right].$$

Proof Integrating the inequality (10) with respect to (x,y) on Δ, we obtain

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{17}$$
$$\times\left(\int_a^b\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right)$$
$$\leqslant\frac{1}{4}\int_a^b\int_c^d[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)]\,dydx$$
$$\leqslant\left(\int_a^b\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right)$$
$$\times\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cosh\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\frac{\rho_2(d-c)}{2}\right]}.$$

Here we have

$$\int_a^b\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx=\frac{2}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right] \tag{18}$$

and similarly

$$\int_c^d\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy=\frac{2}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]. \tag{19}$$

Moreover by change of variable, we get

$$\int_a^b\int_c^d f(x,c+d-y)dydx = \int_a^b\int_c^d f(a+b-x,y)\,dydx$$
$$= \int_a^b\int_c^d f(a+b-x,c+d-y)dydx \tag{20}$$
$$= \int_a^b\int_c^d f(x,y)dydx.$$

If we substitute the equalities (18)-(20) in (17), then we obtain the required result (16). ■

Remark 3.1 For $\rho_1 \to 0$ and $\rho_2 \to 0$ we observe that

$$\lim_{\rho_1\to 0}\frac{2}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right] = b-a, \quad \lim_{\rho_2\to 0}\frac{2}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right] = d-c \tag{21}$$

and

$$\lim_{\rho_1\to 0}\frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right] = \frac{b-a}{2}, \quad \lim_{\rho_2\to 0}\frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right] = \frac{d-c}{2}. \tag{22}$$

Thus, from Theorem 3.1 in the limit $\rho_1 \to 0$ and $\rho_2 \to 0$, we have the inequalities

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{23}$$
$$\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$

which proved by Dragomir in [10].

Theorem 3.2 If $f:\Delta\to\mathbb{R}$ is a co-ordinated hyperbolic ρ-convex function on Δ , then we have the following Hermite-Hadamard type inequalities

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{24}$$
$$\leqslant \frac{1}{(b-a)(d-c)}$$
$$\times\int_a^b\int_c^d f(x,y)\sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx$$
$$\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right].$$

Proof If we divide the inequality (10) by $\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \leqslant & \frac{[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)]}{4} \\ & \times \sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] \\ \leqslant & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]. \end{aligned} \tag{25}$$

Integrating the inequalities (25) on Δ, we have

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \leqslant & \frac{1}{4(b-a)(d-c)} \\ & \times\int_a^b\int_c^d [f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)] \\ & \times \sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \\ \leqslant & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]. \end{aligned} \tag{26}$$

By the change of variable $v=c+d-y$, we have

$$\begin{aligned} & \int_a^b\int_c^d f(x,c+d-y)\sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \\ = & \int_a^b\int_c^d f(x,v)\sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(\frac{c+d}{2}-v\right)\right]dvdx, \\ = & \int_a^b\int_c^d f(x,v)\sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(v-\frac{c+d}{2}\right)\right]dvdx. \end{aligned} \tag{27}$$

Similarly using the change of variable we obtain the following equalities.

$$\int_a^b\int_c^d f(a+b-x,y)\sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \tag{28}$$
$$=\int_a^b\int_c^d f(u,y)\sec h\left[\rho_1\left(u-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydu,$$

and

$$\int_a^b\int_c^d f(a+b-x,y)\sec h\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \tag{29}$$
$$\int_a^b\int_c^d f(u,v)\sec h\left[\rho_1\left(u-\frac{a+b}{2}\right)\right]\sec h\left[\rho_2\left(v-\frac{c+d}{2}\right)\right]dvdu$$

By substituting the equalities (27)-(29) in (26), we establish the desired result (24). ■

Remark 3.2 If we choose $\rho_1 \to 0$ and $\rho_2 \to 0$ in Theorem 3.2, then the inequalities (24) reduce to the inequalities (23).

