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Transient Solution of an $M/M/1$ Variant Working Vacation Queue with Balking

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Abstract. This paper presents the transient solution of a variant working vacation queue with balking. Customers arrive according to a Poisson process and decide to join the queue with probability b or balk with $\bar{b} = 1 - b$. As soon as the system becomes empty, the server takes working vacation. The service times during regular busy period and working vacation period, and vacation times are assumed to be exponentially distributed and are mutually independent. We have obtained the transient-state probabilities in terms of modified Bessel function of the first kind by employing probability generating function, continued fractions and Laplace transform. In addition, we have also obtained some other performance measures.

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1. Introduction

Queueing systems with server vacations have been investigated extensively due to their wide applications in several areas including computer communication systems, manufacturing and production systems. Vacation models are useful in systems where the server wants to utilize the idle time for different purposes. For more detail on this topic the reader may refer to the surveys of Doshi [3], Takagi [18] and Tian and Zhang [20]. In classical vacation queues, the server completely stops service

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during the vacation period. However, there are numerous situations where the server remains active during the vacation period which is called working vacation (WV). Servi and Finn [15] introduced this class of semi-vacation policy. They studied an $M/M/1$ queue with multiple working vacations ($MWVs$). Liu et al. [10] derived the stochastic decomposition results in an $M/M/1$ queue with WV.

The concept of variant multiple vacation policy is relatively a new one where the server is allowed to take a certain fixed number of consecutive vacations, if the system remains empty at the end of a vacation. This kind of vacation schedule is investigated by Zhang and Tian [26] for the $Geo/G/1$ queue with multiple adaptive vacations. Banik [2] studied the infinite-buffer single server queue with variant of multiple vacation policy and batch Markovian arrival process by using matrix analytic method. The literature related to this kind of vacation can be identified in papers by Ke et al. [7] and Wang et al. [22]. Yue et al. [25] analyzed the $M/M/1$ queueing system with impatient customers and the variant of multiple vacation policy. In case of WV, Zhang and Hou [27] analyzed a steady state renewal input $GI/M/1/N$ queue with a variant of multiple working vacation by using matrix analytic method. A finite buffer $M/M/1$ queue with VWV and balking and reneging has been analyzed by Vijaya Laxmi and Jyothsna [21] wherein they obtained the steady state probabilities using matrix form solutions.

In the above literature, we have considered the steady state analysis of different continuous and discrete-time VWV queue models. However, there are areas in computer and communication systems which require time dependent analysis. For example, adaptive routing and load balancing in computer networks and communication systems require transient measures, such as transient queue length distribution. For such systems, one needs to know the transient behavior of the system. There are many systems which are operated only for a specified period of time t. Thus the investigation of the transient behavior of the queueing system is very important, not only from a theoretical point of view but also from its tremendous use in engineering applications.

Parthasarathy and Lenin [11, 12] used continued fractions to analyze the transient behavior of birth-death processes. Krishna Kumar and Arivudainambi [8] studied the transient behavior of an $M/M/1$ queue with catastrophes. Parthasarathy and Selvaraju [13] analyzed the transient behavior of an $M/M/1$ queue in which potential customers are discouraged by the queue length. Tarabia and El-Baz [19] have studied the exact transient solutions to non-empty Markovian queues by using the power series technique. Griffiths et al. [4] have studied the transient behavior in terms of modified Bessel function of the second kind. Parthasarathy and Sudhesh [14] have obtained the transient solution of $M/M/c$ queue with N-policy with the help of modified Bessel function of the second kind. Leonenko [9] studied a new approach for the transient solution of the $M/E_k/1$ queue. Sudhesh [16] has examined the transient behavior of a single server queue with catastrophes and customer impatience. Kalidass and Ramanath [5] have studied the time dependent analysis of a Markovian queue with server vacations and a waiting server. Kalidass et al. [6] have discussed the transient behavior of an $M/M/1$ multiple vacation queue and the possibilities of catastrophes. Sudhesh and Francis Raj [17] have obtained the time dependent system size probabilities of a $M/M/1$ queue with working vacation. Recently, Ammar [1] has investigated the transient solution of a $M/M/1$ multiple vacations queue and impatient customers.

