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A New Iterative Method of Successive Approximation to Solve Nonlinear Urysohn Integral Equations by Haar Wavelet

M. Kazemi[a,*], V. Torkashvand[b] and E. Fathizadeh[c]

[a]*Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran,*
[b]*Department of Mathematics, Hamedan Branch, Islamic Azad University, Hamedan, Iran,*
[c]*Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.*

Abstract. In this paper, a new method for calculating the numerical approximation of the nonlinear Urysohn integral equations is proposed based on Haar wavelets. Also, the convergence analysis and numerical stability of these method are discussed. Conducting numerical experiments confirm the theoretical results of the applied method and endorse the accuracy of the method.

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1. Introduction

In this research, the Haar wavelets constructed on $[a, b]$ are applied for solving the numerical solution of the nonlinear Urysohn Fredholm integral equations of the second kind (NUFIEs) of the form

$$u(s) = f(s) + \lambda \int_a^b K(s, x, u(x))dx, \qquad s \in [a, b], \tag{1}$$

*Corresponding author. Email: univer_ka@yahoo.com; m.kazemi@aiau.ac.ir

http://ijm2c.iauctb.ac.ir

where the functions $f(s)$ and $K(s, x, u(x))$ are known and $u(s)$ is a solution to be determined. The mathematical modeling of physical phenomena, many problems in applied mathematics, engineering, mechanics, mathematical physics and many other fields can be turned into integral equations of the second type [12, 13, 17, 18, 26, 27, 29]. The two most commonly used methods for the numerical solutions of these equations are the Glerkin and collocation methods [2, 14].

Recently, some the numerical methods including, operational matrices [32], block-pulse functions (BPFs)[10, 23], triangular functions (TFs)[15], degenerate kernel method [1], Chebyshev polynomials [35], Least squares approximation method[33], Taylor-series expansion method [22], wavelet method [3, 25] and Bernoulli polynomials [4] have been proposed to obtain approximate solutions of these equations. In [17, 34], the classical theorems on the existence and uniqueness of the solution of nonlinear integral equations can be observed. Existence results for functional integral equations are obtained using the measure of noncompactness and Darbo conditions in [19] and [24] respectively. The method of successive approximations and its iterative methods are used in [6, 20].

In this paper, the iterative method of successive approximations based on the Haar wavelet to obtain the numerical solution of (1) is described. The structure of this article is divided into five sections. The second section, the basic definitions and preliminaries of Haar wavelet are presented. In Section 3, the existence and uniqueness of the solution of (1) is obtained using the fixed point technique. Also, the convergence of the method of successive approximations used to approximate the solution of (1), is described in this section. In order to confirm the theoretical results and show the accuracy of the method, some numerical examples in Section 4 are considered. Section 5 includes the conclusion of the proposed method.

2. Preliminaries

One of the most simple and popular types of wavelet, is the Haar wavelet. Haar function was introduced by Alfred Haar in 1910, [11] and later developed by others. There are different definitions of Haar function and various generalizations have been used [8, 21].

Definition 2.1 ([7]) The Haar scaling function also, called the father wavelet, is defined on the interval $[a, b)$ as

$$\phi(x) = \begin{cases} 1 & , a \leqslant x < b, \\ 0 & , otherwise, \end{cases}$$

Definition 2.2 ([7]) The mother wavelet for Haar wavelets family is also defined on the interval $[a, b)$ as follows

$$\psi(x) = \begin{cases} 1 & , a \leqslant x < \frac{a+b}{2}, \\ -1 & , \frac{a+b}{2} \leqslant x < b, \\ 0 & , otherwise, \end{cases}$$

All the other functions in the Haar wavelets family are defined on subintervals of $[a, b)$ and are generated from $\psi(x)$ by the operations of dilation and translation. Each function in the Haar wavelets family defined for $x \in [a, b)$ except the scaling function can be expressed as

$$h_i(x) = \Psi(2^j - k) = \begin{cases} 1 & ,\alpha \leqslant x < \beta, \\ -1 & ,\beta \leqslant x < \gamma, \\ 0 & ,otherwise, \end{cases}$$

where

$$\alpha = a + (b-a)\frac{k}{n}, \quad \beta = a + (b-a)\frac{k+0.5}{n}, \quad \gamma = a + (b-a)\frac{k+1}{n}, \quad i = 2, 3, ..., 2N$$

In the above difinition the integer $n = 2^j, \quad j = 0, 1, ..., J$ shows the level of the wavelet and $k = 0, 1, ..., n-1$ is the translation parameter. The maximal level of resolution is the integer J.