Theorem 3.3 If $f : \Delta \to \mathbb{R}$ is a co-ordinated hyperbolic ρ-convex function on Δ, then we have the following Hermite-Hadamard type inequalities,

$$\frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\left[(d-c)+\frac{1}{2\rho_2}\sinh\left[\rho_2\,(d-c)\right]\right]\left[(b-a)+\frac{1}{2\rho_1}\sinh\left[\rho_1\,(b-a)\right]\right]$$
$$\leqslant\int_a^b\int_c^d f(x,y)\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \tag{30}$$
$$\leqslant\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{16}\left[(d-c)+\frac{1}{2\rho_2}\sinh\left[\rho_2\,(d-c)\right]\right]$$
$$\times\left[(b-a)+\frac{1}{2\rho_1}\sinh\left[\rho_1\,(b-a)\right]\right]\sec h\left[\frac{\rho_1\,(b-a)}{2}\right]\sec h\left[\frac{\rho_2\,(d-c)}{2}\right].$$

Proof If we multiply the inequality (10) by

$\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$, we obtain

$$\cosh^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leqslant \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right. \tag{31}$$
$$\left.\times\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]\right]$$

$$\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\frac{\cosh^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]}{\cosh\left[\frac{\rho_1(b-a)}{2}\right]}\frac{\cosh^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}{\cosh\left[\frac{\rho_2(d-c)}{2}\right]}.$$

Integrating the inequalities (31) on Δ, we have

$$\int_a^b\int_c^d\cosh^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)dydx$$

$$\leqslant\int_a^b\int_c^d\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]$$
$$\times\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \tag{32}$$

$$\leqslant\int_a^b\int_c^d\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cosh\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\frac{\rho_2(d-c)}{2}\right]}\cosh^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]$$
$$\times\cosh^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx.$$

That is

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\left(\int_a^b\cosh^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d\cosh^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right)$$

$$\leqslant\int_a^b\int_c^d\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]$$
$$\times\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \tag{33}$$

$$\leqslant\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cosh\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\frac{\rho_2(d-c)}{2}\right]}$$

$$\times\left(\int_a^b \cosh^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right)\left(\int_c^d \cosh^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right).$$

A simple calculation we have

$$\begin{aligned}
&\int_a^b \cosh^2\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\\
&=\frac{1}{2}\left[(b-a)+\int_a^b \cosh\left[2\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right]\\
&=\frac{1}{2}\left[(b-a)+\frac{1}{2\rho_1}\sinh\left[2\rho_1\left(x-\frac{a+b}{2}\right)\right]\Bigg|_a^b\right] \qquad (34)\\
&=\frac{1}{2}\left[(b-a)+\frac{1}{2\rho_1}\sinh\left[2\rho_1\left(b-\frac{a+b}{2}\right)\right]-\frac{1}{2\rho_1}\sinh\left[2\rho_1\left(a-\frac{a+b}{2}\right)\right]\right]\\
&=\frac{1}{2}\left[(b-a)+\frac{1}{\rho_1}\sinh\left[\rho_1(b-a)\right]\right]
\end{aligned}$$

and similarly

$$\int_c^d \cosh^2\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy=\frac{1}{2}\left[(d-c)+\frac{1}{\rho_2}\sinh\left[\rho_2(d-c)\right]\right]. \qquad (35)$$

By change of variable $v=c+d-y$, then we have

$$\begin{aligned}
&\int_a^b\int_c^d f(x,c+d-y)\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \qquad (36)\\
&=\int_a^b\int_c^d f(x,v)\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(v-\frac{c+d}{2}\right)\right]dvdx.
\end{aligned}$$

Similarly, using change of variable we have

$$\begin{aligned}
&\int_a^b\int_c^d f(a+b-x,y)\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \qquad (37)\\
&=\int_a^b\int_c^d f(u,y)\cosh\left[\rho_1\left(u-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydu,
\end{aligned}$$

and

$$\int_a^b\int_c^d f(a+b-x,c+d-y)\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \tag{38}$$
$$=\int_a^b\int_c^d f(u,v)\cosh\left[\rho_1\left(u-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(v-\frac{c+d}{2}\right)\right]dvdu.$$

