International Journal of
Mathematical Modelling & Computations
Vol. 09, No. 02, Spring 2019, 101- 111

Fixed Point Results for Cyclic (α, β)-Admissible Type F-Contractions in Modular Spaces

M. Sangurlu Sezen[a,*] and D. Türkoğlu[b]

[a] *Department of Mathematics, Giresun University, Giresun, PO. Code 28100, Turkey,*
[b] *Department of Mathematics, Gazi University, Ankara, PO. Code 06500, Turkey.*

Abstract. In this paper, we prove the existence and uniqueness of fixed points for cyclic (α, β)-admissible type F-contraction and $F-$weak contraction under the setting of modular spaces, where the modular is convex and satisfying the Δ_2-condition. Later, we prove some periodic point results for self-mappings on a modular space. We also give some examples to support our results.

Received: 10 February 2019, Revised: 31 March 2019, Accepted: 10 June 2019.

Keywords: Modular space; Fixed point; F-contraction; Cyclic (α, β)-admissible mapping.

AMS Subject Classification: 47H10, 54H25.

Index to information contained in this paper

1 **Introduction and preliminaries**
2 **Fixed point results for cyclic (α, β)-admissible type F-contractions**
3 **Periodic point results for cyclic (α, β)-admissible type F-contractions**

1. Introduction and preliminaries

Banach has found the most fundamental contraction principle of fixed point theory, which is widely used and considered one of the important issues of mathematics [2]. In its development process, some authors have generalized the concept of Banach fixed point [1, 5–8].

Nakano [15, 16] initiated the notion of modulars on linear spaces and the corresponding theory of modular linear spaces and further it was generalized and redefined by Musielak and Orlicz [13, 14]. In many areas, particularly in applications to integral operators, approximation and fixed point results, modular type conditions have been used. Recently, many comprehensive applications related to the fixed point theory in modular spaces are investigated. Khamsi et al presented

*Corresponding author. Email: muzeyyen.sezen@giresun.edu.tr

a strong interest to study the fixed point property in modular function spaces, which are natural generalization of both function and sequence spaces [9]. Taleb and Hanebaly proved a fixed point theorem and its application to integral equations in modular function spaces [17]. Mongkolkeha and Kumam, considered and proved some fixed point and common fixed point results for generalized contraction mappings in modular spaces [11]. On the other hand, Hajji and Hanebaly presented a generalization of Banach's fixed point theorem in some classes of modular spaces, using some convenient constants in the contraction assumption [4]. For many years, a lot of fixed point theorems and applications have been made in modular spaces and some authors have published many works on this concept in different areas [10, 12]. In this paper, we prove some fixed point theorems in modular spaces. Firstly, we introduce cyclic (α,β)-admissible type F-contraction and $F-$weak contraction in modular spaces benefiting from the Wardowski and Alizadeh et al.'s work [1]. Later, we establish fixed point and periodic point results for such a contraction. We also state some examples to support our results.

Definition 1.1 *[12, 15] Let X be an arbitrary vector space. A functional $\rho : X \to [0,\infty)$ is called a modular if, for any x, y in X, the following conditions hold:*

(a) $\rho(x) = 0$ if and only if $x = 0$,
(b) $\rho(-x) = \rho(x)$,
(c) $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$, whenever $\alpha + \beta = 1$, and $\alpha, \beta \geqslant 0$.

If (c) is replaced with $\rho(\alpha x + \beta y) \leqslant \alpha^s \rho(x) + \beta^s \rho(y)$ where $\alpha^s + \beta^s = 1$, $\alpha, \beta \geqslant 0$, and
$s \in (0, 1]$, then ρ is called s-convex modular. If $s = 1$, then we say that ρ is convex modular.

The following are some consequences of condition (c).

