*International Journal of*
*Mathematical Modelling & Computations*
Vol. 09, No. 02, Spring 2019, 155- 164

# Initial Value Problems for Fourth-Order Fuzzy Differential Equations by Fuzzy Laplace Transform

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**Abstract.** This paper is on the solutions of fuzzy initial value problems for fourth-order fuzzy differential equations with positive and negative fuzzy coefficients by fuzzy Laplace transform. Examples are solved. Conclusions are given.

Received: 30 March 2019, Revised: 10 May 2019, Accepted: 28 June 2019.

**Keywords:** Fuzzy differential equation; Fuzzy Laplace transform; Positive fuzzy number; Negative fuzzy number.

**AMS Subject Classification**: 34A07, 44A10.

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## 1. Introduction

Fuzzy differential equations are important topic. Thus, many researchers study fuzzy differential equation. Especially, to solve fuzzy differential equation is useful by fuzzy Laplace transform. In many papers, fuzzy Laplace transform was used [2, 4, 8, 9, 11].

In this paper, we study the solutions of the fuzzy problem

$$y^{(iv)}(t) + [\lambda]^{\alpha} y''(t) = [\mu]^{\alpha}, \tag{1}$$

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$$y(0) = [A]^{\alpha},\ y^{'}(0) = [B]^{\alpha},\ y^{''}(0) = [C]^{\alpha},\ y^{'''}(0) = [D]^{\alpha} \tag{2}$$

by the fuzzy Laplace transform for positive and negative fuzzy number coefficients, where

$$[\lambda]^{\alpha} = \left[\underline{\lambda}_{\alpha}, \overline{\lambda}_{\alpha}\right],\ [\mu]^{\alpha} = \left[\underline{\mu}_{\alpha}, \overline{\mu}_{\alpha}\right],\ [A]^{\alpha} = \left[\underline{A}_{\alpha}, \overline{A}_{\alpha}\right],$$
$$[B]^{\alpha} = \left[\underline{B}_{\alpha}, \overline{B}_{\alpha}\right],\ [C]^{\alpha} = \left[\underline{C}_{\alpha}, \overline{C}_{\alpha}\right],\ [D]^{\alpha} = \left[\underline{D}_{\alpha}, \overline{D}_{\alpha}\right]$$

are symmetric triangular fuzzy numbers, $y(t)$ is positive fuzzy function, $y$, $y^{'}, y^{''}, y^{'''}$ are (i)-differentiable.

## 2. Preliminaries

**Definition 2.1** ([7]) A fuzzy number is a mapping $u : \mathbb{R} \to [0,1]$ satisfying the properties $\overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact, u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$.

Let $\mathbb{R}_F$ show the set of all fuzzy numbers.

**Definition 2.2** ([6]) Let be $u \in \mathbb{R}_F$. $[u]^{\alpha} = [\underline{u}_{\alpha}, \overline{u}_{\alpha}] = \{x \in \mathbb{R} \mid u(x) \geqslant \alpha\}, 0 < \alpha \leqslant 1$ is $\alpha$-level set of $u$. If $\alpha = 0$, $[u]^{0} = cl\,\{\text{suppu}\} = cl\,\{x \in \mathbb{R} \mid u(x) > 0\}$.

**Remark 2.3** ([6]) The parametric form $[\underline{u}_{\alpha}, \overline{u}_{\alpha}]$ of a fuzzy number satisfying the following requirements is a valid $\alpha$-level set.
$\underline{u}_{\alpha}$ is bounded monotonic increasing (nondecreasing) left-continuous $\forall \alpha \in (0,1]$ and right-continuous for $\alpha = 0$,
$\overline{u}_{\alpha}$ is bounded monotonic decreasing (nonincreasing) left-continuous $\forall \alpha \in (0,1]$ and right-continuous for $\alpha = 0$,
$\underline{u}_{\alpha} \leqslant \overline{u}_{\alpha}$, $0 \leqslant \alpha \leqslant 1$.

**Definition 2.4** ([12]) A fuzzy number $u$ is called positive (negative), denoted by $u > 0$ $(u < 0)$, if its membership function $u(x)$ satisfies $u(x) = 0$, $\forall x < 0$ $(x > 0)$.