By substituting the equalities (34)-(38) in (33), we establish the desired result (30). ■

Remark 3.3 For $\rho_1 \to 0$ and $\rho_2 \to 0$ we observe that

$$\lim_{\rho_1\to 0}\frac{1}{\rho_1}\sinh\left[\rho_1\left(b-a\right)\right]=b-a,\quad \lim_{\rho_2\to 0}\frac{1}{\rho_2}\sinh\left[\rho_2\left(d-c\right)\right]=d-c.$$

Thus, from Theorem 3.3 in the limit $\rho_1 \to 0$ and $\rho_2 \to 0$, we have the inequalities

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leqslant\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{39}$$
$$\leqslant\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$

which are proved by Dragomir in [10].

Now we give the following important inequality for co-ordinated hyperbolic ρ-convex functions.

Theorem 3.4 Suppose that $f:\Delta\to\mathbb{R}$ is a co-ordinated hyperbolic ρ-convex function on Δ. Then for for all $(x,y)\in\Delta$, we have the following inequalities

$$\begin{aligned}
&\frac{4}{\rho_1\rho_2}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\sinh\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leqslant\frac{1}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{1}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
&\leqslant\int_a^b\int_c^d f(x,y)\,dydx\\
&\leqslant\frac{1}{2}\left[\frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b[f(x,c)+f(x,d)]\,dx\right.\\
&\quad\left.+\frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d[f(a,y)+f(b,y)]\,dy\right]\\
&\leqslant\tanh\left[\frac{\rho_1(b-a)}{2}\right]\tanh\left[\frac{\rho_2(d-c)}{2}\right]\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}.
\end{aligned}\tag{40}$$

Proof Since $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is hyperbolic ρ-convex on the co-ordinates on Δ, it follows that the mapping $g_x:[c,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$ is hyperbolic ρ-convex function on $[c,d]$ for all $x\in[a,b]$. By Hermite-Hadamard inequality (6) for hyperbolic ρ-convex functions (see: [9]), we get:

$$\frac{2}{\rho_2}g_x\left(\frac{c+d}{2}\right)\sinh\left[\frac{\rho_2(d-c)}{2}\right]\leqslant\int_c^d g_x(y)dy\leqslant\frac{g_x(c)+g_x(d)}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right].$$

That is,

$$\begin{aligned}
&\frac{2}{\rho_2}f\left(x,\frac{c+d}{2}\right)\sinh\left[\frac{\rho_2(d-c)}{2}\right]\\
&\leqslant\int_c^d f(x,y)\,dy\\
&\leqslant\frac{f(x,c)+f(x,d)}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right].
\end{aligned}\tag{41}$$

Integrating the inequalities (41) with respect to x on $[a,b]$, we have

$$\frac{2}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx \tag{42}$$
$$\leqslant \int_a^b\int_c^d f(x,y)\,dydx$$
$$\leqslant \frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx.$$

By a similar argument applied for mapping $g_y:[a,b]\to\mathbb{R}$, $g_y:=f(x,y)$, we get

$$\frac{2}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy \tag{43}$$
$$\leqslant \int_a^b\int_c^d f(x,y)\,dydx$$
$$\leqslant \frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+f(b,y)]\,dy.$$

Summing the inequalities (42) and (43), we obtain

$$\frac{2}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy \tag{44}$$
$$+\frac{2}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx$$
$$\leqslant 2\int_a^b\int_c^d f(x,y)\,dydx$$
$$\leqslant \frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx$$
$$+\frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+f(b,y)]\,dy.$$

If we divide (42) by 2, then we prove the second and third inequalities in (40).