The wavelet numbers i is calculated according the formula $i = n + k + 1$. In the case of minimal values $n = 1, k = 0$, we have $i = 2$. The maximum of i is $i = 2N = 2^{J+1}$. For $i = 1, 2$, the function $h_1(x)$ is called scaling function whereas $h_2(x)$ is the mother wavelet for the Haar wavelet family. In [3, 16], some authors uniform Haar wavelets for integration of real integrals. Here, we decide to use the method presented to obtain the numerical solution of the equation (NUFIEs). In this paper, suppose for Haar wavelets approximations collocation points $x_i = a + (b-a)\frac{2i-1}{4N}$, $i = 1, 2, ..., 2N$, are considered.

Proposition 2.3 *Let $f : [a, b] \to \mathbb{R}$ be continuous integrable function. Consider the integral*

$$I = \int_a^b f(x)dx$$

over the $[a, b]$. Using the quadrature formula with respect to Haar wavelets the above integral can be approximated as follows:

$$I \simeq \frac{(b-a)}{2N} \sum_{i=1}^{2N} f(a + (b-a)\frac{2i-1}{4N}), \qquad i = 1, 2, ...2N.$$

Definition 2.4 For $L \geqslant 0$, the function $f : \mathbb{R} \to \mathbb{R}$ is L-Lipschitz if

$$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|, \qquad \forall x_1, x_2 \in [a, b]$$

Theorem 2.5 *Let $f : [a, b] \to \mathbb{R}$ be integrable function on $[a, b]$ of L-Lipschitz type. Then the following quadrature formula with respect to Haar wavelets*

$$S_N(f) = \frac{(b-a)}{2N} \sum_{i=1}^{2N} f\big(a + (b-a)\frac{2i-1}{4N}\big) \tag{2}$$

where $N = 2^J$ is the maximal level of resolution of Haar wavelets, approximates the integral $\int_a^b f(x)dx$. Also we have

$$|\int_a^b f(x)dx - S_N(f)| \leqslant L\frac{(b-a)^2}{4N}. \tag{3}$$

Proof

$$\begin{aligned}
\Big|\int_a^b f(x)dx - S_N(f)\Big| &= \Big|\int_a^b f(x)dx - \int_a^b \frac{1}{2N}\sum_{i=1}^{2N} f\big(a+(b-a)\frac{2i-1}{4N}\big)dx\Big| \\
&\leqslant \frac{1}{2N}\int_a^b \sum_{i=1}^{2N} \big|f(x) - f\big(a+(b-a)\frac{2i-1}{4N}\big)\big|dx \\
&\leqslant \frac{1}{2N}\int_a^b \sum_{i=1}^{2N} \big(L|x-(a+(b-a)\frac{2i-1}{4N})|\big)dx
\end{aligned}$$

According to the $x \in [a+(b-a)\frac{2i-1}{4N}, a+\frac{b-a}{4N}(2i))$ we get

$$\Big|\int_a^b f(x)dx - S_N(f)\Big| \leqslant L\frac{(b-a)^2}{4N}.$$

Thus, the proof is complete. ■

3. Main results

3.1 *The sequence of successive approximations*

Here, we consider the nonlinear equation (1), where $\lambda > 0$. We assume that $f : [a,b] \to \mathbb{R}$ and $K : [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ are continuous functions. Let $\mathbf{X} = \{f : [a,b] \to \mathbb{R}; f \;\; is \;\; continuous\}$ be the space of continuous functions with the metric

$$d(f,g) = \big\|f-g\big\| = \sup\{\big|f(s)-g(s)\big|; s \in [a,b],\},$$

Now, we shall prove the existence and uniqueness of the solution of (1) by the method of successive approximations. We define the operators $A : \mathbf{X} \to \mathbf{X}$ by

$$A(u)(s) = f(s) + \lambda\int_a^b K(s,x,u(x))dx, \qquad \forall s \in [a,b],\ \forall u \in \mathbf{X}.$$

Theorem 3.1 *Let $K(s,x,u(x))$ be continuous for $s,x \in [a,b]$, and $f \in \boldsymbol{X}$. Moreover suppose that there exist $\mu > 0, \alpha > 0$, such that*

$$|K(s,x,u) - K(s',x',u')| \leqslant \mu(|s-s'|+|x-x'|) + \alpha|u-u'|, \quad \forall u,u' \in \boldsymbol{X}. \tag{4}$$

