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The Numerical Solution of Klein-Gorden Equation by Using Nonstandard Finite Difference

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Abstract. In this paper we propose a numerical scheme to solve the one dimensional nonlinear Klein-Gorden equation. We describe the mathematical formulation procedure in details. The scheme is three level explicit and based on nonstandard finite difference. It has nonlinear denominator function of the step sizes. Stability analysis of the method has been given and we prove that the proposed method when applied to one dimensional nonlinear Klein-Gorden equation, is unconditionally stable. We illustrate the usefulness of the proposed method by applying it on two examples.

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1. Introduction

Many nonlinear phenomena are modeled by nonlinear Klein-Gorden equation, such as dislocations, ferroelectric and ferromagnetic domain wall. The numerical treatment of one dimensional Klein-Gorden equation

$$u_{tt} - a^2 u_{xx} + g(u) = f(x,t) \quad x \in (L_0, L_1) \quad t > t_0, \tag{1}$$

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subject to the initial conditions

$$u(x, t_0) = \phi(x), \quad u_t(x, t_0) = \psi(x), \quad x \in (L_0, L_1) \tag{2}$$

and with the boundary conditions

$$u(L_0, t) = p_0(t), \quad u(L_1, t) = p_1(t), \quad t > t_0 \tag{3}$$

has been considered, where $u = u(x,t)$ represents the wave displacement at position x and time t,a is an known constant and $g(u)$ is the nonlinear force. The function $\phi(x)$ and $\psi(x)$ are wave modes or kinks and velocity,respectively. When the nonlinear force is given by $g(u) = \sin(u)$ then, the equation (1) is known as Sin-Gorden equation.

The Klein-Gorden equation plays an important role in mathematical physics [21,4,8]. There are a lot of studies on the numerical solution of initial-boundary value problem of the linear and nonlinear Klein-Gorden equation. El-Sayed [9] and Wazwaz et al. [10,22] and Kaya et al. [14] have used Adomian decomposition method for solving linear and nonlinear Klein-Gorden equation. Parkes et al. [18] and Fu et al. [11] used the Jacobi elliptic function expansion method to find double periodic solutions of equation (1). Finite difference methods are known as the first technique for solving such equation. These methods are very effective for solving various kinds of partial differential equations, conditionally stability of explicit finite difference procedures is developed by Dehghan [6]. Jiminez et al. [13] were discussed fourth order finite difference scheme for approximation the nonlinear Klein-Gorden equation. Bratsos [3] used a predictor-corrector (P-C) scheme based on rational approximation of second order three-time level recurrence relations. Abbasbandy [1] obtained numerical solution of nonlinear Klein-Gorden equation by variational iteration method. Ismail et al. [12] were consider spline difference method for solving Klein-Gorden equation.

Yusufoglu [23] presented variational iteration method for studying Klein-Gorden equation. Dehghan et al. [7] proposed a numerical scheme to solve Klein-Gorden equation by using the collocation approximation solution based on Thin Plate Spline (TPS) radial basis functions (RBF). Rashidinia et al. [19] developed a three time level implicit method by using the non-polynomial cubic tension spline function for solving Klein-Gorden equation.

The use of the nonstandard finite difference method has increased in recent years. For example, Areanas et al. [2] constructed nonstandard finite difference schemes to obtain numerical solutions of the susceptible-infected (SI) and susceptible-infected-recovered (SIR) fractional-order epidemic models. Also, Memarbashi et al. [15] developed these method for solving (SEI) Epidemic Model. Dang [5] proposed nonstandard finite difference schemes for a general predator-prey system. In these paper, we will discuss nonstandard finite difference for solving Klein-Gorden equation.

This article organized as following: In section 2, we define nonstandard finite difference preliminaries. In section 3, we present the subequation method which is the basic tool in driving the nonstandard finite difference scheme. In this section, we discuss the application of proposed method to equation (1). In next section, stability analysis has been carried out. In section 5, we illustrate two examples for the efficiency of the proposed method and compare it with standard finite difference schemes. Concluding remarks are given in section 6.

2. Preliminaries [17]

2.1 *Exact finite difference scheme*

Consider the following equation

$$\frac{du}{dt} = f(u, t, \lambda), \quad u(t_0) = u_0, \tag{4}$$

where λ is a set of parameters and $f(u, t, \lambda)$ is such that Eq. 4 has a unique solution over, $t_0 \leqslant t < T$. We denote the solution of (4) by

$$u(t) = \Phi(\lambda, u_0, t_0, t), \tag{5}$$

with $u_0 = \Phi(\lambda, u_0, t_0, t_0)$.