Remark 1 [3]

(a) For $a, b \in \mathbb{R}$ with $|a| < |b|$ we have $\rho(ax) < \rho(bx)$ for all $x \in X$.,
(b) For $a_1, ..., a_n \in \mathbb{R}^+$ with $\sum_{i=1}^n a_i = 1$, we have

$$\rho(\sum_{i=1}^n a_i x_i) = \rho(\sum_{i=1}^n x_i) \text{ for any } x_1, ..., x_n \in X.$$

Remark 2 [11] A modular ρ defines a corresponding modular space, i.e. the space is given by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Definition 1.2 *[9] A sequence $\{x_n\}$ in modular space X_ρ is said to be:*

(a) ρ-convergent to $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$,,
(b) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$,
(c) X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent,
(d) ρ satisfies Δ_2-condition if $\rho(2x_n) \to 0$ as $n \to \infty$, whenever $\rho(x_n) \to 0$ as $n \to \infty$.

Definition 1.3 *Let X_ρ be a modular space and $T : X_\rho \to X_\rho$ be a self-map. We say that T is ρ-continuous when if $\rho(x_n - x) \to 0$, then $\rho(Tx_n - Tx) \to 0$ as $n \to \infty$.*

Definition 1.4 *[1] Let $T : X \to X$ be a mapping and $\alpha, \beta : X \to [0, +\infty)$ be two* *functions. We say that T is a cyclic (α, β)-admissible mapping if*

(a) $\alpha(x) \geqslant 1$ for some $x \in X$ implies $\beta(Tx) \geqslant 1$,
(b) $\beta(x) \geqslant 1$ for some $x \in X$ implies $\alpha(Tx) \geqslant 1$.

DEFINITION 1.5 *[18] Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \longrightarrow \mathbb{R}$ such that*

(F1) F is strictly increasing, that is, for all $\gamma, \delta \in (0, +\infty)$ if $\gamma < \delta$ then $F(\gamma) < F(\delta)$,
(F2) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds: $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

DEFINITION 1.6 *[18] Let (X, d) be a metric space. A map $T : X \to X$ is said to be an $F-$contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)).$$

DEFINITION 1.7 *[19] Let (X, d) be a metric space. A map $T : X \to X$ is said to be an $F-$weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Longrightarrow$$
$$\tau + F(d(Tx, Ty)) \leqslant F(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}).$$

In this paper, we use the following condition instead of the condition (F3) in Definition 1.5:
(F3*) F is continuous on $(0, \infty)$.

2. Fixed point results for cyclic (α, β)-admissible type F-contractions

Let X_ρ be a nonempty set and $T : X_\rho \to X_\rho$ be an arbitrary mapping. We say that $x \in X_\rho$ is a fixed point for T, if $x = Tx$. We denote by $Fix(T)$ the set of all fixed points of T. We give the following conditions. In the sequel, suppose the modular function ρ is convex and satisfies the Δ_2-condition.

DEFINITION 2.1 *Let X_ρ be a ρ-complete modular space. A cyclic (α, β)-admissible mapping $T : X_\rho \to X_\rho$ is said to be cyclic (α, β)-admissible type F-contraction if there exists $\tau > 0$, $F \in \mathcal{F}$ and satisfying $\rho(Tx - Ty) > 0$ such that*

$$\tau + \alpha(x)\beta(y)F(\rho(Tx - Ty)) \leqslant F(\rho(x - y)) \tag{1}$$

for all $x, y \in X_\rho$.

DEFINITION 2.2 *Let X_ρ be a ρ-complete modular space. A cyclic (α, β)-admissible mapping $T : X_\rho \to X_\rho$ is said to be cyclic (α, β)-admissible type F-weak contraction* *if there exists* $\tau > 0$, $F \in \mathcal{F}$ *and satisfying* $\rho(Tx - Ty) > 0$, *such that*

$$\begin{aligned} &\tau + \alpha(x)\beta(y)F(\rho(Tx - Ty)) \\ \leqslant\ & F(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(x-Ty)) + \rho(\frac{1}{2}(y-Tx))}{2}\}) \quad (2) \end{aligned}$$

for all $x, y \in X_\rho$.

Remark 1 Every cyclic (α,β)-admissible type F-contraction is a cyclic (α,β)-admissible type F-weak contraction, but the converse is not necessarily true.