By first inequality in (6), we have

$$\frac{2}{\rho_1\rho_2}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\sinh\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{45}$$
$$\leqslant \frac{1}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$

and

$$\frac{2}{\rho_1\rho_2}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\sinh\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{46}$$
$$\leqslant \frac{1}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy.$$

If we add the inequalities (45) and (46), then we have

$$\frac{4}{\rho_1\rho_2}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\sinh\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{47}$$
$$\leqslant \frac{1}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy$$
$$+\frac{1}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$

which gives the first inequality in (40).

Finally, by the second inequality in (6), we can write

$$\int_a^b f(x,c)\,dx \leqslant \frac{f(a,c)+f(b,c)}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right], \tag{48}$$

$$\int_a^b f(x,d)\,dx \leqslant \frac{f(a,d)+f(b,d)}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right], \tag{49}$$

$$\int_c^d f(a,y)\,dy \leqslant \frac{f(a,c)+f(a,d)}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right], \tag{50}$$

and

$$\int_c^d f(b,y)\,dy \leqslant \frac{f(b,c)+f(b,d)}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]. \tag{51}$$

If we substitute the inequalities (48)-(51) in (44), then we have

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\,dx\right.\\
&\left.+\frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+f(b,y)]\,dy\right]\\
&\leqslant \frac{1}{2}\left[\frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]\tanh\left[\frac{\rho_1(b-a)}{2}\right]\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}\right.\\
&\left.+\frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\tanh\left[\frac{\rho_2(d-c)}{2}\right]\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}\right]\\
&=\tanh\left[\frac{\rho_1(b-a)}{2}\right]\tanh\left[\frac{\rho_2(d-c)}{2}\right]\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}.
\end{aligned}$$

This gives last inequality in (40). The proof is completely completed. ■

Remark 3.4 For $\rho_1 \to 0$ and $\rho_2 \to 0$ we observe that

$$\lim_{\rho_1\to 0}\frac{2}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right]=b-a,\quad \lim_{\rho_2\to 0}\frac{2}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]=d-c$$

and

$$\lim_{\rho_1\to 0}\frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right]=\frac{b-a}{2},\quad \lim_{\rho_2\to 0}\frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]=\frac{d-c}{2}.$$

Thus, from Theorem 3.1 in the limit $\rho_1 \to 0$ and $\rho_2 \to 0$, we have the inequalities (39).

Theorem 3.5 Suppose that $f : \Delta = [a.b] \times [c.d] \to \mathbb{R}$ is a co-ordinated hyperbolic

ρ-convex function on Δ. Then for all $(x,y)\in\Delta$, we have the following inequalities,

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& \leqslant \frac{1}{2}\left[\frac{1}{d-c}\int_a^b f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right. \\
& \left.+\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right] \qquad (52)\\
& \leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \\
& \leqslant \frac{1}{4}\sec\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d\left[f(a,y)+(b,y)\right]\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy \\
& +\frac{1}{4}\sec\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b\left[f(x,c)+(x,d)\right]\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx \\
& \leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec\left[\frac{\rho_1(b-a)}{2}\right]\sec\left[\frac{\rho_2(d-c)}{2}\right].
\end{aligned}
$$

Proof Since $f:\Delta\to\mathbb{R}$ is co-ordinated hyperbolic ρ-convex, by the inequality (7), we have

$$
\begin{aligned}
f\left(\frac{a+b}{2},y\right) &\leqslant \frac{1}{b-a}\int_a^b f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx \qquad (53)\\
&\leqslant \frac{f(a,y)+f(b,y)}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(x,\frac{c+d}{2}\right) &\leqslant \frac{1}{d-c}\int_c^d f(x,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy \qquad (54)\\
&\leqslant \frac{f(x,c)+f(x,d)}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right]
\end{aligned}
$$

multiplying the inequalities (53) and (54) by $\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$ and $\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]$ respectively, and integrating the resultant inequalities on $[c,d]$ and