The discrete model of Eq. (4) can be written as

$$u_{j+1} = g(\lambda, k, u_j, t_j), \tag{6}$$

where $k = \Delta t$ and $t_j = jk$.

Definition 2.1 Equation (4) and (6) are said to have the same general solution if and only if

$$u_j = u(t_j), \tag{7}$$

for arbitrary values of k.

Definition 2.2 An exact finite difference scheme is one for which the solution to the difference equation has the same general solution as the associated differential equation.

Theorem 2.1 The Eq. (4) has an exact finite difference scheme given by

$$u_{j+1} = \Phi[\lambda, u_j, t_j, t_{j+1}], \tag{8}$$

where Φ is given by (5).

Proof [17]. ■

Let $u^{(i)}(t); i = 1, 2, ..., N;$ be the set of linearly independent functions. It is possible to construct an N-th order linear difference equation that has the correspoding discrete functions, $u_j^{(i)} \equiv u^{(i)}(t_j)$ as the solutions for $t_j = (\Delta t)j = jk$, the required equation is given by the following determinante [16].

$$\begin{vmatrix} u_j & u_j^{(1)} & \dots & u_j^{(N)} \\ u_{j+1} & u_{j+1}^{(1)} & \dots & u_{j+1}^{(N)} \\ \vdots & \vdots\vdots & & \\ u_{j+N} & u_{j+N}^{(1)} & \dots & u_{j+N}^{(N)} \end{vmatrix} = 0. \tag{9}$$

To illustrate this procedure, consider the following second order ordinary differential equation

$$\frac{d^2u}{dt^2} + au = 0, \tag{10}$$

where a is a real constant. The two linearly independent solutions are

$$u^{(1)}(t) = e^{i\sqrt{a}t}, \quad u^{(2)}(t) = e^{-i\sqrt{a}t}. \tag{11}$$

Substitution of these function into Eq. (9) gives

$$\begin{vmatrix} u_j & e^{i\sqrt{a}kj} & e^{-i\sqrt{a}kj} \\ u_{j+1} & e^{i\sqrt{a}k(j+1)} & e^{-i\sqrt{a}k(j+1)} \\ u_{j+2} & e^{i\sqrt{a}k(j+2)} & e^{-i\sqrt{a}k(j+2)} \end{vmatrix} = 0, \tag{12}$$

therefore,

$$e^{i\sqrt{a}kj}.e^{-i\sqrt{a}kj} \begin{vmatrix} u_j & 1 & 1 \\ u_{j+1} & e^{i\sqrt{a}k} & e^{-i\sqrt{a}k} \\ u_{j+2} & e^{2i\sqrt{a}k} & e^{-2i\sqrt{a}k} \end{vmatrix} = 0. \tag{13}$$

Evaluation of the determinant

$$u_j(e^{-i\sqrt{a}k} - e^{i\sqrt{a}k}) + u_{j+1}(-e^{-2i\sqrt{a}k} + e^{2i\sqrt{a}k}) + u_{j+2}(e^{-i\sqrt{a}k} - e^{i\sqrt{a}k}) = 0. \tag{14}$$

and $j \to j-1$,we have

$$u_{j+1} - [2\cos(k\sqrt{a})]u_j + u_{j-1} = 0. \tag{15}$$

Using the identity

$$2\cos(k\sqrt{a}) = 2 - 4\sin^2(\frac{k\sqrt{a}}{2}),$$

then Eq. (14) can be written in the form

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{(\frac{4}{a})\sin^2(\frac{k\sqrt{a}}{2})} + au_j = 0. \tag{16}$$

This is the exact finite difference scheme for the Eq. (10).

Another example is the general Logistic differential equation

$$\frac{du}{dt} = \lambda_1 u - \lambda_2 u^2, \quad u(t_0) = u_0, \tag{17}$$

where (λ_1, λ_2) are positive parameters.