**Definition 2.5** ([6, 10]) Let be $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, $w$ is the Hukuhara difference of $u$ and $v$, $w = u \ominus v$.

**Definition 2.6** ([3, 6]) Let be $f : [a,b] \to \mathbb{R}_F$ and $x_0 \in [a,b]$. If there exists $f^{'}(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits hold

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^{'}(x_0),$$

f is Hukuhara differentiable at $x_0$.

**Definition 2.7** ([6]) Let be $f : [a,b] \to \mathbb{R}_F$ and $x_0 \in [a,b]$. If there exists $f^{'}(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits hold

$$\lim_{h \to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f^{'}(x_0),$$

f is (i)-differentiable at $x_0$. If there exists $f'(x_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h)$, $f(x_0-h) \ominus f(x_0)$ and the limits hold

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = \lim_{h\to 0} \frac{f(x_0-h) \ominus f(x_0)}{-h} = f'(x_0),$$

f is (ii)-differentiable.

**Theorem 2.1** ([5]) Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy function and denote $[f(x)]^\alpha = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right]$, for each $\alpha \in [0,1]$.

1) If f is (i)-differentiable, $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable, $\left[f'(x)\right]^\alpha = \left[\underline{f}'_\alpha(x), \overline{f}'_\alpha(x)\right]$,

2) If f is (ii)-differentiable, $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable, $\left[f'(x)\right]^\alpha = \left[\overline{f}'_\alpha(x), \underline{f}'_\alpha(x)\right]$.

**Theorem 2.2** ([5]) Let $f' : [a,b] \to \mathbb{R}_F$ be fuzzy function and denote $[f(x)]^\alpha = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right]$, for each $\alpha \in [0,1]$, the function f is (i)-differentiable or (ii)-differentiable.

1) If f and f$'$ are (i)-differentiable, $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable, $\left[f''(x)\right]^\alpha = \left[\underline{f}''_\alpha(x), \overline{f}''_\alpha(x)\right]$,

2) If f is (i)-differentiable and $f'$ is (ii)-differentiable, $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable, $\left[f''(x)\right]^\alpha = \left[\overline{f}''_\alpha(x), \underline{f}''_\alpha(x)\right]$,

3) If f is (ii)-differentiable and $f'$ is (i)-differentiable, $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable, $\left[f''(x)\right]^\alpha = \left[\overline{f}''_\alpha(x), \underline{f}''_\alpha(x)\right]$,

4) If f and f$'$ are (ii)-differentiable, $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable, $\left[f''(x)\right]^\alpha = \left[\underline{f}''_\alpha(x), \overline{f}''_\alpha(x)\right]$.

**Definition 2.8** ([11]) Let $f : [a,b] \to \mathbb{R}_F$ be fuzzy function. The fuzzy Laplace transform of f is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \left[\lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}(t)\,dt, \lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}(t)\,dt\right].$$

$$F(s,\alpha) = L\left((f(t))^\alpha\right) = \left[L\left(\underline{f}_\alpha(t)\right), L\left(\overline{f}_\alpha(t)\right)\right].$$

$$L\left(\underline{f}_\alpha(t)\right) = \int_0^\infty e^{-st}\underline{f}_\alpha(t)\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}_\alpha(t)\,dt,$$

$$L\left(\overline{f}_\alpha(t)\right) = \int_0^\infty e^{-st}\overline{f}_\alpha(t)\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}_\alpha(t)\,dt.$$

**Theorem 2.3** ([1]) Suppose that $f, f', ..., f^{(n-1)}$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $f^{(n)}$ is piecewise continuous fuzzy-valued function on $[0,\infty)$ . Then

$$L\left(f^{(n)}(t)\right) = s^n L(f(t)) \ominus s^{n-1} f(0) \ominus s^{n-2} f'(0) \ominus s^{n-3} f''(0) \ominus ... \ominus f^{(n-1)}(0),$$

if $f, f', ..., f^{(n-1)}$ are (i)-differentiable.