Example 2.3 Let $X_\rho = [0,2]$ and $\rho(x) = |x|$ for all $x \in X_\rho$. Define $T : X_\rho \to X_\rho$ by

$$Tx = \begin{cases} 0\ if \ \ x \in [0,2) \\ \frac{1}{2}\ if\ otherwise. \end{cases}$$

Define $\alpha, \beta : X_\rho \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1\ ,\ \ x \in [0,2) \\ 0\ ,\ otherwise. \end{cases} \qquad \text{and} \quad \beta(x) = 1 \text{ for } x \in [0,2].$$

$F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(a) = lna$. It is clear that F satisfies (F1)-(F3)-(F3*) for any $k \in (0,1)$. Then for $x = \frac{7}{4}$, $y = 2$, we have

$$\tau + \alpha(\frac{7}{4})\beta(2)F(\rho(T(\frac{7}{4}), T(2))) = \tau + \ln(\frac{1}{2}) \text{ and } F(\rho(\frac{7}{4}, 2)) = \ln\frac{1}{4}.$$

Then, we get

$$\tau + \ln(\frac{1}{2}) \nleq \ln\frac{1}{4}.$$

So T is not a cyclic (α,β)-admissible type F-contraction. However, since for $x \in [0,2)$, $y = 2$

$$\max\{\rho(x-2), \rho(x-0), \rho(2-\frac{1}{2}), \frac{\rho(\frac{1}{2}(x-\frac{1}{2})) + \rho(\frac{1}{2}(2-0))}{2}\} = \frac{3}{2}.$$

Therefore, T is a cyclic (α,β)-admissible type F-weak contraction.

Remark 2 Definition 1.6 (respectively, Definition 1.7) reduces to an F-contraction (respectively, an F-weak contraction) for $\alpha(x)\beta(y) = 1$.

Theorem 2.4 *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α,β)-admissible type F-weak contraction satisfying the following conditions:*

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geqslant 1$ and $\beta(x_0) \geqslant 1$,
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geqslant 1$ for all n, then $\beta(x) \geqslant 1$, or
(c) T is continuous,

then T has a fixed point. Moreover, if $\alpha(x) \geqslant 1$ and $\beta(y) \geqslant 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a cyclic (α,β)-admissible type F-weak contraction and $\alpha(x_0) \geqslant 1$ then $\beta(x_1) =$

$\beta(Tx_0) \geqslant 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geqslant 1$. By continuing this process, we get $\alpha(x_{2n}) \geqslant 1$ and $\beta(x_{2n-1}) \geqslant 1$ for all $n \in \mathbb{N}$. Again, since T is a cyclic (α, β)-admissible type F-weak contraction and $\beta(x_0) \geqslant 1$, by the similar method, we have $\beta(x_{2n}) \geqslant 1$ and $\alpha(x_{2n-1}) \geqslant 1$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geqslant 1$ for all $n \in \mathbb{N}$. We assume that

$$0 < \rho(x_n - Tx_n) = \rho(Tx_{n-1} - Tx_n), \ \forall n \in \mathbb{N}.$$

Since T is a cyclic (α, β)-admissible type F-weak contraction for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\tau + F(\rho(Tx_{n-1} - Tx_n)) &\leqslant \tau + \alpha(x_{n-1})\beta(x_n)F(\rho(Tx_{n-1} - Tx_n)) \\ &\leqslant F(\max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \frac{\rho(\frac{1}{2}(x-Ty)) + \rho(\frac{1}{2}(y-Tx)}{2}\}).\end{aligned}$$

Repeating this process, we get

$$\begin{aligned}&F(\rho(Tx_n - Tx_{n-1})) \\ &\leqslant F(\max\{\rho(x_n - x_{n-1}), \rho(x_n - Tx_n), \rho(x_{n-1} - Tx_{n-1}) \\ &\quad , \frac{\rho(\frac{1}{2}(x_n - Tx_{n-1})) + \rho(\frac{1}{2}(x_{n-1} - Tx_n))}{2}\}) - \tau \\ &= F(\max\{\rho(x_n - x_{n-1}), \rho(x_n - x_{n+1}), \rho(x_{n-1} - x_n) \\ &\quad , \frac{\rho(x_n - x_n) + \rho(x_{n-1} - x_{n+1})}{2}\}) - \tau \\ &= F(\max\{\rho(x_n - x_{n-1}), \rho(x_n - x_{n+1})\}) - \tau. \qquad (3)\end{aligned}$$