If $\alpha\lambda(b-a) < 1$, then equation (1) has a unique solution $u^* \in \boldsymbol{X}$, which can be obtained by the following successive approximations method

$$\begin{aligned}
u_0(s) &= f(s), \\
u_m(s) &= f(s) + \lambda\int_a^b K(s,x,u_{m-1}(x))dx, \qquad m \geqslant 1
\end{aligned} \tag{5}$$

Also, the sequence of successive approximations, $(u_m)_{m\geqslant 1}$ *converges to the solution* u^*. *Furthermore, the following error estimates hold*

$$d(u^*, u_m) \leqslant \frac{(\alpha\lambda(b-a))^m}{(1-\alpha\lambda(b-a))} d(u_0, u_1) \tag{6}$$

and choosing $u_0 = f \in \boldsymbol{X}$, *the inequality* (6) *becomes*

$$d(u^*, u_m) \leqslant \frac{(\alpha\lambda(b-a))^{m+1}}{\alpha(1-\alpha\lambda(b-a))} M_0 \tag{7}$$

where

$$M_0 = \max\{\left|K(s,x,u_0)\right|; s,x \in [a,b], u_0 \in R\}.$$

Proof Firstly, we prove that $A(\mathbf{X}) \subset \mathbf{X}$. To this purpose, we see that for all $\varepsilon > 0$ there are $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 + \lambda(b-a)\varepsilon_2 < \varepsilon$. Since f is continuous on compact set of $[a,b]$, we infer that it is uniformly continuous and therefore for $\varepsilon_1 > 0$ exists $\delta^{'} > 0$ such that

$$\mid f(s_1) - f(s_2) \mid < \varepsilon_1 \quad \forall s_1, s_2 \in [a,b],$$

with $|s_1 - s_2| < \delta^{'}$.
As mentioned above, K also is uniformly continuous thus, for $\varepsilon_2 > 0$ exists $\delta^{''} > 0$ such that

$$\left|K(s_1, x, u(x)) - K(s_2, x, u(x))\right| < \varepsilon_2 \quad \forall s_1, s_2 \in [a,b],$$

with $|s_1 - s_2| < \delta^{''}$.
Let $\delta = \min\{\delta^{'}, \delta^{''}\}$ and $s_1, s_2 \in [a,b]$, with $|s_1 - s_2| < \delta$. We obtain

$$\begin{aligned}\mid A(u)(s_1) - A(u)(s_2) \mid &\leqslant \mid f(s_1) - f(s_2) \mid + \lambda \int_a^b |K(s_1,x,u(x)) - K(s_2,x,u(x))| dx \\ &\leqslant \varepsilon_1 + \lambda(b-a)\varepsilon_2 < \varepsilon,\end{aligned}$$

we derive

$$\mid A(u)(s_1) - A(u)(s_2) \mid \leqslant \varepsilon.$$

This shows that $A(u)$ is uniformly continuous for any $u \in \mathbf{X}$, and so continuous on $[a,b]$, and hence A maps $\mathbf{X}$ into $\mathbf{X}$, (i.e. $A(\mathbf{X}) \subset \mathbf{X}$).
Now, we show that the operator A is a contraction map. So, for $u, F \in \mathbf{X}$ and $s \in [a,b]$, we have

$$\begin{aligned}|A(u)(s) - A(F)(s)| &\leqslant \lambda \int_a^b \mid K(s,x,u(x)) - K(s,x,F(x)) \mid dx \\ &\leqslant \lambda\alpha(b-a) \parallel F - K \parallel\end{aligned}$$

Consequently

$$\parallel A(F) - A(N) \parallel \leqslant \lambda\alpha(b-a) \parallel F - K \parallel .$$

Since $\lambda\alpha(b-a) < 1$, the operator A is a contraction on Banach space $(\mathbf{X}, \parallel . \parallel)$. Using the Banach's fixed point principle implies that (1) has a unique solution u^* in $\mathbf{X}$.
The same Banach's fixed point principle leads to the estimates (6).
Choosing $u_0 = f$, we have

$$\begin{aligned}
|u_0 - u_1| &\leqslant \lambda \int_a^b \mid K(s, x, u_0(x)) \mid dx \\
&\leqslant \lambda \int_a^b \max_{a \leqslant s \leqslant b} \mid K(s, x, u_0(x)) \mid dx \\
&\leqslant \lambda(b-a)M_0.
\end{aligned}$$

taking supremum from the above inequality we get

$$\parallel u_0 - u_1 \parallel \leqslant \lambda(b-a)M_0.$$

In this way we obtain the inequality (7), which completes the proof. ∎

Remark 3.2 Indeed, in previous theorem the existence and uniqueness of solution of (1) and the convergence of the sequence of successive approximations $(u_m)_{m\in N}$ to its exact solution in $(\mathbf{X}, \parallel . \parallel)$ are proved.