By use of exact solution and Eq. (9), the exact scheme is

$$\frac{u_{j+1} - u_j}{\frac{e^{k\lambda_1 - 1}}{\lambda_1}} = \lambda_1 u_j - \lambda_2 u_{j+1} u_j. \tag{18}$$

2.2 *Nonstandard finite difference scheme*

A Nonstandard Finite Difference scheme is a discrete model of a differential equation that is constructed according to the following rules.
Rule 1. Denominator functions for the discrete derivatives must, in general, be expressed in term of more complicated function of the step-sizes than those conventionally used.[e.g. $(\frac{4}{a})\sin^2(\frac{k\sqrt{a}}{2})$]
Rule 2. Nonlinear terms should, in general, be replaced by nonlocal discrete representations. For example

$$u^2(t_j) \approx u_{j+1}u_j, u_j\frac{y_{j+1}+u_{j-1}}{2}, u_{j-1}u_{j+1}, \tag{19}$$

$$u^3(t_j) \approx u_j^2u_{j+1}, u_{j-1}u_ju_{j+1}, u_j^2\frac{u_{j+1}+u_{j-1}}{2}. \tag{20}$$

Rule 3. For differential equation having $N(\geqslant 3)$ terms, it is generally useful to construct finite difference scheme for various sub-equation composed of M terms, where $M < N$, and then combined all the schemes together in an overall consistent finite difference model.

In general nonstandard schemes are not exact scheme; however, they do offer the prospect of obtaining finite difference scheme that do not possess the usual numerical instabilities. The application of nonstandard modeling rules does not necessarily lead to a unique discrete model for a given differential equation.

3. Implementation of the method

To illustrate the analysis of the previous section, we consider the nonlinear Klein-Gorden equation [3]

$$u_{tt} - a^2u_{xx} + au - bu^3 = 0, \tag{21}$$

where a and b are constant. We approximate solution of Eq. (21) on a spatial interval $[L_0, L_1]$, over the time interval $[0, T]$. We fix the step size $h = \frac{L_1-L_0}{n}$ in x direction and time step size $k = \frac{T}{m}$, where n and m are integers. Denote $x_l = L_0 + lh, l = 0(1)n$ and $t_j = t_0 + jk, j = 0(1)m$. The approximate value of u at the point x_l and time t_j will be denoted by u_l^j.

The (standard) finite difference scheme used to approximate the solution of Eq. (21) is

$$\delta_t^2u_l^j = \lambda^2a^2\delta_x^2u_l^j - ak^2u_l^j - bk^2(u_l^j)^3, \tag{22}$$

where

$$\begin{aligned}\delta_t^2u_l^j &= u_l^{j+1} - 2u_l^j + u_l^{j-1},\\ \delta_x^2u_l^j &= u_{l+1}^j - 2u_l^j + u_{l-1}^j,\\ \lambda &= \frac{k}{h}, \quad l = 1(1)n-1.\end{aligned}$$

In the following, we used the subequation method to obtain a nonstandard finite difference (NSFD) scheme. A subeqaution is an ordinary differential equation or partial differential equation obtain by dropping one or more terms appearing in the full equation. In contrast the full equation, these subequation have known exact scheme, therefore, we can construct a NSFD scheme for the original partial differential equation. Thus, we have from Eq. (21) the following two useful subequations:

$$u_{tt} + au = 0, \tag{23}$$

$$-a^2 u_{xx} + au = 0. \tag{24}$$

The exact scheme of these equations are:

$$\frac{u^{j+1} - 2u^j + u^{j-1}}{\varphi_1} + au^j = 0, \quad \varphi_1(k) = \frac{4}{a}\sin^2(\frac{k}{2}\sqrt{a}), \tag{25}$$

$$\frac{u_{l+1} - 2u_l + u_{l-1}}{\varphi_2} - \frac{1}{a}u_l = 0, \quad \varphi_2(h) = 4a\sinh^2(\frac{h}{2}\sqrt{\frac{1}{a}}). \tag{26}$$

We will now give a novel scheme that incorporates the exact scheme of the above two subequations.

$$\frac{u_l^{j+1} - 2u_l^j + u_l^{j-1}}{\varphi_1(k)} - a^2\frac{u_{l+1}^j - 2u_l^j + u_{l-1}^j}{\varphi_2(h)} + au_l^j = 0. \tag{27}$$

The nonlinear terms in Eq. (21) may be approximate by an expression, which contains

$$(u_l^j)^3 = (u_l^j)^2\frac{u_{l+1}^j + u_{l-1}^j}{2}.$$

Therefore, we have the NSFD scheme of Eq. (21)

$$\frac{u_l^{j+1} - 2u_l^j + u_l^{j-1}}{\varphi_1(k)} - a^2\frac{u_{l+1}^j - 2u_l^j + u_{l-1}^j}{\varphi_2(h)} + au_l^j - b(u_l^j)^2\frac{u_{l+1}^j + u_{l-1}^j}{2} = 0, \quad l = 1(1)n-1. \tag{28}$$

The proposed scheme is explicit scheme, to start any computation, it is necessary to know the solution of u at first time level. We can obtain u_l^{-1} by using Taylor expansion of u_l^{-1} about u_l^0 and using the differential equation in (2).