If there exists $n \in \mathbb{N}$ such that $\max\{\rho(x_n - x_{n-1}), \rho(x_n - x_{n+1})\} = \rho(x_n - x_{n+1})$ then (3) becomes

$$F(\rho(x_{n+1} - x_n)) \leqslant F(\rho(x - x_n)) - \tau < F(\rho(x_{n+1} - x_n)).$$

It is a contradiction. Therefore,

$$\max\{\rho(x_n - x_{n-1}), \rho(x_n - x_{n+1})\} = \rho(x_n - x_{n-1})$$

for all $n \in \mathbb{N}$. Thus, from (3), we have

$$F(\rho(x_{n+1} - x_n)) \leqslant F(\rho(x_n - x_{n-1})) - \tau$$

for all $n \in \mathbb{N}$. It implies that

$$F(\rho(x_{n+1} - x_n)) \leqslant F(\rho(x_1 - x_0)) - n\tau \qquad (4)$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in (14), we get

$$\lim_{n\to\infty} F(\rho(x_{n+1} - x_n)) = -\infty$$

that together with (F2) gives

$$\lim_{n\to\infty} \rho(x_{n+1} - x_n) = 0. \qquad (5)$$

Now, we want to show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$,

$$\rho(x_{n(k)} - x_{m(k)}) \geqslant \varepsilon \text{ and } \rho(2(x_{n(k)-1} - x_{m(k)})) < \varepsilon. \tag{6}$$

Now for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leqslant \rho(x_{n(k)} - x_{m(k)}) \\ &\leqslant \rho(2(x_{n(k)} - x_{n(k)-1})) + \rho(2(x_{n(k)-1} - x_{m(k)})) \\ &< \rho(2(x_{n(k)} - x_{n(k)-1})) + \varepsilon. \end{aligned} \tag{7}$$

Taking the limit as $k \to +\infty$ in (7), using (5) from Δ_2-condition, we get

$$\lim_{k\to\infty} \rho(x_{n(k)} - x_{m(k)}) = \varepsilon. \tag{8}$$

Then, we get

$$\begin{aligned} \rho(2(x_{n(k)} - x_{m(k)})) &\leqslant \rho(2(x_{n(k)} - x_{n(k)+1} + x_{n(k)+1} - x_{m(k)})) \\ &\leqslant \rho(4(x_{n(k)} - x_{m(k)}) + \rho(4(x_{n(k)+1} - x_{m(k)+1} + x_{m(k)+1} - x_{m(k)})) \\ &\leqslant \rho(4(x_{n(k)} - x_{m(k)}) + \rho(8(x_{n(k)+1} - x_{m(k)+1})) + \rho(8(x_{m(k)+1} - x_{m(k)})). \end{aligned} \tag{9}$$

By taking the limit as $k \to +\infty$ in (9), using (5) and (8), we deduce that

$$\lim_{k\to\infty} \rho(x_{n(k)+1} - x_{m(k)+1}) = \varepsilon. \tag{10}$$

Now, by (2), we get

$$\begin{aligned} \tau + F(\rho(x_{n(k)+1} - x_{m(k)+1})) &\leqslant \tau + \alpha(x_{n(k)})\beta(x_{m(k)})F(\rho(x_{n(k)+1} - x_{m(k)+1})) \\ &\leqslant F(\max\{\rho(x_{n(k)} - x_{m(k)}) \\ &\quad , \rho(x_{n(k)} - x_{n(k)+1}), \rho(x_{m(k)} - x_{m(k)+1}) \\ &\quad , \frac{\rho(\frac{1}{2}(x_{n(k)} - x_{m(k)+1})) + \rho(\frac{1}{2}(x_{m(k)} - x_{n(k)+1}))}{2}\})). \end{aligned} \tag{11}$$