In this paper, we consider an $M/M/1$ queue with variant working vacations and balking. On arrival, customers arrive according to a Poisson process and decide to join the queue with probability b or balk with $\bar{b} = 1 - b$. If there is no customer at the instant of a service completion, the server begins a WV of random length.

During the vacation period, the arriving customers are served generally at a lower rate. When a WV ends, the server inspects the system and switches to normal busy period, if there are customers in the queue; otherwise, takes another WV and continues so till K consecutive vacations have been taken. This policy refers to the phenomena of variant working vacations and one may note that this VWV generates MWV when $K \to \infty$ and single working vacation (SWV) when K is equal to 1. After the end of the Kth vacation, the server switches to normal busy period and stays idle or busy depending on the availability of the customers in the system. Further, we have obtained the explicit expressions for the transient-state probabilities in terms of modified Bessel function of the first kind by using probability generating function, continued fractions and Laplace transform when the server is in different states. We have derived closed-form expressions for some other performance measures.

The rest of the paper is organized as follows. Model description is presented in Section 2. In Section 3, we have obtained the explicit expressions for the transient state probabilities in terms of modified Bessel function of the first kind by using generating function, continued fractions and Laplace transform. The closed-form expressions of the performance measures are presented in Section 4. Finally, Section 5 concludes the paper.

2. Description of the model

We consider an $M/M/1$ queueing system with variant working vacations and balking. Customers arrive according to a Poisson process with rate λ, and the service is provided by a single server with exponential service rate μ. On arrival, customers decide to join the queue with probability b or balk with probability $\bar{b} = 1-b$. At the end of a service, if there is no customer in the system, the server begins a WV of random length which is exponentially distributed with rate ϕ. During WV, service is provided to the customers according to a Poisson process with rate η ($< \mu$). If the server finds customer at a WV completion instant, it returns to regular busy period; otherwise, the server takes WVs sequentially until 'K' consecutive WVs are complete; after which the server switches to normal busy period staying idle or busy. The customers are served according to $FCFS$ queue discipline. In addition, we assume that inter-arrival times, service times during vacations and normal busy period, and vacation times are mutually independent.

3. Analysis of the model

In this section, we derive the transient solutions for the model under consideration by employing generating functions, Laplace transforms, continued fractions and modified Bessel function.

Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ denote the state of the server at time t, which is defined as follows:

$$J(t) = \begin{cases} j, & \text{the server is on } (j+1)^{th} \text{ working vacation at time } t \\ & \text{for } j = 0, 1, \ldots, K-1, \\ K, & \text{the server is idle or busy at time } t, \end{cases}$$

The process $\{(L(t),\, J(t))\,,\ t \geq 0\}$ defines a continuous-time Markov process with state space $\Omega = \{(n,j) : n \geq 0,\ j = 0, 1, \ldots, K\}$. Let $P_{n,j}(t) = \text{Prob.}\{L(t) = n,\, J(t) = j\}\,,\ n \geq 0,\ j = 0, 1, \ldots, K$ denote the transient state probabilities. These probabilities satisfy the following forward Kolmogorov differential difference equations:

$$P'_{0,0}(t) = -(\lambda+\phi)P_{0,0}(t) + \eta P_{1,0}(t) + \mu P_{1,K}(t), \quad (1)$$

$$P'_{1,0}(t) = -(\lambda b+\phi+\eta)P_{1,0}(t) + \lambda P_{0,0}(t) + \eta P_{2,0}(t), \quad (2)$$

$$P'_{n,0}(t) = -(\lambda b+\phi+\eta)P_{n,0}(t) + \lambda b P_{n-1,0}(t) + \eta P_{n+1,0}(t), \ n \geq 2, \quad (3)$$

$$P'_{0,j}(t) = -(\lambda+\phi)P_{0,j}(t) + \eta P_{1,j}(t) + \phi P_{0,j-1}(t), \ 1 \leq j \leq K-1, \quad (4)$$

$$P'_{1,j}(t) = -(\lambda b+\phi+\eta)P_{1,j}(t) + \lambda P_{0,j}(t) + \eta P_{2,j}(t), \ 1 \leq j \leq K-1, \quad (5)$$

