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A Novel Finite Difference Method of Order Three for the Third Order Boundary Value Problem in ODEs

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Abstract. In this article we have developed third order exact finite difference method for the numerical solution of third order boundary value problems. We constructed our numerical technique without change in structure of the coefficient matrix of the second-order method in [11]. We have discussed convergence of the proposed method. Numerical experiments on model test problems approves the simply high accuracy and efficiency of the method.

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1. Introduction

In this article we consider a direct method for the numerical solution of the third order boundary value problems of the following form

$$u'''(x) = f(x, u), \quad a < x < b, \tag{1}$$

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subject to the boundary conditions

$$u(a) = \alpha, \quad u'(a) = \beta, \quad \text{and} \qquad u'(b) = \gamma.$$

where α, β and γ are real constant.

We find the third order system of differential equations/equation in the study of many subjects in particular fluid dynamics, obstacle problems, moving and free boundary value problems, etc.. The solution of these problems is important, but it is not possible to find an analytical solution for these problems for an arbitrary forcing function $f(x, u)$, so we rely on an approximate solution in study of solution of these problems.

The existence and uniqueness of the solution to problem (1) are assumed. However the theory on existence and uniqueness of the solution of higher order boundary value problems can be found in [1] and for specific problem (1) in [3, 4, 9] and references there. In recent years, a numerical techniques for solution of third order boundary value problems were reported in the literature by many researchers. For instance we refer some literary work in finite difference method [2], Non polynomial spline method [6], Quintic Splines [8] and references therein.

Recently there appeared a higher order finite difference method for the numerical solution of third order boundary value problems in the literature [13]. Based on the idea [11], the purpose of this article is to develop third order finite difference method to deal with about numerical solution of the mentioned above boundary value problem which is more accurate, less expensive and simpler in computational efforts.

In this article, we have organised our work as follows. In the following section we derived a finite difference method. In Section 3, under proper condition, we have discussed and analysed the convergence of the proposed method. The computational accuracy of the proposed method on the model problems and illustrative results so produced in Section 4. Discussion on computational performance of the proposed method are presented in Section 5.

2. The difference method

We define N finite numbers of nodal points of the domain $[a, b]$, in which the solution of the problem (1) is desired, as $a \leqslant x_0 < x_1 < x_2 < \cdots < x_N \leqslant b$ using uniform step length h such that $x_i = a + ih$, $i = 0, 1, \cdots, N$. Suppose that we wish to determine the numerical approximation of the theoretical solution $u(x)$ of the problem (1) at the nodal point $x_i, i = 1, 2, \cdots, N$. We denote the numerical approximation of $u(x)$ at node $x = x_i$ as u_i. Let us denote f_i as the approximation of the theoretical value of the source function $f(x, u(x))$ at node $x = x_i$, $i = 0, 1, \cdots, N$. Thus the boundary value problem (1) at node $x = x_i$ may be written as

$$u_i''' = f_i, \quad a \leqslant x_i \leqslant b, \tag{2}$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u_0' = \beta, \quad \text{and} \qquad u_N' = \gamma.$$

Let we define nodes $x_{i\pm\frac{1}{2}} = x_i \pm \frac{h}{2}, \quad i = 1, 2, ..., N-1$ and denote the solution of the problem (1) at these nodes as $u_{i\pm\frac{1}{2}}$. Following the idea in [11] and simplify using method of undetermined coefficients and Taylor's series expansion, we descretize problem (2) at these nodes in $[a, b]$ as follows,

$$9u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} = 8u_{i-1} + 3hu'_{i-1} - \frac{3h^3}{160}(2f_{i-1} + 17f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}) + T_i, \qquad i = 1 \quad (3)$$

$$-15u_{i-\frac{3}{2}} + 10u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} = -8u_{i-2} - \frac{h^3}{16}(14f_{i-\frac{3}{2}} + 27f_{i-\frac{1}{2}} - f_{i+\frac{1}{2}}) + T_i, \quad i = 2$$

$$u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} = -\frac{h^3}{2}(f_{i-\frac{3}{2}} + f_{i-\frac{1}{2}}) + T_i, \quad 3 \leqslant i \leqslant N-1$$

$$u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} = hu'_i + \frac{h^3}{1920}(31f_{i-\frac{5}{2}} - 1062f_{i-\frac{3}{2}} - 809f_{i-\frac{1}{2}}) + T_i, \quad i = N$$

where $T_i, i = 1, 2, ..., N$ is truncation error. Also in discretization we have used boundary conditions in a natural way.