$$L\left(f^{(n)}(t)\right) = \ominus\left(f^{(n-1)}(0)\right) \ominus (-s^n) L(f(t)) \ominus s^{n-1} f(0)$$
$$\ominus s^{n-2} f'(0) \ominus ... \ominus s^{n-(n-1)} f^{(n-2)}(0),$$

if $f, f', ..., f^{(n-2)}$ are (i)-differentiable and $f^{(n-1)}$ is (ii)-differentiable.

$$L\left(f^{(n)}(t)\right) = \ominus(s^{n-(n-1)} f^{(n-2)}(0)) \ominus f^{(n-1)}(0) \ominus (-s^n) L(f(t)) \ominus s^{n-1} f(0)$$
$$\ominus s^{n-2} f'(0) \ominus ... \ominus \left(s^{n-(n-2)}\right) f^{(n-3)}(0),$$

if $f, f', ..., f^{(n-3)}$ are (i)-differentiable and $f^{(n-1)}$, $f^{(n-2)}$ are (ii)-differentiable. Similarly

$$L\left(f^{(n)}(t)\right) = \ominus(s^{n-1} f(0)) \ominus (-s^n) L(f(t)) \ominus s^{n-2} f'(0) \ominus ... \ominus f^{(n-1)}(0),$$

if $f', ..., f^{(n-1)}$ are (i)-differentiable and $f$ is (ii)-differentiable.

Continuing the process until we obtain $2^n$ system of differential equations, hence according to [2] the last equation is

$$L\left(f^{(n)}(t)\right) = s^n L(f(t)) \ominus s^{n-1} f(0) \ominus s^{n-2} f'(0) \ominus s^{n-3} f''(0) ... - f^{(n-1)}(0),$$

if $f, f', ..., f^{(n-1)}$ are (ii)-differentiable.

**Theorem 2.4** ([2]) Let $f(t)$, $g(t)$ be continuous fuzzy-valued functions and $c_1$ and $c_2$ constants, then

$$L(c_1 f(t) + c_2 g(t)) = (c_1 L(f(t))) + (c_2 L(g(t))).$$

## 3. Main results

### 3.1 *The problem with positive fuzzy coefficient*

Let be $[\lambda]^\alpha$ is positive fuzzy number. Taking the fuzzy Laplace transform of the equation (1), we have the equations

$$s^4 \underline{Y}_\alpha(s) - s^3 \underline{y}_\alpha(0) - s^2 \underline{y}'_\alpha(0) - s\underline{y}''_\alpha(0) - \underline{y}'''_\alpha(0)$$

$$+\underline{\lambda}_\alpha\left(s^2 \underline{Y}_\alpha(s) - s\underline{y}_\alpha(0) - \underline{y}'_\alpha(0)\right) = \frac{\underline{\mu}_\alpha}{s},$$

$$s^4\overline{Y}_\alpha(s) - s^3\overline{y}_\alpha(0) - s^2\overline{y}_\alpha^{'}(0) - s\overline{y}_\alpha^{''}(0) - \overline{y}_\alpha^{'''}(0)$$

$$+\overline{\lambda}_\alpha\left(s^2\overline{Y}_\alpha(s) - s\overline{y}_\alpha(0) - \overline{y}_\alpha^{'}(0)\right) = \frac{\overline{\mu}_\alpha}{s}.$$

Using the initial conditions (2),

$$\underline{Y}_\alpha(s) = \frac{\underline{\mu}_\alpha}{s^3\left(s^2+\underline{\lambda}_\alpha\right)} + \frac{\underline{A}_\alpha}{s} + \frac{\underline{B}_\alpha}{s^2} + \frac{\underline{C}_\alpha}{s\left(s^2+\underline{\lambda}_\alpha\right)} + \frac{\underline{D}_\alpha}{s^2\left(s^2+\underline{\lambda}_\alpha\right)},$$

$$\overline{Y}_\alpha(s) = \frac{\overline{\mu}_\alpha}{s^3\left(s^2+\overline{\lambda}_\alpha\right)} + \frac{\overline{A}_\alpha}{s} + \frac{\overline{B}_\alpha}{s^2} + \frac{\overline{C}_\alpha}{s\left(s^2+\overline{\lambda}_\alpha\right)} + \frac{\overline{D}_\alpha}{s^2\left(s^2+\overline{\lambda}_\alpha\right)}$$

