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Modeling and Analysis of Vehicles Flows on the Road

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Abstract. This study is carried out to describe the behaviour of vehicles flow on the road, in the presence of blocking effects. A non-linear three dimensional system of ordinary differential equations is used to describe vehicles flow on the road. The study classify total vehicles population on the road into three compartments as Free-Slow-Released vehicles. The formulated model is well-posed. The blocking free equilibrium point is globally asymptotically stable. Further, effects of blocking are described using concept of retardation number. That is, blocking effect decrease whenever retardation number is less than one and the blocking effects increase if retardation number is greater than one.

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1. Introduction

Traffic flow is one of the critical issues for both developing and developed countries. It is well known that, if the disease in the population is not managed properly, then it can invade the whole population. Likewise, poor management of blocking effects on the road can bring both economical and human life destruction [6,7]. In in all over the world, we hear accident's news that caused by blocking effects on the road. Thus, this paper is done to mathematically describe how the flow of vehicles carried out on the road, so that this study creates some understanding for those involved in the traffic flows. The deterministic approach was applied to human population in order to study the persistence and extinction of a disease entered the population [4,5,8,9,11]. This study describes a mathematical model of three compartments that describe the flow of vehicles flow on the road in terms of blocking effects similar to the works done in [6,7]. Blocking is the act of preventing the vehicles to flow freely on the road. This is the strongest controller of vehicles flow on the road and needs great transportation management. Most of the time people with epidemics disease are treated and cured with proper medication and assistance given by doctors. But, most of the time the accident caused by this blocking does not give time to anyone to be treated. Thus, it requires high cares from the beginning to the end of the driving [1,2,3,10,12].

Motivation of the study:

Comparison of model of epidemiology with model of traffic flow

Commonly, in epidemiology the dynamics of diseases can be described using deterministic model of ODE [6]. Now this study extended the deterministic model approach or compartmental

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or compartmental approach to the vehicles population. This study is the modification of works done in [7]. The previous study is focused on four compartments of vehicles population whereas the current work concentrated on three compartmental model of vehicles.

Now, let us describe characteristics of variables in epidemiology and traffic flow which is new approach to describe dynamics of vehicles on the road. In epidemiology, deterministic models can be formulated by dividing total human population into three compartments i.e., Susceptible-Infected-Recovered (SIR).

(i) The Susceptible compartment encloses disease free people but can get disease if exposed to it.
(ii) The Infectious compartment encloses people who infected by the disease and can propagate disease to others in allowable contact.
(iii) The Recovered compartment encloses people got affected by the disease recovered from the diseseaset by some means like natural cure, treatment etc.

In analogous methodology used to construct epidemiological deterministic models in this study the vehicles flow on a road (or drivers driving vehicles) are categorized to three compartments i.e., Free-Slow-Released (FSR).

(i) The Free compartment contains unblocked vehicles bot can be blocked if they face blocking vehicles.
(ii) The Slow compartment contains vehicles with low blocking. These vehicles can block others in allowable distance of blocking if they meet vehicles on the road.
(iii) The Released compartment contains blocking free vehicles after blockage.

Thus, observing that the blocking effects in traffic flow is continuously persisting with human life this study is aimed to put platform for further study of traffic flows on the road so that the advanced concepts of ODE can describe better understanding than that of PDE description of traffic flow models. In this study, in addition to what described above about vehicles flow related to blocking, different terms are used in the following sections.

2. Model formulation

The deterministic model formulated in this study divides total vehicles population available on the road into three compartments. These are described as: Free vehicles $F(t)$, Slow vehicles $S(t)$, and Released vehicles $R(t)$. So that,

$$N(t) = F(t) + S(t) + R(t) \quad (1)$$

Here in (1), (i) $N(t)$ denotes total population size of vehicles (ii) $F(t)$ denotes population size of Free Vehicles. These are vehicles that move without the influence of blocking on the road but have a chance or possibility to be blocked in the future. (ii) $S(t)$ denotes the population size of slow vehicles which are partially blocked and are moving under the influence of blockings and can block others in possible allowable distance between vehicles. (iii) $R\ (t)$ denotes the population size of Released vehicles which are just released from blockings and have a chance of experiencing blockings again.