$$P'_{n,j}(t) = -(\lambda b+\phi+\eta)P_{n,j}(t) + \lambda b P_{n-1,j}(t) + \eta P_{n+1,j}(t), \quad 1 \leq j \leq K-1, \ n \geq 2, \quad (6)$$

$$P'_{0,K}(t) = -\lambda P_{0,K}(t) + \phi P_{0,K-1}(t), \quad (7)$$

$$P'_{1,K}(t) = -(\lambda b+\mu)P_{1,K}(t) + \lambda P_{0,K}(t) + \mu P_{2,K}(t) + \phi \sum_{j=0}^{K-1} P_{1,j}(t), \quad (8)$$

$$P'_{n,K}(t) = -(\lambda b+\mu)P_{n,K}(t) + \lambda b P_{n-1,K}(t) + \mu P_{n+1,K}(t) + \phi \sum_{j=0}^{K-1} P_{n,j}(t), \quad n \geq 2. \quad (9)$$

Let us suppose that initially there is no customer in the system, i.e., $P_{0,0}(0) = 1$ and $P_{n,j}(0) = 0, \ n \geq 1, \ 0 \leq j \leq K$. Let the probability generating function be

$$G_K(t,z) = \sum_{n=0}^{\infty} P_{n,K}(t) z^n, |z| \leq 1, \quad (10)$$

with an initial condition $G_K(0,z) = 0$. Now, multiplying equations (7), (8) and (9) by z^n, and summing over all possible values of n and rearranging the terms, we get

$$\begin{aligned}\frac{\partial G_K(t,z)}{\partial t} &= \left[(\lambda b z + \frac{\mu}{z}) - (\lambda b+\mu)\right] G_K(t,z) + (\lambda \bar{b} - \frac{\mu}{z})(1-z)P_{0,K}(t) \\ &- \mu P_{1,K}(t) + \phi \left[\sum_{j=0}^{K-1}\sum_{m=0}^{\infty} P_{m,j}(t) z^m - \sum_{j=1}^{K-1} P_{0,j-1}(t)\right].\end{aligned} \quad (11)$$

Equation (11) is a first order linear differential equation in $G_K(t,z)$ for fixed z. Using the initial condition its solution can be given as below:

$$\begin{aligned}G_K(t,z) =& (\lambda \bar{b} - \frac{\mu}{z})(1-z)\int_0^t P_{0,K}(u) e^{A(t-u)} du - \mu \int_0^t P_{1,K}(u) e^{A(t-u)} du \\ &+ \phi \sum_{j=0}^{K-1} \int_0^t \sum_{m=0}^{\infty} P_{m,j}(u) z^m e^{A(t-u)} du - \phi \sum_{j=1}^{K-1} \int_0^t P_{0,j-1}(u,) e^{A(t-u)} du,\end{aligned} \quad (12)$$

where $A = (\lambda b z + \frac{\mu}{z}) - (\lambda b + \mu)$. It is well known that if $\alpha = 2\sqrt{\lambda b \mu}$ and $\beta = \sqrt{\frac{\lambda b}{\mu}}$, then

$$\exp\left[(\lambda bz+\frac{\mu}{z})t\right]=\sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha t),$$

where $I_n(\alpha t)$ is the modified Bessel function of the first kind. Comparing the coefficients of z^n on both sides of equation (12) for $n = 1, 2, 3, \ldots$, we get

$$\begin{aligned}P_{n,K}(t) =&\lambda\bar{b}\int_0^t P_{0,K}(u)\beta^n\big[I_n(\alpha(t-u))-\beta^{-1}I_{n-1}(\alpha(t-u))\big]e^{-(\lambda b+\mu)(t-u)}du\\&+\mu\int_0^t P_{0,K}(u)\beta^n\big[I_n(\alpha(t-u))-\beta I_{n+1}(\alpha(t-u))\big]e^{-(\lambda b+\mu)(t-u)}du\\&-\mu\int_0^t P_{1,K}(u)\beta^n I_n(\alpha(t-u))e^{-(\lambda b+\mu)(t-u)}du\\&+\phi\sum_{j=0}^{K-1}\int_0^t\sum_{m=0}^{\infty}P_{m,j}(u)\beta^{n-m}I_{n-m}(\alpha(t-u))e^{-(\lambda b+\mu)(t-u)}du\\&-\phi\sum_{j=1}^{K-1}\int_0^t P_{0,j-1}(u)\beta^n I_n(\alpha(t-u))e^{-(\lambda b+\mu)(t-u)}du.\end{aligned}\tag{13}$$