After neglecting the terms Ti in (3), at nodal points $x_{i-\frac{1}{2}}, i = 1, 2, ..., N$, we will obtain the N linear or nonlinear system of equations in N unknown namely $u_{i-\frac{1}{2}}$ which depends on the source function $f(x, u)$. To obtain an approximate solution of problem (1) we have to solve a system of equations by an appropriate method. However we have applied an iterative method either Gauss Seidel or Newton-Raphson to solve a system of equations respectively for linear and nonlinear system of equations.

We computed numerical value of $u_i, i = 1, 2, ..., N$ using following third order approximation,

$$u_i = \begin{cases} -3u_{i-1} + 4u_{i-\frac{1}{2}} - hu'_{i-1}, & i = 1 \\ \frac{1}{8}(-u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}}), & i = 2, ..., N-1 \\ \frac{1}{8}(-u_{i-\frac{3}{2}} + 9u_{i-\frac{1}{2}} + 3hu'_i), & i = N \end{cases} \quad (4)$$

3. Convergence analysis

We will consider following linear test equation for convergence analysis of the proposed method (3).

$$u'''(x) = f(x, u), \qquad a < x < b. \quad (5)$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u'_0 = \beta, \quad \text{and} \quad u'_N = \gamma.$$

Let $U_{i-\frac{1}{2}}$ denote the approximation of $u_{i-\frac{1}{2}}$ for $i = 1, 2, \cdots, N$. Thus N-dimensional vector $\mathbf{U} = (U_{\frac{1}{2}}, U_{\frac{3}{2}}, \cdots, U_{N-\frac{1}{2}})$ denote approximation of N-dimensional vector $\mathbf{u} = (u_{\frac{1}{2}}, u_{\frac{3}{2}}, \cdots, u_{N-\frac{1}{2}})$. Let us define error there in approximate solution of the problem (1),

$$\epsilon_{i-\frac{1}{2}} = U_{i-\frac{1}{2}} - u_{i-\frac{1}{2}}, \quad i = 1, 2, \cdots, N.$$

Thus we can now define N-dimensional error vector $\mathbf{E} = \mathbf{U} - \mathbf{u}$. Also let $\mathbf{T} = (T_1, T_2, \cdots, T_N)$ be the N-dimensional truncation error that associated with the proposed difference method (3). We can linearize source function $f(x, u)$ by application of Taylor series expansion i.e. $f(x, u) - f(x, U) = (u - U)\frac{\partial f}{\partial U}$. Thus we can write the error in proposed method (3) in the matrix form as

$$\mathbf{JE} = \mathbf{T} \tag{6}$$

where $\mathbf{J} = \mathbf{A} + \mathbf{B}$ and matrices $\mathbf{A}$ and $\mathbf{B}$ are,

$$\mathbf{A} = \begin{pmatrix} 9 & -1 & & & & & 0 \\ -15 & 10 & -3 & & & & \\ 1 & -3 & 3 & -1 & & & \\ & 1 & -3 & 3 & -1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -3 & 3 & -1 \\ 0 & & & & 1 & -3 & 2 \end{pmatrix}_{N\times N},$$

$$\mathbf{B} = \frac{h^3}{1920} \begin{pmatrix} 612\delta_{\frac{1}{2}} & 36\delta_{\frac{3}{2}} & & & & 0 \\ 1680\delta_{\frac{1}{2}} & 3240\delta_{\frac{3}{2}} & -120\delta_{\frac{5}{2}} & & & \\ & 960\delta_{\frac{3}{2}} & 960\delta_{\frac{5}{2}} & & & \\ & \ddots & \ddots & & & \\ & & & 960\delta_{N-\frac{5}{2}} & 960\delta_{N-\frac{3}{2}} & \\ 0 & & & -31\delta_{N-\frac{5}{2}} & 1062\delta_{N-\frac{3}{2}} & 809\delta_{N-\frac{1}{2}} \end{pmatrix}_{N\times N},$$

where $\delta = \frac{\partial f}{\partial U}$ and

$$T_i = \begin{cases} \frac{h^6}{640} u^{(6)}_{i-\frac{1}{2}}, & i = 1 \\ -\frac{h^6}{28} u^{(6)}_{i-\frac{1}{2}}, & i = 2 \\ o(h^7), & 3 \leqslant i \leqslant N-1 \\ \frac{159h^6}{11520} u^{(6)}_{i-\frac{1}{2}}, & i = N \end{cases}$$