Now, we consider a uniform partition $D : a = s_0 < s_1 < s_2 < ... < s_{2n-1} < s_{2N} = b$ of $[a, b]$ with $s_i = a + i\frac{b-a}{2N}$ and $i = \overline{0, 2N}$. Applying the quadrature rule (2) and (3) in the computation of the integrals from (5) we obtain,

$$\begin{aligned}
&\bar{u}_0(s) = f(s) \\
&\bar{u}_m(s) = f(s) + \frac{b-a}{2N} \sum_{i=1}^{2N} K\big(s, a + \frac{b-a}{4N}(2i-1), \bar{u}_{m-1}(a + \frac{b-a}{4N}(2i-1))\big) \qquad (8)
\end{aligned}$$

3.2 *Convergence analysis*

In this section, we investigate the convergence of the iterative proposed method to the solution of equation (1).

Lemma 3.3 *Consider the iterative procedure 8. Moreover suppose there exists* $\beta > 0$ *such that*

$$\mid f(s) - f(s^{'}) \mid \leqslant \beta |s - s^{'}|, \qquad \forall s, s^{'} \in [a, b].$$

Under all assumptions of Theorem 3.1, the functions $K(s, x, u_m(x))$ *are Lipschitzian.*

Using the conditions (4) we obtain

$$\begin{aligned}| K(s,x,u_m(x)) - K(s,x^{'},u_m(x^{'}) \mid &\leqslant\mid K(s,x,u_m(x)) - K(s,x,u_m(x^{'}) \mid \\ &+ \mid K(s,x,u_m(x')) - K(s,x^{'},u_m(x^{'}) \mid \\ &\leqslant \alpha \mid u_m(x) - u_m(x^{'}) \mid + \mu|x-x^{'}|.\end{aligned}$$

and $\forall m \geqslant 1$

$$\begin{aligned}\mid u_m(x) - u_m(x^{'}) \mid &\leqslant \beta|x-x^{'}| + \lambda(b-a)|x-x^{'}| \\ &\leqslant (\beta + \lambda(b-a))|x-x^{'}|.\end{aligned}$$

Then, for any $x, x^{'} \in [a,b]$ we have

$$| K(s,x,u_m(x)) - K(s,x^{'},u_m(x^{'}) |\leqslant \big(\alpha(\beta+\lambda(b-a))+\mu\big)|x-x^{'}|.$$

On the other hand,

$$\begin{aligned}| K(s,x,u_0(x)) - K(s,x^{'},u_0(x^{'}) \mid &\leqslant \alpha \mid f(x) - f(x^{'}) \mid + \mu|x-x^{'}| \\ &\leqslant \alpha\beta|x-x^{'}| + \mu|x-x^{'}| \\ &\leqslant (\alpha\beta+\mu)|x-x^{'}|.\end{aligned}$$

Supposing

$$L = \max\{\big(\alpha(\beta+\lambda(b-a))+\mu\big), \alpha\beta+\mu\},$$

we have

$$| K(s,x,u_m(x)) - K(s,x^{'},u_m(x^{'}) |\leqslant L|x-x^{'}|$$

Thus, the functions $K(s,x,u_m(x))$ for all m are Lipschitzian.