where

$$\begin{aligned} \rho(\frac{1}{2}(x_{n(k)} - x_{m(k)+1})) &= \rho(\frac{1}{2}(x_{n(k)} - x_{m(k)} + x_{m(k)} - x_{m(k)+1})) \\ &\leqslant \rho((x_{n(k)} - x_{m(k)}) + \rho(x_{m(k)} - x_{m(k)+1}) \\ &\leqslant \varepsilon + \rho(x_{m(k)} - x_{m(k)+1}) \end{aligned} \tag{12}$$

and

$$\begin{aligned} \rho(\frac{1}{2}(x_{m(k)} - x_{n(k)+1}) &= \rho(\frac{1}{2}(x_{m(k)} - x_{n(k)} + x_{n(k)} - x_{n(k)+1})) \\ &\leqslant \rho(x_{m(k)} - x_{n(k)}) + \rho(x_{n(k)} - x_{n(k)+1}) \\ &\leqslant \varepsilon + \rho(x_{n(k)} - x_{n(k)+1}). \end{aligned} \tag{13}$$

By taking the limsup on both sides of (11), applying (5), (10), (12), (13) and from (F3*), we obtain

$$\tau + F(\varepsilon) \leqslant F(\varepsilon)$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X_ρ is complete, there exists x^* is a point such that $x_n \to x^*$. Since X_ρ is complete, there exists $x^*$is a fixed point of T by two following cases.
F is continuous. In this case, we consider two following cases.
Case 1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_n+1} = Tx^*$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then we have

$$x^* = \lim_{n\to\infty} x_{i_n+1} = \lim_{n\to\infty} Tx^* = Tx^*.$$

This proves that x^* is a fixed point of T.
Case 2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geqslant n_0$. That is $\rho(Tx_n - Tx^*) > 0$ for all $n \geqslant n_0$. Assume that (c) is held. That is $\alpha(x_n)\beta(x^*) \geqslant 1$. It follows from (14) and (F1)

$$\begin{aligned}
\tau + F(\rho(x_{n+1} - Tx^*)) &\leqslant \tau + \alpha(x_n)\beta(x^*)F(\rho(Tx_n - Tx^*)) \\
&\leqslant F(\max\{\rho(x_n - x^*), \rho(x_n - Tx_n), \rho(x^* - Tx^*), \\
&\quad \frac{\rho(x_n - Tx^*) + \rho(x^* - Tx_n)}{2}\}) \\
&= F(\max\{\rho(x_n - x^*), d(x_n - x_{n+1}), d(x^* - Tx^*), \\
&\quad \frac{\rho(x_n - Tx^*) + \rho(x^* - x_{n+1})}{2}\}) \\
&= F(\max\{\rho(x_n - x^*), d(x_n - x_{n+1}), d(x^* - Tx^*), \\
&\quad \frac{\rho(x_n - Tx^*) + \rho(x^* - Tx^*) + \rho(x^* - x_{n+1})}{2}\}). \qquad (14)
\end{aligned}$$

If $\rho(x^* - Tx^*) > 0$ then by the fact

$$\lim_{n\to\infty} \rho(x_n - x^*) = \lim_{n\to\infty} \rho(x^* - x_{n+1}) = 0,$$

there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$, we have

$$\begin{aligned}
&\max\{\rho(x_n - x^*), \rho(x_n - x_{n+1}), \rho(x^* - Tx^*), \frac{\rho(x_n - Tx^*) + \rho(x^* - Tx^*) + \rho(x^* - x_{n+1})}{2}\} \\
&= \rho(x^* - Tx^*)).
\end{aligned}$$

From (14), we get

$$\tau + F(\rho(x_{n+1} - Tx^*)) \leqslant \tau + \alpha(x_n)\beta(x^*)F(\rho(x_{n+1} - Tx^*)) \leqslant F(\rho(x^* - Tx^*)), \quad (15)$$

for all $n \geqslant \max\{n_0, n_1\}$. Since F is continuous, taking the limit as $n \to \infty$ in (15), we obtain

$$\tau + F(\rho(x^* - Tx^*)) \leqslant \tau + \alpha(x^*)\beta(x^*)F(\rho(x^* - Tx^*)) \leqslant F(\rho(x^* - Tx^*)).$$

It is contradiction. Therefore, $\rho(x^* - Tx^*) = 0$, that is, x^* is a fixed point of T. Now, if T is continuous, we have

$$\rho(x^* - Tx^*) = \lim_{n\to\infty} \rho(x_n - Tx_n) = \lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0.$$

This proves that x^* is a fixed point of T. By two above cases, T has a fixed point x^*.