For any square matrix $\mathbf{S}$ such that $||\mathbf{S}|| < 1$ then matrix $(\mathbf{I} + \mathbf{S})$ is invertible [5, 7, 12], and

$$||(\mathbf{I} + \mathbf{S})^{-1}|| < \frac{1}{1 - ||\mathbf{S}||}$$

where $\mathbf{I}$ is an identity matrix and same order of $\mathbf{S}$. Let us assume

$$||\mathbf{A}^{-1}||||\mathbf{B}|| < 1$$

Thus from (6), we have

$$||\mathbf{E}|| < \frac{1}{1-||\mathbf{A}^{-1}||||\mathbf{B}||}||\mathbf{A}^{-1}||||\mathbf{T}|| \tag{7}$$

Let

$$M = \max_{x\in[a,b]}|u^{(6)}(x)|, \quad D = \max_{x\in[a,b]}\delta_{i-\frac{1}{2}}, \quad \text{and} \quad D > 0.$$

Also we have $||\mathbf{A}^{-1}|| < \frac{(b-a)^3}{12h^3}$ in [11]. Thus from (7) we obtained,

$$||\mathbf{E}|| < \frac{53(b-a)^3Mh^3}{5(9216-D(b-a)^3)} \tag{8}$$

It follows from equation (8) that $||\mathbf{E}|| \to 0$ as $h \to 0$. This establishes that our proposed method (3) is convergent and the order of convergence of the method is at least $O(h^3)$.

4. Numerical results

To test the computational efficiency and validity of theoretical development of proposed method, we have considered four model problems. In each model problem, we took uniform step size h. In Table 1 - Table 6, we have shown MAU the maximum absolute error in the solution $u(x)$ of the problems (1) for different values of N. We have used the following formula in computation of MAU,

$$MAU = \max_{1\leqslant i\leqslant N}|u(x_i) - u_i|.$$

We have used an iterative method to solve system of equations arise from equation (3). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compilers (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iterations reached 10^3.

Problem 1. The model linear problem in [13] and given by

$$u'''(x) = f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 2, \qquad u'(0) = 20\pi, \qquad \text{and} \quad u'(1) = \exp(1) + 20\pi + 1,$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = \exp(x) + 2\sin(10\pi x) - cos(10\pi x) + x(x-1) + 2$. The MAU computed by a method (3) for different values of N are presented in Tables 1-2.

Problem 2. The nonlinear model problem given by

$$u'''(x) = u(x)(u(x) + 1.0) + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 0, \qquad u'(0) = 1, \qquad \text{and} \quad u'(1) = 0,$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = x\exp(-x)$. The MAU computed by a method (3) for different values of N are presented in Table 3.

Problem 3. The nonlinear model problem given by

$$u'''(x) = u(x)(x^4 - u(x)) + f(x), \quad 0 < x < 1$$

subject to boundary conditions

$$u(0) = 0, \qquad u'(0) = -1, \qquad \text{and} \quad u'(1) = \sin(1).$$

The analytical solution of the problem is $u(x) = (x-1)\sin(x)$. The MAU computed by a method (3) for different values of N are presented in Table 4.

Problem 4. Consider the following third-order obstacle problems [10],

$$u'''(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{4} \\ u(x) - 1, & \frac{1}{4} \leqslant x \leqslant \frac{3}{4} \\ 0, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases}$$

subject to boundary conditions

$$u(0) = 0, \qquad u'(0) = 0, \qquad \text{and} \quad u'(1) = 0.$$

The analytical solution of the problem is

$$u(x) = \begin{cases} \frac{1}{2}a_1 x^2, & 0 \leqslant x \leqslant \frac{1}{4} \\ 1 + a_2\exp(x) + \exp(\frac{-x}{2})(a_3\cos(\frac{\sqrt{3}}{2}x) + a_4\sin(\frac{\sqrt{3}}{2}x)), & \frac{1}{4} \leqslant x \leqslant \frac{3}{4} \\ \frac{1}{2}a_5 x(x-2) + a_6, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases}$$

where the constants $a_i, i = 1, 2, ..., 6$ can be determined by the solving a system of linear equations which can be obtained by applying the continuity conditions of $u(x)$, $u'(x)$ and $u''(x)$ at $x = \frac{1}{4}$ and $\frac{3}{4}$. We have computed the numerical value of $MAUI$, $MAUM$ and $MAUE$ respectively in interval $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$ computed by a method (3) for different values of N are presented in Table 5. We have also presented $MAU = \max\{MAUI, MAUM, MAUE\}$ in Table 5 and compared with some higher order method reported in literature in Table 6.