By the help of Taylor expansion, a third-order approximation to u at $t = -k$ can be written as

$$u_l^{-1} = u_l^0 - k(\frac{\partial u}{\partial t})_l^0 + \frac{k}{2}(\frac{\partial^2 u}{\partial t^2})_l^0 + O(k^3). \tag{29}$$

Using the Eq. (2), we have

$$u_l^{-1} = \phi(lh) - k\psi(lh) + \frac{k}{2}u_{tt}(lh, 0). \tag{30}$$

We know $u_{tt} = a^2 u_{xx} - au + bu^3$, thus,

$$u_l^{-1} = \phi(lh) - k\psi(lh) + \frac{k}{2}[a^2\phi_{xx}(lh) - a\phi(lh) + b(\phi(lh))^2(\frac{\phi((l+1)h + \phi((l-1)h)}{2})]. \tag{31}$$

4. Stability

For stability of scheme (28), we use the Von Neumann's method [20]. To investigate the stability analysis, we may rewrite (28) as

$$A\delta_t^2 u_l^j - B\delta_x^2 u_l^j + au_l^j - 3b(u_l^j)^3 = 0, \tag{32}$$

where

$$A = \frac{1}{\varphi_1(k)},$$
$$B = \frac{1}{\varphi_2(h)},$$
$$(u_l^j)^2 \frac{u_{l+1}^j + u_{l-1}^j}{2} \simeq (u_l^j)^3.$$

Furthermore, the exact value $U_l^j = u(x_l, t_j)$ satisfies

$$A\delta_t^2 U_l^j - B\delta_x^2 U_l^j + aU_l^j - 3b(U_l^j)^3 = 0. \tag{33}$$

We assume that there exists an error $\varepsilon_l^j = U_l^j - u_l^j$ at grid point (x_l, t_j). Subtracting (33) from (32)

$$A\delta_t^2 \varepsilon_l^j - B\delta_x^2 \varepsilon_l^j + a\varepsilon_l^j - 3b((U_l^j)^3 - (u_l^j)^3) = 0,$$

where

$$(U_l^j)^3 - (u_l^j)^3 = (U_l^j - u_l^j)((U_l^j)^2 + (u_l^j)^2 + U_l^j u_l^j) = \varepsilon_l^j(3(u_g)^2),$$

where $(u_g)^2$ is a typical value of $u_l^j; l = 0, 1, ...N$ used for the linearization of the nonlinear term $(u_l^j)^3$. We obtain the error equation

$$A\delta_t^2 \varepsilon_l^j - B\delta_x^2 \varepsilon_l^j + a\varepsilon_l^j - 3b\varepsilon_l^j(u_g)^2 = 0. \tag{34}$$

To establish stability for the scheme (34), we assume that the solution of (34) at the grid point (l, j) is of the form $\varepsilon_l^j = \xi^j e^{i\theta l}$, where $i = \sqrt{-1}$, θ is real and ξ in general is complex. Subsituting $\varepsilon_l^j = \xi^j e^{i\theta l}$ in the error equation and simplifying, we have the following characteristic equation

$$\xi^2 - 2\xi(1 - \frac{2B}{A}\sin^2(\frac{\theta}{2}) + \frac{3b(u_g)^2 - a}{2A}) + 1 = 0. \tag{35}$$

Equation (35) is of the general form $p\xi^2 - 2q\xi + p = 0$, with $p, q \in R$ and $p > 0$.

The Von Neumann's stability criterion for stability $\mid \xi \mid \leqslant 1$ will always be satisfied, where $\mid q \mid \leqslant p$, otherwise

$$-p \leqslant q \leqslant p \tag{36}$$

The right hand side of inequality (36), gives

$$1 - \frac{2B}{A}\sin^2(\frac{\theta}{2}) + \frac{3b(u_g)^2 - a}{2A} \leqslant 1.$$

After simplifying this criterion, it gives the following restriction for the space step

$$\sinh^2(\frac{h}{2}\sqrt{\frac{1}{a}}) \leqslant \frac{a\sin^2(\frac{\theta}{2})}{3b(u_g)^2 - a}. \tag{37}$$

The left hand side of (36), gives

$$-1 \leqslant 1 - \frac{2B}{A}\sin^2(\frac{\theta}{2}) + \frac{3b(u_g)^2 - a}{2A},$$

which is satisfied when $2 + \frac{3b(u_g)^2 - a}{2A} \geqslant 0$, otherwise it gives the following restriction for the time step

$$\sin^2(\frac{k}{2}\sqrt{a}) \leqslant \frac{a}{a - 3b(u_g)^2}. \tag{38}$$

We can conclude that the presented method is stable as long as criterion (37) and (38) is satisfied.