$[a,b]$ respectively, we obtain

$$\int_c^d f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy$$

$$\leqslant \frac{1}{b-a}\int_a^b\int_c^d f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx \quad (55)$$

$$\leqslant \frac{1}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+f(b,y)]\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy$$

and

$$\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx$$

$$\leqslant \frac{1}{d-c}\int_a^b\int_c^d f(x,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dydx \quad (56)$$

$$\leqslant \frac{1}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+f(x,d)]\sec\rho_1\left(x-\frac{a+b}{2}\right)dx.$$

By dividing the inequalities (55) and (56) by $(b-a)$ and $(d-c)$ respectively, then by adding the resultant inequalities, we obtain

$$\frac{1}{2}\left[\frac{1}{d-c}\int_a^b f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy\right.$$
$$\left.+\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx\right] \quad (57)$$

$$\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dydx$$

$$\leqslant \frac{1}{4}\sec\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d [f(a,y)+(b,y)]\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy$$

$$+\frac{1}{4}\sec\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b [f(x,c)+(x,d)]\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx$$

the second and third inequalities in (52).

From the first inequality in (7), we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx \tag{58}$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leqslant\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy \tag{59}$$

by summing the inequalities (58) and (59), we establish first inequality in (52).

By using the second inequality in (7), we get

$$\frac{1}{d-c}\int_c^d f(a,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy$$
$$\leqslant\frac{f(a,c)+f(a,d)}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right] \tag{60}$$

$$\frac{1}{d-c}\int_c^d f(b,y)\sec\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]dy$$
$$\leqslant\frac{f(b,c)+f(b,d)}{2}\sec\left[\frac{\rho_2(d-c)}{2}\right] \tag{61}$$

$$\frac{1}{b-a}\int_a^b f(x,c)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx$$
$$\leqslant\frac{f(a,c)+f(a,d)}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right] \tag{62}$$

and

$$\frac{1}{b-a}\int_a^b f(x,d)\sec\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]dx$$
$$\leqslant\frac{f(a,d)+f(b,d)}{2}\sec\left[\frac{\rho_1(b-a)}{2}\right] \tag{63}$$

If we substitute the inequalities (60)-(63) in (57) then we obtain the last inequality in (52).The proof is completed. ■

Remark 3.5 By choosing $\rho_1 = 0$ and $\rho_2 = 0$ Theorem 3.5, then the inequalities (52) reduce to the inequalities (3) obtained by Dragomir in [10].

4. Conclusions

This paper introduced the co-ordinated hyperbolic ρ-convex functions. Using this concept, we presented some Hermite-Hadamard type inequalities which generalize some inequalities given earlier works. The next step in the research direction proposed here is to establish the weighted versions of the inequalities obtained in this paper.

References

[1] A. Akkurt, M. Z. Sarikaya, H. Budak and H. Yildirim, On the Hadamard's type inequalities for co-ordinated convex functions via fractional integrals, Journal of King Saud University - Science, **29** (2017) 380-387.

[2] T. Ali, M. A. Khan, A. Kilicman and Q. Din, On the refined Hermite-Hadamard inequalities, Mathematical Sciences & Applications E-Notes, **6 (1)** (2018) 85-92.

[3] A. G. Azpeitia, Convex functions and the Hadamard inequality, Revista Colombiana de Matemticas, **28** (1994) 7-12.

[4] M. K. Bakula, An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates, Australian Journal of Mathematical Analysis and Applications, **11 (1)** (2014) 1-7.

[5] F. Chen, A note on the Hermite-Hadamard inequality for convex functions on the co-ordinates, Journal of Mathematical Inequalities, **8 (4)** (2014) 915-923.

[6] F. Chen, A note on Hermite-Hadamard inequalities for products of convex functions, Journal of Applied Mathematics, **2013** (2013), Article ID 935020.

[7] F. Chen and S. Wu, Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions, Journal of Nonlinear Sciences and Applications, **9 (2)** (2016), 705-716.

[8] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, (2000).