Theorem 3.4 *Consider the NUFIEs* (1) *with the hypotheses of Theorem 3.1. If* $\alpha\lambda(b-a) < 1$*, then the iterative procedure (8) converges to the unique solution of* (1)*,* u^**, and its error estimate is as follows*

$$d(u^*, \bar{u}_m) \leqslant \frac{(\alpha\lambda(b-a))^{m+1}}{\alpha(1-\alpha\lambda(b-a))}M_0 + \frac{L(b-a)^2}{4N(1-\alpha\lambda(b-a))}$$

where

$$L = \max\{\big(\alpha(\beta+\lambda(b-a))+\mu\big), \alpha\beta+\mu\},$$

Proof. Using (6) we have

$$\begin{aligned}d(u^*, \bar{u}_m) &\leqslant d(u^*, u_m) + d(u_m, \bar{u}_m) \\ &\leqslant \frac{(\alpha\lambda(b-a))^{m+1}}{\alpha(1-\alpha\lambda(b-a))}M_0 + \|u_m(s) - \bar{u}_m(s)\| \qquad (9)\end{aligned}$$

therefore, we shall to obtain the estimates for $\|u_m(s) - \bar{u}_m(s)\|$. Computing the integrals from (5) we apply the quadrature formula (3) and obtain

$$\begin{aligned}
&u_0(s) = f(s) \\
&u_m(s) = f(s) + \frac{b-a}{2N}\sum_{i=1}^{2N} K\big(s, a + \frac{b-a}{4N}(2i-1), u_{m-1}(a + \frac{b-a}{4N}(2i-1))\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + E_m(s). \qquad (10)
\end{aligned}$$

with

$$| E_m(s) | \leqslant \frac{L(b-a)^2}{4N}.$$

Form (8) and (10), for $m = 1$, we obtain

$$| u_1(s) - \bar{u}_1(s) | \leqslant | E_1(s) | \leqslant \frac{L(b-a)^2}{4N} \qquad (11)$$

Now, from (11) for $m = 2$ it follow that

$$\begin{aligned}
| u_2(s) - \bar{u}_2(s) | &\leqslant \frac{L(b-a)^2}{4N} + \lambda\alpha\frac{b-a}{2N}\sum_{i=1}^{2N} | u_1\big(a + \frac{b-a}{4N}(2i-1)\big) \\
&\qquad\qquad\qquad\qquad - \bar{u}_1\big(a + \frac{b-a}{4N}(2i-1)\big) | \\
&\leqslant \frac{L(b-a)^2}{4N} + \lambda\alpha\frac{b-a}{2N}\sum_{i=1}^{2N}\frac{L(b-a)^2}{4N} \\
&= \big(1 + \lambda\alpha(b-a)\big)\frac{L(b-a)^2}{4N}.
\end{aligned}$$

By induction, for $m \in N$, $m \geqslant 3$, we obtain

$$\begin{aligned}
| u_m(s) - \bar{u}_m(s) | &\leqslant [1 + \lambda\alpha(b-a)... + (\lambda\alpha(b-a))^{m-1}]\frac{L(b-a)^2}{4N} \\
&\leqslant \frac{1 - (\lambda\alpha(b-a))^m}{1 - \lambda\alpha(b-a)}\frac{L(b-a)^2}{4N} \qquad (12) \\
&\leqslant \frac{1}{1 - \lambda\alpha(b-a)}\frac{L(b-a)^2}{4N} \\
&= \frac{L(b-a)^2}{4N(1 - \alpha\lambda(b-a))}.
\end{aligned}$$

Hence, from (9), (11) and (12) we conclude that

$$d(u^*, \bar{u}_m) \leqslant \frac{(\alpha\lambda(b-a))^{m+1}}{\alpha(1 - \alpha\lambda(b-a))}M_0 + \frac{L(b-a)^2}{4N(1 - \alpha\lambda(b-a))}$$

Remark 3.5 Since $\alpha\lambda(b-a) < 1$, it is easy to see that

$$\lim_{\substack{m\to\infty \\ \delta_x\to 0}} d(u^*, \bar{u}_m) = 0,$$

that shows the convergence of the method.

3.3 *The numerical stability analysis*

With the purpose of studying the numerical stability of the iterative method (8), considering the small changes in the first iteration, an another first iteration term $v_0(s) = g(s) \in C([a,b], R)$ is considered in such a way that there exists $\varepsilon > 0$ for which $\left|v_0(s) - u_0(s)\right| < \varepsilon, \forall s \in [a,b]$. Let β', M_0' such that

$$| g(s) - g(s') | \leqslant \beta' | s - s' | \quad \forall s, s' \in I,$$

and let $L' > 0$ be a Lipschitz constant having similar meaning as L similarly, as in the proof of Lemma 3.3.
The new sequence of successive approximations is:

$$v_m(s) = g(s) + \lambda \int_a^b K(s, x, v_{m-1}(x))dx, \quad m \geqslant 1$$

using the same iterative method (8) to solve (1) we have

$$\bar{v}_0(s) = g(s)$$

$$\bar{v}_m(s) = g(s) + \frac{b-a}{2N} \sum_{i=1}^{2N} K\big(s, a + \frac{b-a}{4N}(2i-1), \bar{v}_{m-1}(a + \frac{b-a}{4N}(2i-1))\big)$$