Table 1. Description of Model Variables.

Variable	Description pertaining to traffic flow	Description pertaining to epidemiology
$F(t)$	Population size of Free vehicles	Size of Susceptible population
$S(t)$	Population size of Slow vehicles	Size of Infectious population
$R(t)$	Population size of Released vehicles	Size of Recovered population

Table 2. Description of Model Parameters.

Parameter	Description pertaining to traffic flow (FSBD)	Description pertaining to epidemiology (SEIR)
τ	Recruitment rate is the rate at which new vehicles joining the road (OR) growth rate of free vehicles	Constant birth rate of susceptible population
β	Rate of free vehicles becoming Slow vehicles. That is, with this rate the free vehicles are experiencing blockings and are running reduced speed.	Transmission rate of infection to Susceptible population
γ	Transferring rate of Slow vehicles to Released vehicles.	Recovery rate
ω	Transferring rate of Released vehicles to Susceptible vehicles.	Transferring rate of recovered population to susceptible compartment.
ν	Natural outflow	Natural death rate

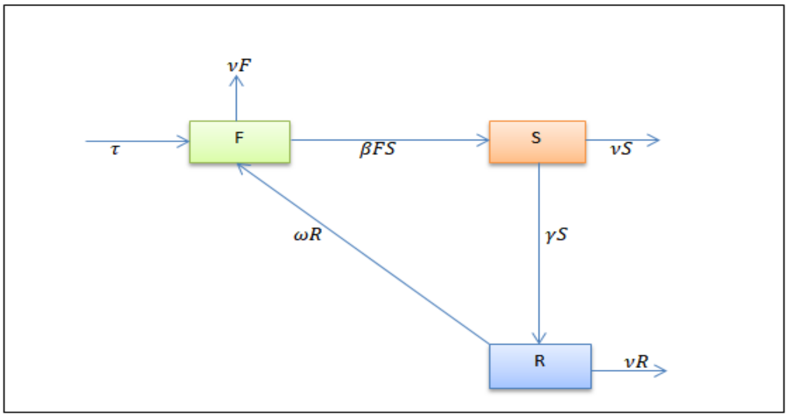

Figure 1. Schematic diagram of vehicles dynamics with blocking on the road.

Based on the assumptions listed above and the diagram given in Figure 1, the mathematical model describing the dynamics of population sizes of various vehicles pertaining to a traffic flow on a road can be expressed as a system of nonlinear differential equations as

$$dF/dt = \tau - \beta FS - \nu F + \omega R$$

$$dS/dt = \beta FS - \gamma S - \nu S$$

$$dR/dt = \gamma S - \nu R - \omega R$$

with initial conditions, $F(0)$, $S(0)$, and $R(0)$.

3. Analysis of the Model

In this section mathematical analysis of the model (2)-(4) is carried out. The analysis comprises of the following features: (i) Existence, positivity and boundedness of solutions (ii) Equilibrium points (iii) Blocking Free equilibrium points (iv) Endemic equilibrium points (v) Basic retardation number (vi) Stability analysis of the blocking free equilibrium points (vii) Local stability of blocking free equilibrium point (viii) Global stability of blocking free equilibrium point. These mathematical aspects are presented and explained in the following sub-sections respectively.

3.1 *Existence, positivity and boundedness of solution*

In order that the model equations (2)-(4) have physically valid meaning for the modeled problem of traffic flow and Mathematically well posed the following theorems are stated and proved.

3.1.1 Positivity of the solutions

Theorem 3.1 *If the initial conditions $F(0)$, $S(0)$, and $R(0)$ are non-negative then the solution region $R = \{F(t), S(t), R(t)\}$ of the system of equations* (2)-(4) *is non-negative.*

Proof To show that the solution of (2)-(4) is non-negative here each model equation of the dynamical system is considered separately and shown that it has a non-negative solution as follows:

Positivity of $F(t)$: Consider the model equation (2) given by $dF/dt = \tau - \beta FS - \nu F + \omega R$ which without loss of generality, after discarding the positive terms τ and ωR, it can be expressed as an inequality as $dF/dt \geq -(\beta S + \nu)F$. This differential inequality, being first order and linear, can be solved easily to find its solution as $F(t) \geq F(0)\ Exp\left\{-\int_0^t [\beta S + \nu]dt\right\}$. Here the integral constant $F(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\left\{-\int_0^t [\beta S + \nu]dt\right\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $F(t)$ is a non-negative quantity i.e., $F(t) \geq 0$.