Now, we prove that the fixed point of T is unique. Let x_1^*, x_2^* be two fixed points of T. Suppose to the contrary that $x_1^* \neq x_2^*$. Then $Tx_1^* \neq Tx_2^*$, therefore, $\rho(Tx-Ty) = \rho(x-y) > 0$. Since $\alpha(x_1^*)\beta(x_2^*) \geqslant 1$, it follows from (2) that

$$\begin{aligned}
\tau + F(\rho(x_1^* - x_2^*)) &\leqslant \tau + \alpha(x_1^*)\beta(x_2^*)F(\rho(x_1^* - x_2^*)) = \tau + \alpha(x_1^*)\beta(x_2^*)F(\rho(Tx_1^* - Tx_2^*)) \\
&\leqslant F(\max\{\rho(x_1^* - x_2^*), \rho(x_1^* - Tx_1^*), \rho(x_2^* - Tx_2^*), \\
&\quad \frac{\rho(x_1^* - Tx_2^*), \rho(x_2^* - Tx_1^*)}{2}\}) \\
&= F(\max\{\rho(x_1^* - x_2^*), \rho(x_1^* - Tx_1^*), \rho(x_2^* - Tx_2^*), \\
&\quad \frac{\rho(x_1^* - x_2^*), \rho(x_2^* - x_1^*)}{2}\}) \\
&= F(\rho(x_1^* - x_2^*)).
\end{aligned}$$

It is a contradiction. Then $\rho(x_1^* - x_2^*) = 0$, that is $x_1^* = x_2^*$. This proves that the fixed point of T is unique. ■

Example 2.5 The map T in Example 2.3 satisfies all the hypotheses of Theorem **??**, hence T has a unique fixed point $x = 0$.

COROLLARY 2.6 *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a cyclic (α, β)-admissible type F-contraction satisfying the following conditions:*

(a) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geqslant 1$ and $\beta(x_0) \geqslant 1$,
(b) T is continuous,

then T has a fixed point. Moreover, if $\alpha(x) \geqslant 1$ and $\beta(y) \geqslant 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof The proof of Corollary 2.6 because of the Remark *2* is proved as in Theorem **??**. ■

COROLLARY 2.7 *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ a cyclic (α, β)-admissible mapping satisfying the following conditions:*

(a) for all $x, y \in X_\rho$ where $a, b, c \geqslant 0$ and $a + b + c + 2e < 1$

$$\begin{aligned}
\rho(Tx - Ty) > 0 \Rightarrow \tau + \alpha(x)\beta(y)F(\rho(Tx - Ty)) \\
&\leqslant a\rho(x - y) + b\rho(x - Tx) \\
&+ c\rho(y - Ty) + e[\rho(x - Ty) + \rho(y - Tx)],
\end{aligned} \tag{16}$$

(b) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geqslant 1$ and $\beta(x_0) \geqslant 1$, or
(c) T is continuous,

then T has a fixed point. Moreover, if $\alpha(x) \geqslant 1$ and $\beta(y) \geqslant 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Proof For all $x, y \in X_\rho$, we have

$$\begin{aligned}
&a\rho(x-y)+b\rho(x-Tx)\\
&+cd(y-Ty)+e[\rho(x-Ty)+\rho(y-Tx)]\\
\leqslant\ &(a+b+c+2e)\max\{\rho(x-y),\rho(x-Tx),\rho(y-Ty),\frac{\rho(x-Ty),\rho(y-Tx)}{2}\}\\
\leqslant\ &\max\{\rho(x-y),d(x,Tx),\rho(y-Ty),\frac{\rho(x-Ty),\rho(y-Tx)}{2}\}.
\end{aligned}$$

Then, by (F1) we see that (2) is a consequence of (16). Than the corollary is proved. ■

3. Periodic point results for cyclic (α, β)-admissible type F-contractions

In this section, we prove some periodic point results for self-mappings on a modular space.