Table 1. Maximum absolute error (Problem 1).

	Maximum absolute error				
	$N = 80$	$N = 160$	$N = 320$	$N = 640$	$N = 1280$
MAU	.10687590(-1)	.13031960(-2)	.16021729(-3)	.20027161(-4)	.45299530(-5)

Table 2. Comparison of maximum absolute error (Problem 1).

	Maximum absolute error				
	$N = 80$	$N = 160$	$N = 320$	$N = 640$	$N = 1280$
(3)	.1068(-1)	.1303(-2)	.1602(-3)	.2002(-4)	.4529(-5)
[13]	.1218(1)	.1431(0)	.1750(-1)	.2175(-2)	.2714(-3)

Table 3. Maximum absolute error (Problem 2).

	Maximum absolute error				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
MAU	.94715506(-6)	.11874363(-6)	.14668331(-6)	.89174137(-7)	.14551915(-6)

Table 4. Maximum absolute error (Problem 3).

	Maximum absolute error				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
MAU	.59137983(-6)	.89406967(-7)	.74505806(-7)	.44703484(-7)	.74505806(-7)

Table 5. Maximum absolute error (Problem 4).

N	Maximum absolute error			
	$MAUI$	$MAUM$	$MAUE$	MAU
16	.47538197(-5)	.18196437(-4)	.18044375(-8)	.18196437(-4)
32	.16210543(-5)	.32767859(-5)	.37252903(-8)	.32767859(-5)
64	.35636913(-6)	.57252339(-6)	.37252903(-8)	.57252339(-6)
128	.12665623(-7)	.82158941(-7)	.18198989(-8)	.82158941(-7)
256	.28580871(-8)	.58417232(-7)	.18205810(-8)	.58417232(-7)
512	.78565563(-8)	.62699833(-7)	.18205810(-8)	.62699823(-7)

Table 6. Comparison of maximum absolute error(Problem 4).

	Maximum absolute error			
	$N = 16$	$N = 32$	$N = 64$	$N = 128$
(3)	.181(-4)	.327(-5)	.572(-6)	.821(-7)
[13]	.223(-3)	.281(-4)	.518(-5)	.125(-5)
[2]	.196(-3)	.489(-4)	.122(-4)	.306(-5)
[6]	.712(-3)	.405(-3)	.222(-3)	.115(-3)
[10]	.126(-2)	.560(-3)	.310(-3)	.167(-3)

We have tested our numerical method for numerical solution four linear/nonlinear model problems including an obstacle problem considered. The numerical result for model problems for different values of N are presented in table

1-6. Observing the numerical result in the tables, we found maximum absolute error in solution decreases with decrease in step size h. The order of accuracy in model problem 1 and problem 2 are cubic and in other problems less than three. From the numerical results in Table 5, it is observed that the maximum absolute error occurs in interval $[\frac{1}{4}, \frac{3}{4}]$ for different value of N. The comparative numerical result in Table 2 and Table 6 show that our method outperforms the other existing methods. On the other hand we can conclude that our method is convergent and consistent with the theoretical development.

5. Conclusion

We have developed a cubic order novel finite difference method for the numerical solution of third order boundary value problems. We discretized the problem at discrete nodal points $x = x_{i-\frac{1}{2}}$, $i = 1, 2, \cdots, N$ in the domain of a considered problem. So we have obtained $N \times N$ a system of algebraic equations (3). The system of equations (3) is linear if source function $f(x, u)$ is linear otherwise nonlinear. Our proposed method (3) produced good numerical solution for model problems. Thus, we arrived at conclusion that our method is computationally efficient and accurate. The idea presented in this article is simple and leads to the possibility to develop higher order finite difference methods. Works in these directions are in progress.

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