Positivity of $S(t)$: Consider the model equation (3) given by $dS/dt = \beta FS - \gamma S - \nu S$ which without loss of generality, after discarding the positive term βFS, it can be expressed as an inequality as $dS/dt \geq -(\gamma + \nu)S$. This differential inequality, being first order and linear, can be solved easily to find its solution as $S(t) \geq S(0)\ Exp\{-(\gamma + \nu)t\}$. Here, the integral constant $S(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\{-(\gamma + \nu)t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $S(t)$ is a non-negative quantity i.e., $S(t) \geq 0$ for all $t \in [0, \infty)$.

Positivity of $R(t)$: Consider the model equation (4) given by $dR/dt = \gamma S - \nu R - \omega R$ which without loss of generality, after discarding the positive term γS can be expressed as an inequality as $dR/dt \geq -(\nu + \omega)R$. This differential inequality, being first order and linear, can be solved easily to find its solution as $R(t) \geq R(0)\ Exp\{-(\nu + \omega)t\}$. Here the integral constant $R(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\{-(\nu + \omega)t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $R(t)$ is a non-negative quantity i.e., $R(t) \geq 0$ for all $t \in [0, \infty)$.
Since, obviously exponential expressions are positive and initial conditions are non-negative it can be concluded that the solutions region D is a set containing non-negative quantities. Thus, solution region can be written as,

$$D = \{(F(t), S(t), R(t))\ \exists\ F(t) \geq 0,\ S(t) \geq 0,\ R(t) \geq 0,\ \forall t \in [0, \infty)\}. \quad \blacksquare$$

3.1.2 Boundedness of the solutions region

In order to make the formulated model is valid and well posed it is also necessary to show that the solutions region is bounded. This fact has been stated as Theorem 3.2 and verified in its proof following [7].

Theorem 3.2 *The non-negative solutions region* $D = \{F(t), S(t), R(t)\}$ *of the system of equations* (2)-(4) *is bounded i.e.,* $N(t) \leq \tau/\nu$.

Proof To show that the solutions region of the system of equations (2)-(4) is bounded, it is sufficient if it is shown that the total population is bounded. Now adding all terms on the right and left sides of equations in dynamical system (2)-(4) gives the resultant equation as

$$(dF/dt) + (dS/dt) + (dR/dt)$$
$$= \tau - \beta FS - \nu F + \omega R + \beta FS - \gamma S - \nu S + \gamma S - \nu R - \omega R$$

After discarding the zero-valued-pairs and using the fact $N(t) = F(t) + S(t) + P(t) + R(t)$ as given in (1), the foregoing equation reduces to a simplified form as follows:

$$dN/dt = \tau - \nu N$$

Now, for the first order linear equation with constant coefficients $dN/dt = \tau - \nu N$. It is straight forward to find the complete solution as $N(t) = (\tau/\nu) - [(\tau/\nu) - N(0)]e^{-\nu t}$. Here $N(0)$ is the initial size of all categories of vehicles on the road. It follows that $N(t)$ is bounded as $t \to \infty$ i.e., $N(t) \leq (\tau/\nu)$ provided that the condition $N(0) \leq (\tau/\nu)$ is satisfied. Thus, it can be concluded that the solutions region $D = \{F(t), S(t), R(t)\}$ is bounded i.e., $N(t) \leq \tau/\nu$. ■

3.1.3 Existence and uniqueness of the solutions

Here it is to show that a solution for the system (2)-(4) exists and is unique following the procedure given in Derric and Grossman (1976). The existence and uniqueness of the solution can be stated as shown in Theorem 3.3.