are obtained. From this, the solution is

$$\begin{aligned}\underline{y}_\alpha(t) &= \underline{A}_\alpha + \frac{\underline{D}_\alpha}{\underline{\lambda}_\alpha} - \frac{\underline{\mu}_\alpha}{\underline{\lambda}_\alpha^2} + \left(\underline{B}_\alpha + \frac{\underline{C}_\alpha}{\underline{\lambda}_\alpha}\right)t + \frac{\underline{\mu}_\alpha}{\underline{\lambda}_\alpha}t^2 \\ &+ \left(\frac{\underline{\mu}_\alpha}{\underline{\lambda}_\alpha^2} - \frac{\underline{D}_\alpha}{\underline{\lambda}_\alpha}\right)\cos\left(\underline{\lambda}_\alpha t\right) - \frac{\underline{C}_\alpha}{\sqrt{\left(\underline{\lambda}_\alpha\right)^3}}\sin\left(\underline{\lambda}_\alpha t\right),\end{aligned}$$

$$\begin{aligned}\overline{y}_\alpha(t) &= \overline{A}_\alpha + \frac{\overline{D}_\alpha}{\overline{\lambda}_\alpha} - \frac{\overline{\mu}_\alpha}{\overline{\lambda}_\alpha^2} + \left(\overline{B}_\alpha + \frac{\overline{C}_\alpha}{\overline{\lambda}_\alpha}\right)t + \frac{\overline{\mu}_\alpha}{\overline{\lambda}_\alpha}t^2 \\ &+ \left(\frac{\overline{\mu}_\alpha}{\overline{\lambda}_\alpha^2} - \frac{\overline{D}_\alpha}{\overline{\lambda}_\alpha}\right)\cos\left(\overline{\lambda}_\alpha t\right) - \frac{\overline{C}_\alpha}{\sqrt{\left(\overline{\lambda}_\alpha\right)^3}}\sin\left(\overline{\lambda}_\alpha t\right),\end{aligned}$$

$$[y(t)]^\alpha = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right].$$

**Example 3.1** Consider the problem

$$y^{(iv)}(t) + [1]^\alpha y^{''}(t) = [2]^\alpha,$$

$$y(0) = [0]^\alpha, \quad y^{'}(0) = [1]^\alpha, \ y^{''}(0) = [2]^\alpha, \ y^{'''}(0) = [3]^\alpha$$

by fuzzy Laplace transform, where $[0]^\alpha = [-1+\alpha, 1-\alpha], [1]^\alpha = [\alpha, 2-\alpha],$ $[2]^\alpha = [1+\alpha, 3-\alpha], [3]^\alpha = [2+\alpha, 4-\alpha].$

The solution of the fuzzy problem is

$$\underline{y}_\alpha(t) = -1+\alpha+\left(\frac{\alpha^2+\alpha+1}{\alpha}\right)t+\frac{1+\alpha}{\alpha}t^2 + \left(\frac{1-\alpha-\alpha^2}{\alpha^2}\right)(\cos(\alpha t)-1)-\left(\frac{1+\alpha}{\sqrt{\alpha^3}}\right)\sin(\alpha t),$$

$$\overline{y}_\alpha(t) = 1-\alpha+\left(\frac{(2-\alpha)^2+3-\alpha}{2-\alpha}\right)t+\frac{3-\alpha}{2-\alpha}t^2 + \left(\frac{5\alpha-\alpha^2-5}{(2-\alpha)^2}\right)(\cos((2-\alpha)t)-1)-\left(\frac{3-\alpha}{\sqrt{(2-\alpha)^3}}\right)\sin((2-\alpha)t),$$