THEOREM 3.1 *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be is cyclic (α, β)-admissible mapping satisfying the following conditions:*

(a) there exists $\tau > 0$, $F \in \mathcal{F}$ and satisfying $\rho(Tx - Ty) > 0$ such that

$$\tau + \alpha(x)\beta(Tx)F(\rho(Tx - T^2x)) \leqslant F(\rho(x - Tx))$$

hold for all $x \in X_\rho$,

(b) there exists $x_0 \in X_\rho$ such that $\alpha(x_0) \geqslant 1$ and $\beta(x_0) \geqslant 1$,

(c) T is continuous,

(d) if $x \in Fix(T^n)$ and $x \notin Fix(T)$, then $\alpha(T^{n-1}x)\beta(T^n x) \geqslant 1$.

Then $Fix(T^n) = Fix(T)$ for all $n \in \mathbb{N}$.

Proof Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is a cyclic (α, β)-admissible mapping and $\alpha(x_0) \geqslant 1$ then $\beta(x_1) = \beta(Tx_0) \geqslant 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geqslant 1$. By continuing this process, we get $\alpha(x_{2n}) \geqslant 1$ and $\beta(x_{2n-1}) \geqslant 1$ for all $n \in \mathbb{N}$. Again, since T is a cyclic (α, β)-admissible mapping and $\beta(x_0) \geqslant 1$, by the similar method, we have $\beta(x_{2n}) \geqslant 1$ and $\alpha(x_{2n-1}) \geqslant 1$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_{n-1}) \geqslant 1$ for all $n \in \mathbb{N}$.

We assume that

$$0 < \rho(Tx_{n-1} - T^2x_{n-1}),\ \forall n \in \mathbb{N}.$$

From (a), we have

$$\begin{aligned}
\tau + F(\rho(x_n - x_{n+1})) &= \tau + F(\rho(Tx_{n-1} - T^2x_{n-1}))\\
&\leqslant \tau + \alpha(x_{n-1})\beta(Tx_{n-1})F(\rho(Tx_{n-1} - T^2x_{n-1}))\\
&\leqslant F(\rho(x_{n-1}, Tx_{n-1})),
\end{aligned}$$

or equivalently,

$$F(\rho(x_n - x_{n+1})) \leqslant F(\rho(x_{n-1} - x_n) - \tau.$$

By using a similar reasoning to the proof of Theorem 2.4, we see that the sequence $\{x_n\}$ is a Cauchy sequence. By completeness of X_ρ, $\{x_n\}$ converges to some point $x \in X_\rho$. By the condition (c), we have

$$\rho(x - Tx) = \lim_{n\to\infty} \rho(x_n - Tx_n) = \lim_{n\to\infty} \rho(x_n - x_{n+1}) = 0.$$

That is, $x = Tx$. Hence, T has a fixed point and $Fix(T^n) = Fix(T)$ is true for $n = 1$. Let $n > 1$ and assume, by contradiction, that $x \in Fix(T^n)$ and $x \notin Fix(T)$, such that $\rho(x - Tx) > 0$. Now, applying (d) and (a), we have

$$\begin{aligned}\tau + F(\rho(x - Tx)) &\leqslant \tau + F(\rho(T(T^{n-1}x) - T^2(T^{n-1}x)))\\ &\leqslant \alpha(T^{n-1}x)\beta(T^n x)F(\rho(T(T^{n-1}x) - T^2(T^{n-1}x)))\\ &\leqslant F(\rho(T^{n-1}x - T^n x)))\end{aligned}$$