Theorem 3.6 *Let the conditions of Theorem 3.4 are fulfilled. Then the iterative approach (8) is numerically stable with respect to the selection of the first iteration.*

Proof We reintroduce the proof of Theorem 3.4, we obtain

$$| v_m(s) - \overline{v}_m(s) | \leqslant \frac{L'(b-a)^2}{4N(1-\alpha\lambda(b-a))}.$$

We have

$$\begin{aligned} | \overline{u}_m(s) - \overline{v}_m(s) | &\leqslant | \overline{u}_m(s) - u_m(s) | + | u_m(s) - v_m(s) | + | v_m(s) - \overline{v}_m(s) | \\ &\leqslant | u_m(s) - v_m(s) | + \frac{L(b-a)^2}{4N(1-\alpha\lambda(b-a))} + \frac{L'(b-a)^2}{4N(1-\alpha\lambda(b-a))}. \end{aligned}$$

also,

$$| u_0(s) - v_0(s) | < \varepsilon, \qquad \forall s \in [a,b],$$

and

$$\begin{aligned}\big|u_1(s)-v_1(s)\big| &\leqslant \big|u_0(s)+\lambda\int_a^b K(s,x,v_0(x))dx \\ &\quad - v_0(s)-\lambda\int_a^b K(s,x,v_0(x))dxdydz\big| \\ &\leqslant \varepsilon+\alpha\lambda\int_a^b \mid u_0(s)-v_0(s)\mid dx \\ &\leqslant (1+\alpha\lambda(b-a))\varepsilon=(1+\alpha\lambda(b-a))\varepsilon,\end{aligned}$$

for $m\geqslant 2$, by induction, we have

$$\begin{aligned}\big|u_m(s)-v_m(s)\big| &\leqslant \big|u_0(s)-v_0(s)\big| \\ &\quad +\lambda\int_a^b \big|K(s,x,u_{m-1}(x))-K(s,x,v_{m-1}(x))\big|dx \\ &\leqslant \varepsilon+\alpha\lambda\int_a^b \mid u_{m-1}(s)-v_{m-1}(s)\mid dxdy \\ &\leqslant (1+\alpha\lambda(b-a)+...+(\alpha\lambda(b-a))^m)\varepsilon,\end{aligned}$$

for all $s\in[a,b]$ and $m\geqslant 0$. Then,

$$d(u_m(s),v_m(s))\leqslant\frac{1}{1-\alpha\lambda(b-a)}\varepsilon.$$

Now, we get,

$$\mid\overline{u}_m(s)-\overline{v}_m(s)\mid\leqslant\frac{1}{1-\alpha\lambda(b-a)}\varepsilon+\frac{(L+L')(b-a)^2}{4N(1-\alpha\lambda(b-a))}$$

■

Remark 3.7 Since $\alpha\lambda(b-a)<1$, it is easy to see that

$$\lim_{\delta_x,\varepsilon\to 0} d(\overline{u}_m,\overline{v}_m)=0.$$

this shows the stability of the method.

3.4 *Algorithm of the approach*

The iterative procedure 8 gives the following algorithm of computation for the solution of (1):

Step 1: The data are introduced as, $a,b,\lambda,\varepsilon,N$ and the functions K,f.

Step 2: For $j = \overline{0, 2N}$ comput $\bar{u}_m(s_j)$ by

$$\bar{u}_m(s) = f(s) + \frac{b-a}{2N}\sum_{i=1}^{2N} K\big(s, a + \frac{b-a}{4N}(2i-1), \bar{u}_{m-1}(a + \frac{b-a}{4N}(2i-1))\big).$$

Step 3: Compute $\mid \bar{u}_m(s_j) - \bar{u}_{m-1}(s_j) \mid$.
Step 4: If $\mid \bar{u}_m(s_j) - \bar{u}_{m-1}(s_j) \mid < \varepsilon$, Print $\bar{u}_m(s_j)$, $j = \overline{0, 2N}$. STOP.