$$[y(t)]^\alpha = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right].$$

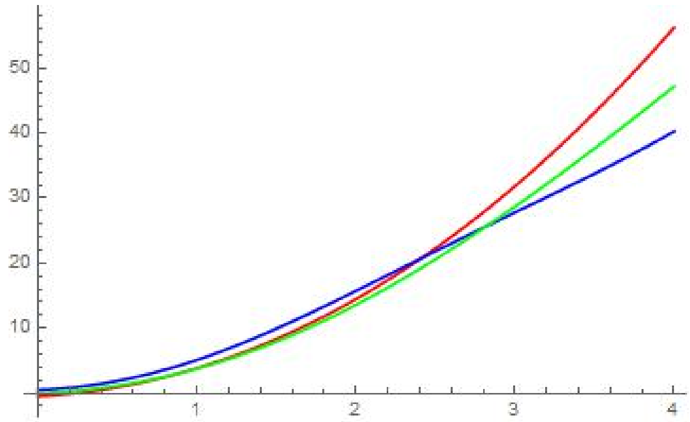


Figure 1. Graphic of solution for $\alpha = 0.5$; Blue: $\overline{y}_\alpha(t)$, Red: $\underline{y}_\alpha(t)$, Green: $\overline{y}_1(t) = \underline{y}_1(t)$.

According to Remark 1, $y(t)$ is a valid fuzzy function for $t \in (0.240824, 2.40719)$ in Figure **??**.

### 3.2 *The problem with negative fuzzy coefficient*

Let be $[\lambda]^\alpha$ is negative fuzzy number. Taking the fuzzy Laplace transform of the equation (1), we have the equations

$$s^4\underline{Y}_\alpha(s) - s^3\underline{y}_\alpha(0) - s^2\underline{y}'_\alpha(0) - s\underline{y}''_\alpha(0) - \underline{y}'''_\alpha(0)$$

$$+s^2\underline{\lambda}_\alpha\overline{Y}_\alpha(s) - s\underline{\lambda}_\alpha\overline{y}_\alpha(0) - \underline{\lambda}_\alpha\overline{y}'_\alpha(0) = \frac{\underline{\mu}_\alpha}{s},$$

$$s^4\overline{Y}_\alpha(s) - s^3\overline{y}_\alpha(0) - s^2\overline{y}'_\alpha(0) - s\overline{y}''_\alpha(0) - \overline{y}'''_\alpha(0)$$

$$+s^2\overline{\lambda}_\alpha \underline{Y}_\alpha(s) - s\overline{\lambda}_\alpha \underline{y}_\alpha(0) - \overline{\lambda}_\alpha \underline{y}'_\alpha(0) = \frac{\overline{\mu}_\alpha}{s}$$

Using the initial conditions (2),

$$s^2\underline{Y}_\alpha(s) + \underline{\lambda}_\alpha \overline{Y}_\alpha(s) = \frac{\underline{\mu}_\alpha}{s^3} + s\underline{A}_\alpha + \underline{B}_\alpha + \frac{\underline{C}_\alpha}{s} + \frac{\underline{D}_\alpha}{s^2} + \frac{\underline{\lambda}_\alpha \overline{A}_\alpha}{s} + \frac{\underline{\lambda}_\alpha \overline{B}_\alpha}{s^2}, \quad (3)$$

$$s^2\overline{Y}_\alpha(s) + \overline{\lambda}_\alpha \underline{Y}_\alpha(s) = \frac{\overline{\mu}_\alpha}{s^3} + s\overline{A}_\alpha + \overline{B}_\alpha + \frac{\overline{C}_\alpha}{s} + \frac{\overline{D}_\alpha}{s^2} + \frac{\overline{\lambda}_\alpha \underline{A}_\alpha}{s} + \frac{\overline{\lambda}_\alpha \underline{B}_\alpha}{s^2} \quad (4)$$

are obtained. If $\overline{Y}_\alpha(s)$ in the equation (4) is replaced by the equation (3) and making the necessary operations, we have

$$\underline{Y}_\alpha(s) = -\frac{\overline{\mu}_\alpha \underline{\lambda}_\alpha}{s^3\left(s^4 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)} - \frac{\left(\overline{D}_\alpha \underline{\lambda}_\alpha + \underline{\lambda}_\alpha \overline{\lambda}_\alpha \underline{B}_\alpha\right)}{s^2\left(s^4 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)} + \frac{\underline{\mu}_\alpha - \overline{C}_\alpha \underline{\lambda}_\alpha - \underline{\lambda}_\alpha \overline{\lambda}_\alpha \underline{A}_\alpha}{s\left(s^4 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)}$$
$$+\frac{\underline{D}_\alpha}{\left(s^4 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)} + \frac{\underline{C}_\alpha s}{\left(s^4 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)} + \frac{\underline{B}_\alpha s^2}{\left(s^4 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)} + \frac{\underline{A}_\alpha s^3}{\left(s^4 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)}.$$