Consequently we have

$$\begin{aligned}F(\rho(x - Tx)) &\leqslant F(\rho(T^{n-1}x - T^n x))) - \tau\\ &\leqslant F(\rho(T^{n-2}x - T^{n-1}x))) - 2\tau\\ &\quad ...\\ &\leqslant F(\rho(x - Tx))) - n\tau.\end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality, we have $F(\rho(x-Tx)) = -\infty$, hence $\rho(x - Tx) = 0$, which is a contradiction. So, $x = Tx$. Hence $Fix(T^n) = Fix(T)$. Taking $\alpha(x)\beta(y) = 1$ for all $x, y \in X_\rho$ in Theorem 3.1, we get the following result. ∎

Corollary 3.2 *Let X_ρ be a ρ-complete modular space and $T : X_\rho \to X_\rho$ be a continuous mapping satisfying $\rho(Tx - T^2x) > 0$ such that*

$$\tau + F(\rho(Tx - T^2x)) \leqslant F(\rho(x - Tx))$$

for some $\tau > 0$ and for all $x \in X_\rho$. Then $Fix(T^n) = Fix(T)$ for all $n \in \mathbb{N}$.

References

[1] S. Alizadeh, F. Moradlou and P. Salimi, Some fixed point results for $(\alpha, \beta) - (\psi, \varphi)$ -contractive mappings, Filomat, **28 (3)** (2014) 635-647.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales, Fundamenta Mathematicae, **3 (1)** (1922) 133-182.

[3] M. Beygmohammadi and A. Razani, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type in the modular space, nternational Journal of Mathematics and Mathematical Sciences, **2010** (2010), Article ID 317107, doi:10.1155/2010/317107.

[4] A. Hajji and E. Hanebaly, Perturbed integral equations in modular function spaces, Electronic Journal of Qualitative Theory of Diferential Equations, **2003 (3)** (2003) 1-7.

[5] H. Isik, N. Hussain and M. A. Kutbi, Generalized rational contractions endowed with a graph and an application to a system of integral equations, Journal of Computational Analysis and Applications, **22 (6)** (2017) 1158-1175.

[6] H. Isik and P. Kumam, Fixed points under new contractive conditions via cyclic (α, β, r)-admissible mappings, Journal of Advanced Mathematical Studies, **11 (1)** (2018) 17-23.

[7] H. Isik and S. Radenovic, A new version of coupled fixed point results in ordered metric spaces with applications, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, **79 (2)** (2017) 131-138.

[8] H. Isik and D. Turkoglu, Generalized weakly α-contractive mappings and applications to ordinary differential equations, Miskolc Mathematical Notes, **17 (1)** (2016) 365-379.

[9] M. A. Khamsi, W. M. Kozlowski and S. Reich, Fixed point theory in modular function spaces, Nonlinear Analysis: Theory, Methods & Applications **14 (11)** (1990) 935-953.

[10] W. M. Kozlowski, Modular function spaces. Marcel Dekker. 1988, MR 99d:46043.
[11] C. Mongkolkeha and P. Kumam, Some fixed point results for generalized weak contraction mappings in modular spaces, International Journal of Analysis, **2013** (2013), Article ID 247378, doi:10.1155/2013/247378.
[12] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Mathematics, Springer-Verlag, Berlin, (1983).
[13] J. Musielak and W. Orlicz, On modular spaces, Studia Mathematica, **18 (1)** (1959) 49-65.
[14] J. Musielak and W. Orlicz, Some remarks on modular spaces, Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys., **7** (1959) 661-668.
[15] H. Nakano, Modulared semi-ordered linear spaces. Maruzen Co., Tokyo, (1950).
[16] H. Nakano, Modulared sequence spaces, Proceedings of the Japan Academy, **27 (9)** (1951) 508-512.
[17] A. A. Taleb and E. Hanebaly, A fixed point theorem and its application to integral equations in modular function spaces, Proceedings of the American Mathematical Society, **128 (2)** (2000) 419-426.
[18] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Applications, **2012 (1)** (2012), doi:10.1186/1687-1812-2012-94.
[19] D. Wardowski and N. Van Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstratio Mathematica, **47 (1)** (2014) 146-155.