Theorem 3.3 *Consider a system of* n *first order differential equations of the type* $x_i' = f_i(x_1, x_2, x_3 \ldots, x_n, t)$ *together with the initial conditions* $x_i(t_0) = x_{i0}$ *where* $i = 1, \ldots, n$. *Let D denotes a region in* $(n+1)$*-dimensional space among which one dimension is for* t *and* n *dimensions are for the vector* x. *If all the partial derivatives* $\partial f_i/\partial x_j$ *for all* $i, j = 1,2, \ldots n$ *are continuous in* $D = \{(x,t), |t - t_0| \leq a, |x - x_0| \leq b\}$ *then there is exists a constant* $\delta > 0$ *such that there a unique continuous vector solution* $x^* = [x_1(t), x_2(t), \ldots, x_n(t)]$ *in the interval* $|t - t_0| \leq \delta$ *for the system of* n *equations.*

Now accordingly let state the theorem and prove

$$dF/dt = \tau - \beta FS - \nu F + \omega R \tag{2}$$
$$dS/dt = \beta FS - \gamma S - \nu S \tag{3}$$
$$dR/dt = \gamma S - \nu R - \omega R \tag{4}$$

Theorem 3.4 *There exists a unique solution to the system of equations* (2)-(4).

Proof The statement here is proved following the procedure given in Theorem 3. Now, the system of equations (2)-(4) together with the initial conditions can be expressed as

$$dF/dt = \tau - \beta FS - \nu F + \omega R \equiv f_1, \qquad F(t_0) = F_0$$
$$dS/dt = \beta FS - \gamma S - \nu S \equiv f_2, \qquad S(t_0) = S_0$$
$$dR/dt = \gamma S - \nu R - \omega R \equiv f_3, \qquad R(t_0) = R_0$$

To show existence and boundedness let us show that partial derivatives of functions f_i with respect to variables be continuous and bounded, where $i = 1,2,3$.

Table 3. Verification of continuity and boundedness of the partial derivatives.

Function	Continuity	Boundedness
f_1	$\partial f_1/\partial F = -\beta S - \nu$ $\partial f_1/\partial S = -\beta F$ $\partial f_1/\partial R = \omega$	$\lvert\partial f_1/\partial F\rvert = \lvert-\beta S - \nu\rvert < \infty$ $\lvert\partial f_1/\partial S\rvert = \beta F < \infty$ $\lvert\partial f_1/\partial R\rvert = \omega < \infty$
f_2	$\partial f_2/\partial F = \beta S$ $\partial f_2/\partial S = \beta F - \gamma - \nu$ $\partial f_2/\partial R = 0$	$\lvert\partial f_2/\partial F\rvert = \lvert\beta S\rvert < \infty$ $\lvert\partial f_2/\partial S\rvert = \lvert\beta F - \gamma - \nu\rvert < \infty$ $\lvert\partial f_2/\partial R\rvert = 0 < \infty$
f_3	$\partial f_3/\partial F = 0$ $\partial f_3/\partial S = \gamma$ $\partial f_3/\partial R = -(\nu + \omega)$	$\lvert\partial f_3/\partial F\rvert = 0 < \infty$ $\lvert\partial f_3/\partial S\rvert = \gamma < \infty$ $\lvert\partial f_3/\partial R\rvert = \nu + \omega < \infty$

Hence, by Derric and Grossman 1976 the solution exists and is unique. ■

3.2 *Equilibrium points*

In order to have a better understanding about the dynamics of a model, the equilibrium points of the solution region are to be identified and their stability analysis is to be conducted. In this section such identification and analysis are conducted.

An equilibrium solution is a steady state solution of the model equations (2)-(4). That is, if the system begins at such a state, it will remain there all the times for any disturbance occurs. In other words, the population sizes remain unchanged and thus the rate of change for each population vanishes. Equilibrium points of the model are found, categorized, stability analysis is conducted and the results have been presented in the following:

3.2.1 Blocking free equilibrium BFE

At blocking free equilibrium vehicles flow freely without any interference of any kind of blockings. That is, at this equilibrium point vehicles will run freely with speeds as per the wish of their drivers. Furthermore, at this equilibrium no vehicle is forced either to run with slower speeds or to stop completely. That is, $S = 0$. Thus, under this assumption the solutions of system of equations (2)-(4) is given by:

$$F = \tau/\nu\,, \;\; S = 0, \quad R = 0$$

Thus, blocking free equilibrium BFE of the model is obtained as

$$E_0 = (\tau/\nu\,, 0,0)$$

3.3 *Derivation of basic retardation number* $(\boldsymbol{R_0})$

Basic retardation number has similar meaning to that of basic reproduction number in Epidemiology. It is described as the average number of secondary blocked vehicles generated by one blocked vehicle in fully freely flowing vehicles. Calculating retardation number is important to analyze the local stability of nonlinear system of equations (2)-(4).