From this, the lower solution is obtained as

$$\underline{y}_\alpha(t) = \frac{\overline{\mu}_\alpha}{\overline{\lambda}_\alpha}t^2 + \frac{\overline{D}_\alpha}{\overline{\lambda}_\alpha}t + \underline{A}_\alpha + \underline{B}_\alpha + \frac{\overline{C}_\alpha}{\overline{\lambda}_\alpha} - \frac{\underline{\mu}_\alpha}{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}$$
$$+\left(\frac{\underline{C}_\alpha \overline{\lambda}_\alpha - \overline{\mu}_\alpha}{2\sqrt{\underline{\lambda}_\alpha \left(\overline{\lambda}_\alpha\right)^3}}\right)\left(\cosh\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right) - \cos\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right)\right)$$
$$-\left(\frac{\overline{D}_\alpha}{2\sqrt[4]{\underline{\lambda}_\alpha \left(\overline{\lambda}_\alpha\right)^5}}\right)\left(\sinh\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right) + \sin\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right)\right)$$
$$+\left(\frac{\underline{\mu}_\alpha - \overline{C}_\alpha \underline{\lambda}_\alpha}{2\underline{\lambda}_\alpha \overline{\lambda}_\alpha}\right)\left(\cosh\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right) + \cos\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right)\right)$$
$$+\left(\frac{\underline{D}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha \overline{\lambda}_\alpha\right)^3}}\right)\left(\sinh\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right) - \sin\left(\sqrt[4]{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}t\right)\right).$$

Similarly, the upper solution is obtained as

$$\begin{aligned}
\overline{y}_\alpha(t) &= \frac{\mu_\alpha}{\underline{\lambda}_\alpha}t^2 + \frac{D_\alpha}{\underline{\lambda}_\alpha}t + \overline{A}_\alpha + \overline{B}_\alpha + \frac{C_\alpha}{\underline{\lambda}_\alpha} - \frac{\overline{\mu}_\alpha}{\underline{\lambda}_\alpha\overline{\lambda}_\alpha} \\
&+ \left(\frac{\overline{C}_\alpha\underline{\lambda}_\alpha - \underline{\mu}_\alpha}{2\sqrt{(\underline{\lambda}_\alpha)^3\,\overline{\lambda}_\alpha}}\right)\left(\cosh\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right) - \cos\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right)\right) \\
&- \left(\frac{\underline{D}_\alpha}{2\sqrt[4]{(\underline{\lambda}_\alpha)^5\,\overline{\lambda}_\alpha}}\right)\left(\sinh\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right) + \sin\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right)\right) \\
&+ \left(\frac{\overline{\mu}_\alpha - \underline{C}_\alpha\overline{\lambda}_\alpha}{2\underline{\lambda}_\alpha\overline{\lambda}_\alpha}\right)\left(\cosh\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right) + \cos\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right)\right) \\
&+ \left(\frac{\overline{D}_\alpha}{2\sqrt[4]{\left(\underline{\lambda}_\alpha\overline{\lambda}_\alpha\right)^3}}\right)\left(\sinh\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right) - \sin\left(\sqrt[4]{\underline{\lambda}_\alpha\overline{\lambda}_\alpha}t\right)\right).
\end{aligned}$$

Consequently, the solution is

$$[y(t)]^\alpha = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right].$$

**Example 3.2** Consider the problem

$$y^{(iv)}(t) + [-1]^\alpha y''(t) = [2]^\alpha,$$

$$y(0) = [0]^\alpha,\ \ y'(0) = [1]^\alpha,\ y''(0) = [2]^\alpha,\ y'''(0) = [3]^\alpha$$

by fuzzy Laplace transform, where $[-1]^\alpha = [-2+\alpha, -\alpha]$, $[0]^\alpha = [-1+\alpha, 1-\alpha]$, $[2]^\alpha = [1+\alpha, 3-\alpha]$, $[3]^\alpha = [2+\alpha, 4-\alpha]$.