[9] S. S. Dragomir, Some inequalities of Hermite-Hadamard type for hyperbolic ρ-convex functions, Preprint RGMIA Res. Rep. Coll. 21 (2018), Art 13.

[10] S. S. Dragomir, On Hadamards inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Mathematics, **5** (2001) 775-788.

[11] S. S. Dragomir, Some inequalities of Fejer type for hyperbolic ρ-convex functions, Preprint RGMIA Res. Rep. Coll. 21 (2018), Art 14.

[12] S. S. Dragomir, Some inequalities of Ostrowski and trapezoid type for hyperbolic ρ-convex functions, Preprint RGMIA Res. Rep. Coll. 21 (2018), Art 15.

[13] S. S. Dragomir, Some inequalities of Jensen type for hyperbolic ρ-convex functions, Preprint RGMIA Res. Rep. Coll. 21 (2018), Art 17.

[14] S. S. Dragomir and B. T. Torebek, Some Hermite-Hadamard type inequalities in the class of hyperbolic ρ-convex functions, arXiv preprint arXiv:1901.06634, 2019.

[15] J. Hadamard, Etude sur les proprietes des fonctions entieres en particulier d'une fonction consideree par Riemann, Journal de Mathmatiques Pures et Appliques, **58** (1893) 171-215.

[16] U. S. Kırmacı, M. K. Bakula, M. E. Özdemir and J. Pečarić, Hadamard-tpye inequalities for s-convex functions, Applied Mathematics and Computation, **193 (1)** (2007) 26-35.

[17] M. A. Latif and M. Alomari, Hadamard-type inequalities for product two convex functions on the co-ordinetes, International Mathematical Forum, **4 (47)** (2009) 2327-2338.

[18] M. A. Latif, S. S. Dragomir and E. Momoniat, Generalization of some Inequalities for differentiable co-ordinated convex functions with applications, Moroccan Journal of Pure and Applied Analysis, **2 (1)** (2016) 12-32.

[19] LM. A. Latif and S. S. Dragomir, On some new inequalities for differentiable co-ordinated convex functions, Journal of Inequalities and Applications, **2012** (2012), doi:10.1186/1029-242X-2012-28.

[20] M. E. Ozdemir, C. Yildiz and A. O. Akdemir, On the co-ordinated convex functions, Applied Mathematics & Information Sciences, **8 (3)** (2014) 1085-1091.

[21] B. G. Pachpatte, On some inequalities for convex functions, RGMIA Res. Rep. Coll. 6 (E) (2003).

[22] Z. Pavic, Improvements of the Hermite-Hadamard inequality, Journal of Inequalities and Applications (2015) 2015:222.

[23] J. E. Pečarić, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, (1992).

[24] M. Z. Sarikaya, E. Set, M. E. Ozdemir and S. S. Dragomir, New some Hadamard's type inequalities for coordinated convex functions, Tamsui Oxford Journal of Information and Mathematical Sciences, **28 (2)** (2012) 137-152.

[25] E. Set, M. E. Özdemir and S. S. Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, J. Inequal. Appl. (2010) 9. Article ID 148102.

[26] K. L. Tseng and S. R. Hwang, New Hermite-Hadamard inequalities and their applications, Filomat, 30(14), 2016, 3667-3680.. Simon Stevin, **20** (2013) 655-666.

[27] D. Y. Wang, K. L. Tseng and G. S. Yang, Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane. Taiwan. J. Math., **11** (2007) 63-73.
[28] R. Xiang and F. Chen, On some integral inequalities related to Hermite-Hadamard-Fejér inequalities for coordinated convex functions, Chinese Journal of Mathematics, **2014** (2014), Article ID 796132.
[29] G. S. Yang and K. L. Tseng, On certain integral inequalities related to Hermite-Hadamard inequalities, J. Math. Anal. Appl., **239** (1999) 180-187.
[30] G. S. Yang and M. C. Hong, A note on Hadamard's inequality, Tamkang J. Math., **28** (1997) 33-37.