The retardation number is the largest eigenvalue of the matrix $K = \mathcal{F}V^{-1}$, where,

$$\mathcal{F} = \left(\partial f/\partial x_j\right)|_{E_0}, \qquad V = \left(\partial v/\partial x_j\right)|_{E_0} \tag{5}$$

Here, f is the newly blocking terms and v is non-singular matrix of the remaining transfer terms. Now, the basic retardation number R_0 of the model (2)-(4) is computed using the next generation matrix in similar procedure as reproduction number in epidemiological concept used to be computed. Thus, the next generation matrices $\mathcal{F}\,V^{-1}$can straightforwardly constructed and obtained follows:

$$K = \mathcal{F}\,V^{-1} = \frac{\beta\tau}{\nu(\gamma + \nu)}$$

Hence, the Retardation number of the model is given by

$$R_0 = \rho(\mathcal{F}V^{-1}) = \frac{\beta\tau}{\nu(\gamma + \nu)}$$

3.4 *Stability analysis of equilibrium*

In the absence of blockings, the traffic flow model will have a unique blocking free steady state E_0. To find the local stability of E_0 , the Jacobian matrix of the model equations valued at blocking free equilibrium point$E_0$is used. It is already shown that the BFE of model (2)-(4) is given by $E_0 = (\tau/\nu\,, 0{,}0)$. Now, the stability analysis of BFE is conducted and the results are presented in the form of theorems.

Theorem 3.5 *Let E_0 is a BFE of the system of equations and all eigenvalues of Jacobian matrix at E_0 are negative, then E_0 is locally asymptotically stable if $R_0 = \rho(FV^{-1}) < 1$, but unstable if $R_0 > 1$.*

Theorem 3.6 *If $(V - \mathcal{F})$ is non-singular M-matrix and $(F_0 - F)B \geq 0$, then the blocking free equilibrium is globally asymptotically stable for $\rho(\mathcal{F}V^{-1}) < 1$.*

Proof Following procedures done in [7,9] and the computations done above, it is observable that $\mathcal{F}$ is non-negative and V is non-singular M-matrix. Further, it is clear that $(V - \mathcal{F})$ is non-singular M-matrix for $\rho(\mathcal{F}V^{-1}) < 1$. Now, to show that the blocking-free equilibrium is globally asymptotically stable for $R_0 < 1$, it is sufficient to show that $F \leq F_0$. From the total population $N(t)$ we have, $N(t) = F(t) + E(t) + S(t) + A(t) + R(t)$ which satisfies $N'(t) = \tau - \nu N(t)$, so that $N(t) = F_0 - (F_0 - N(0))e^{-\nu t}$, with $F_0 = \tau/\nu$. If $N(0) \leq F_0$, then $F(t) \leq N(t) \leq F_0$ for all time. If, on the other hand, $N(0) > F_0$, then N(t) decays exponentially to F_0, and either $F(t) \to F_0$, or there is some time T after which $F(t) < F_0$. Thus, the blocking free equilibrium is globally asymptotically stable for $\rho(\mathcal{F}V^{-1}) < 1$. ■

4. Results and discussion

The blocking has high effect on the flow of the vehicles. The more blocked vehicles available on the road the more time usage it takes for the passengers and drivers on the road. Less blocked vehicles the more free flowing vehicles on the road.

5. Conclusions

In this study, modeling vehicles flow on the road is described with non-linear three dimensional system of differential equations. Moreover, existence, positivity and boundedness of the formulated model is verified to illustrate that the model is physically meaningful and mathematically well posed. In particular, the stability analyses of the model were investigated using the basic reproduction number and Routh Hurwitz criterion. Blocking reduces number of freely flowing vehicles.

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