The solution of the fuzzy problem is

$$\begin{aligned}
\underline{y}_\alpha(t) &= \left(\frac{\alpha-3}{\alpha}\right)(t^2+1) + \left(\frac{\alpha-4}{\alpha}\right)t + \frac{5\alpha^2 - 2\alpha^3 - 3\alpha - 1}{\alpha(\alpha-2)} \\
&- \left(\frac{\alpha^2+3}{2\sqrt{\alpha^3(2-\alpha)}}\right)\left(\cosh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) - \cos\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right) \\
&+ \left(\frac{\alpha-4}{2\sqrt[4]{\alpha^5(2-\alpha)}}\right)\left(\sinh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) + \sin\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right) \\
&+ \left(\frac{\alpha^2-4\alpha+7}{2\alpha(2-\alpha)}\right)\left(\cosh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) + \cos\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right) \\
&+ \left(\frac{2+\alpha}{2\sqrt[4]{(\alpha(2-\alpha))^3}}\right)\left(\sinh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) - \sin\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right).
\end{aligned}$$

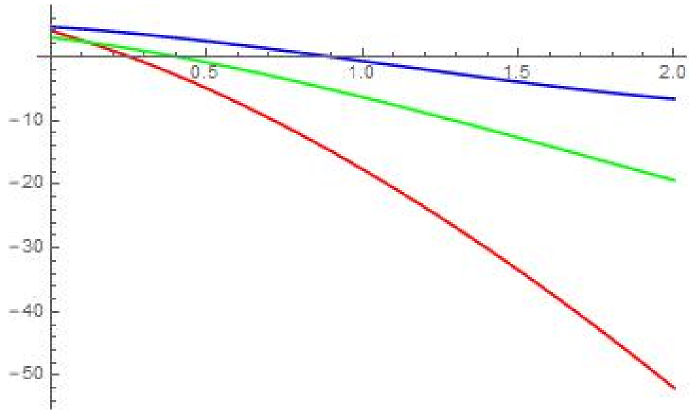


Figure 2. Graphic of solution for $\alpha = 0.5$; Blue: $\overline{y}_\alpha(t)$, Red: $\underline{y}_\alpha(t)$, Green: $\overline{y}_1(t) = \underline{y}_1(t)$.

$$\begin{aligned}
\overline{y}_\alpha(t) = &-\left(\frac{1+\alpha}{2-\alpha}\right)\left(t^2+1\right) - \left(\frac{2+\alpha}{2-\alpha}\right)t + \frac{2\alpha^3 - 7\alpha^2 + 7\alpha - 3}{\alpha(\alpha-2)} \\
&+ \left(\frac{4\alpha - \alpha^2 - 7}{2\sqrt{\alpha(2-\alpha)}^3}\right)\left(\cosh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) - \cos\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right) \\
&- \left(\frac{2+\alpha}{2\sqrt[4]{\alpha(2-\alpha)}^5}\right)\left(\sinh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) + \sin\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right) \\
&+ \left(\frac{\alpha^2+3}{2\alpha(2-\alpha)}\right)\left(\cosh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) + \cos\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right) \\
&+ \left(\frac{4-\alpha}{2\sqrt[4]{(\alpha(2-\alpha))^3}}\right)\left(\sinh\left(\sqrt[4]{\alpha(2-\alpha)}t\right) - \sin\left(\sqrt[4]{\alpha(2-\alpha)}t\right)\right),
\end{aligned}$$

$$[y(t)]^\alpha = \left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right].$$

According to Remark 1 and since $y(t)$ is positive fuzzy function, $y(t)$ is a valid fuzzy function for $t \in (0, 0.891621)$ in Figure 2.

## 4. Conclusions

In this paper, fuzzy initial value problems for fourth-order fuzzy differential equations with positive and negative fuzzy number coefficients are studied by fuzzy Laplace transform. Examples are solved. Graphics of solutions are drawn using the Mathematica program. It is found that the solutions are valid fuzzy functions in different interval. Also, when initial values are triangular fuzzy numbers, the solutions are triangular fuzzy numbers for any $t > 0$ time.

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