4. Numerical experiments

We have applied our method on some numerical examples, to observe the accuracy and efciency of the present method for solving NUFIEs. Also, we compare the numerical solutions obtained by using the proposed method with the exact solutions. In order to analyze the error of the method we introduce notations

$$e_{2N} = |u^*(s) - \bar{u}(s)|,$$

and

$$\|e_{2N}\|_\infty := max\{|e_{2N}(s_j)|, j = 0, 1, 2, ..., 2N\},$$

where $\bar{u}(s), u^*(s)$ are the approximate solution and the exact solution of integral equations, respectively. Moreover, the number of iterations, $NI = m$, and the errors $\|e_{2N}\|_\infty$ are inserted in the tables. In this section, points are proposed as $s_j = \frac{j}{10}$, for $j = 1, 2, ..., 9$. and we assumed that $[a, b] = [0, 1]$, $\lambda = 1$. The computations have been done using Maple 17.

Example 4.1 The following nonlinear Fredholm integral equation has been considered by other authors as a numerical test [5, 9, 28, 30, 31],

$$u(s) = f(s) + \int_0^1 K(s, x, u(x))dx,$$

where

$$f(s) = \sin(\pi s)$$
$$K(s, x, u(x)) = \frac{1}{5}\cos(\pi s)\sin(\pi x)(u(x))^3,$$

with the exact solution

$$u(s) = \sin(\pi s) + \frac{20 - \sqrt{391}}{3}\cos(\pi s),$$

Ezquerro et al. studied existence of the solutions of the above equation (4.1) in [9]. Moreover, Rashidinia et al. in [30] analytically solutions for the mentioned eqution , including $u_1(s) = \sin(\pi s) + \frac{20-\sqrt{391}}{3}\cos(\pi s)$ and $u_2(s) = \sin(\pi s) + \frac{20+\sqrt{391}}{3}\cos(\pi s)$. But in [5] just $u(s) = \sin(\pi s) + \frac{20-\sqrt{391}}{3}\cos(\pi s)$ has been considered. In [5] the minimum absolute errors of approximation is 3.6765×10^{-7} and the error just in the point $s = 0.5$ is zero. Also, in [31] the minimum absolute errors of approximation is 7.796×10^{-3}. By using the proposed method, we can present the approximate solution for this example. Table 1 shows that the numerical results for this example.

Table 1. Numerical results for $2N = 10, 2N = 20, 2N = 40$, in Example 1.

s_j	exact	$e_{2N=10}$	$e_{2N=20}$	$e_{2N=40}$
0.1	0.380752038	$4.0580526341 \times 10^{-4}$	$5.2713913659 \times 10^{-5}$	$6.330998663 \times 10^{-6}$
0.2	0.648806725	$3.4519857540 \times 10^{-4}$	$4.4841133266 \times 10^{-5}$	$5.290888307 \times 10^{-6}$
0.3	0.853351689	$2.5080144563 \times 10^{-4}$	$3.2578990321 \times 10^{-5}$	$3.673434132 \times 10^{-6}$
0.4	0.974364644	$1.3185412293 \times 10^{-4}$	$1.7127788813 \times 10^{-5}$	$1.640715164 \times 10^{-6}$
0.5	1	0	0	0
0.6	0.9277483875	$1.3185412293 \times 10^{-4}$	$1.7127788813 \times 10^{-5}$	$1.640715164 \times 10^{-6}$
0.7	0.7646822990	$2.5080144563 \times 10^{-4}$	$3.2578990321 \times 10^{-5}$	$3.673434132 \times 10^{-6}$
0.8	0.5267637791	$3.4519857540 \times 10^{-4}$	$4.4841133266 \times 10^{-5}$	$5.290888307 \times 10^{-6}$
0.9	0.2372819503	$4.0580526341 \times 10^{-4}$	$5.2713913659 \times 10^{-5}$	$6.330998663 \times 10^{-6}$
NI		12	12	12
$\|e_n\|_\infty$		7.1664×10^{-5}	5.2714×10^{-6}	6.3310×10^{-7}

Example 4.2 Consider the following nonlinear Fredholm integral equation

$$u(s) = f(s) + \int_0^1 K(s, x, u(x))dx,$$

where

$$f(s) = \cos(s) - \frac{1}{9}\sqrt{s}(\cos(1)^3 + 3\cos(1)^2\sin(1) + 6\cos(1) + 6\sin(1) - 7)$$
$$K(s, x, u(x)) = \sqrt{s}x(u(x))^3,$$

and exact solution $u(s) = \cos(s)$. We have to obtain the absolute error, for the grid points s_j, for $j = 1, 2, ..., 9$. Numerical results (error between exact and approximate value of $\bar{u}(s)$) with $2N = 10, 2N = 20, 2N = 40$ are given in Table 2.