5. Numerical illustrations

In this section we present numerical result for our scheme for nonlinear Klein-Gorden equation.

Example 5.1 Eq. (21) with the initial data

$$\begin{aligned} u(x,0) &= \sqrt{\frac{a}{b}}\tanh[\sqrt{\frac{a}{2(c^2-a^2)}}x], \\ u_t(x,t) &= -c\sqrt{\frac{a}{b}}\sqrt{\frac{a}{2(c^2-a^2)}}sech^2[\sqrt{\frac{a}{2(c^2-a^2)}}x],\ x \in [0,1], \end{aligned} \tag{39}$$

and Eq. (21) has the following Kink-solution

$$u(x,t) = \sqrt{\frac{a}{b}}\tanh[\sqrt{\frac{a}{2(c^2-a^2)}}(x-ct)], \tag{40}$$

where $a, b, c^2 - a^2 > 0$.
We consider Eq. (21) along with initial condition (39) and exact solution (40), for $a = 0.01, b = 1, c = 0.3$. We solve this problem with different values of h by scheme (28) and (22). Computed solution is compared with exact solution at grid points.

Table 1. Maximum absolute error for Example 1.

x	scheme (22)		
	$h = 0.05$	$h = 0.02$	$h = 0.01$
0.2	1.30812(−6)	1.30818(−6)	1.30819(−6)
0.4	3.92575(−6)	3.92606(−6)	3.92611(−6)
0.6	6.47712(−6)	6.47504(−6)	6.47511(−6)
0.8	1.11092(−3)	4.32944(−5)	8.91215(−6)
x	scheme (28)		
	$h = 0.05$	$h = 0.02$	$h = 0.01$
0.2	2.52017(−8)	2.4970(−8)	2.49365(−8)
0.4	5.13999(−8)	5.07277(−8)	5.06304(−8)
0.6	7.66362(−8)	7.55891(−8)	7.54367(−8)
0.8	4.38845(−8)	9.9136(−8)	9.89450(−8)

In Table 1, we take $k = 0.01$. The absolutely maximum error for different values of mesh size $h = 0.05, 0.02, 0.01$ have been calculated. The table show the NSFD in comparison with standard method are more accurate.

Example 5.2 Eq. (21) with the initial data

$$u(x,0) = \sqrt{\frac{2a}{b}} sech[\sqrt{\frac{a}{(a^2-c^2)}}x],$$
$$u_t(x,t) = c\sqrt{\frac{2a}{b}}\sqrt{\frac{a}{(a^2-c^2)}} sech[\sqrt{\frac{a}{2(c^2-a^2)}}x]\tanh[\sqrt{\frac{a}{2(c^2-a^2)}}x],\ x \in [0,1], \tag{41}$$

and Eq. (21) has the following Soliton-solution

$$u(x,t) = \sqrt{\frac{2a}{b}} sech[\sqrt{\frac{a}{(a^2-c^2)}}(x-ct)]. \tag{42}$$

We apply NSFD method to Eq. (21) along with initial conditions (41) and exact solution (42) for values $a = 0.003, b = 1, c = 0.25$.

Table 2. Maximum absolute error for Example 2.

x	$t = 1$	$t = 2$
0.1	1.15268(−4)	2.33113(−4)
0.2	2.30721(−4)	4.62646(−4)
0.3	3.46455(−4)	6.94723(−4)
0.4	4.62615(−4)	9.27667(−4)
0.5	5.79352(−4)	1.16179(−3)
0.6	6.96823(−4)	1.39740(−3)
0.7	8.15196(−4)	1.63486(−3)
0.8	9.34648(−4)	1.87452(−3)
0.9	1.05537(−3)	2.11678(−3)

In Table 2, we take $h = k = 0.01$. The absolutely maximum error for different times have been calculated.

6. Conclusion

In this article, we have outlined a new idea for solving Klein-Gorden equation by using nonstandard finite difference method. The time step restriction of the NSFD scheme is usually much less restrictive than for standard finite difference schemes. We know that some numerical method, leads to numerical instabilities, Mickens suggest what is known as the NSFD method. It has been found that the present algorithm gives accuracy numerical results and it is more efficient than the standard method. Our future works will deal with hybrid of some of numerical method (such as Spline) and NSFD in system of nonlinear partial differential equation. It is possible to determine the denominator function for the discretizations of the partial differential equations.

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