Table 2. Numerical results for $2N = 10, 2N = 20, 2N = 40$, in Example 2.

s_j	$e_{2N=10}$	$e_{2N=20}$	$e_{2N=40}$
0.1	$1.994941976 \times 10^{-6}$	$1.43383339 \times 10^{-7}$	9.3314031×10^{-9}
0.2	$3.941226833 \times 10^{-6}$	$2.83269524 \times 10^{-7}$	1.8435212×10^{-8}
0.3	$5.840613257 \times 10^{-6}$	$4.19784957 \times 10^{-7}$	2.7319650×10^{-8}
0.4	$7.694776196 \times 10^{-6}$	$5.53050023 \times 10^{-7}$	3.5992555×10^{-8}
0.5	$9.505311771 \times 10^{-6}$	$6.83179439 \times 10^{-7}$	4.4461391×10^{-8}
0.6	$1.127374186 \times 10^{-5}$	$8.10282591 \times 10^{-7}$	5.2733278×10^{-8}
0.7	$1.300151840 \times 10^{-5}$	$9.34463832 \times 10^{-7}$	6.0815007×10^{-8}
0.8	$1.469002728 \times 10^{-5}$	$1.05582277 \times 10^{-6}$	6.8713059×10^{-8}
0.9	$1.634059214 \times 10^{-5}$	$1.17445454 \times 10^{-6}$	7.6433628×10^{-8}
NI	9	9	9
$\|e_N\|_\infty$	1.63406×10^{-5}	1.17445×10^{-6}	7.64336×10^{-8}

Example 4.3 ([1, 4]) Consider the following nonlinear Fredholm integral equation

$$u(s) = f(s) + \int_0^1 K(s, x, u(x))dx, s \in [0, 1]$$

where

$$f(s) = s - \frac{1}{2}(-\cos(1) + \sin(1))e^{(s-1)} - \frac{1}{2}e^{(s)}$$
$$K(s, x, u(x)) = e^{s-x}\cos(u(x)),$$

and exact solution $u(s) = s$.
By using the proposed method, we can present the approximate solution for this example. Table 3 shows that the numerical results for this example.

Table 3. Numerical results for $2N = 10, 2N = 20, 2N = 40$, in Example 3.

s_i	$e_{2N=10}$	$e_{2N=20}$	$e_{2N=40}$
0.1	$2.1552794792 \times 10^{-5}$	$4.357367721 \times 10^{-6}$	$2.91759361 \times 10^{-7}$
0.2	$4.2579911663 \times 10^{-5}$	$8.608458180 \times 10^{-6}$	$5.76402643 \times 10^{-7}$
0.3	$6.3100351018 \times 10^{-5}$	$1.275711272 \times 10^{-5}$	$8.54187040 \times 10^{-7}$
0.4	$8.3132208484 \times 10^{-4}$	$1.680698978 \times 10^{-5}$	$1.12535753 \times 10^{-6}$
0.5	$1.0269272812 \times 10^{-4}$	$2.076157561 \times 10^{-5}$	$1.39014753 \times 10^{-6}$
0.6	$1.2179835196 \times 10^{-4}$	$2.462419433 \times 10^{-5}$	$1.64877963 \times 10^{-6}$
0.7	$1.4046476605 \times 10^{-4}$	$2.839801722 \times 10^{-5}$	$1.90146617 \times 10^{-6}$
0.8	$1.5870694346 \times 10^{-4}$	$3.208607140 \times 10^{-5}$	$2.14840982 \times 10^{-6}$
0.9	$1.7653918430 \times 10^{-4}$	$3.569124796 \times 10^{-5}$	$2.38980419 \times 10^{-6}$
NI	11	11	11
$\|e_N\|_\infty$	1.76539×10^{-4}	3.56912×10^{-5}	2.38980×10^{-6}

5. Conclusions

In this work a computational method has been presented for numerical solution of nonlinear Urysohn integral equations based on Haar wavelet series. This method is very simple and involves lower computation. In the Theorem 3.1 sufficint conditions for the existence and uniquness solution of the (NUFIEs) are presented. Proof of the convergence and the error estimation of the proposed method in terms of Lipschitz condition are provided in the Theorem 3.4. To illustrate the efficiency of the presented method, three examples are given.

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