Equation (12) holds for $n = -1, -2, -3, \ldots$, with left hand side replaced by zero. Using $I_{-n}(x) = I_n(x)$ for $n = 1, 2, 3, \ldots$, we get

$$\begin{aligned}0 = \lambda\bar{b}&\int_0^t P_{0,K}(u)\beta^{-n}\left[I_n(\alpha(t-u))-\beta^{-1}I_{n+1}(\alpha(t-u))\right]E\,du\\&+\mu\int_0^t P_{0,K}(u)\beta^{-n}\left[I_n(\alpha(t-u))-\beta I_{n-1}(\alpha(t-u))\right]E\,du\\&-\mu\int_0^t P_{1,K}(u)\beta^{-n}I_n(\alpha(t-u))E\,du\\&+\phi\sum_{j=0}^{K-1}\int_0^t\sum_{m=0}^{\infty}P_{m,j}(u)\beta^{-n-m}I_{n+m}(\alpha(t-u))E\,du\\&-\phi\sum_{j=1}^{K-1}\int_0^t P_{0,j-1}(u)\beta^{-n}I_n(\alpha(t-u))E\,du,\end{aligned}\tag{14}$$

where $E = e^{-(\lambda b+\mu)(t-u)}$. From equations (13) and (14), for $n = 1, 2, 3, \ldots$,

$$\begin{aligned}P_{n,K}(t) =& \left(\mu\beta^{n+1}-\lambda\bar{b}\beta^{n-1}\right)\int_0^t P_{0,K}(u)\big[I_{n-1}(\alpha(t-u))-I_{n+1}(\alpha(t-u))\big]E\,du\\&+\phi\sum_{j=0}^{K-1}\int_0^t\sum_{m=1}^{\infty}P_{m,j}(u)\beta^{n-m}\big[I_{n-m}(\alpha(t-u))-I_{n+m}(\alpha(t-u))\big]E\,du.\end{aligned}\tag{15}$$

From equation (7), we get

$$P_{0,K}(t) = \phi \int_0^t P_{0,K-1}(u) e^{-\lambda(t-u)} du. \tag{16}$$

Equations (15) and (16) show the transient state probabilities of the system during busy period and probability of idle server during busy state, respectively.

Now, we evaluate $P_{n,j}(t)$ for $j = 0, 1, 2, \ldots, K-1$, which represent the transient state probabilities during WV. Let $\tilde{P}_{n,j}(s)$ be the Laplace transform of $P_{n,j}(t)$ for $j = 0, 1, \ldots, K-1$. On taking the Laplace transform of equation (1), we get

$$\tilde{P}_{0,0}(s) = \frac{1}{(s+\lambda+\phi) - \eta\frac{\tilde{P}_{1,0}(s)}{\tilde{P}_{0,0}(s)} - \mu\frac{\tilde{P}_{1,K}(s)}{\tilde{P}_{0,0}(s)}}, \tag{17}$$

and from equations (2) and (3) we get the expression

$$\frac{\tilde{P}_{n,0}(s)}{\tilde{P}_{n-1,0}(s)} = \frac{\lambda}{(s+\lambda b+\phi+\eta) - \eta\frac{\tilde{P}_{n+1,0}(s)}{\tilde{P}_{n,0}(s)}}, \text{for } n = 1, 2, 3, \ldots. \tag{18}$$

We observe that equation (18) represents a continued fraction which takes the following form:

$$\tilde{P}_{n,0}(s) = \tilde{V}_n(s)\ \tilde{P}_{0,0}(s), \text{ for } n = 1, 2, 3, \ldots. \tag{19}$$

where

$$\tilde{V}_n(s) = \frac{1}{b}\left(\sqrt{\frac{\lambda b}{\eta}}\right)^n \left(\frac{d_1 - \sqrt{{d_1}^2 - (2\sqrt{\eta\lambda b})^2}}{2\sqrt{\eta\lambda b}}\right)^n, \ d_1 = (s+\lambda b+\phi+\eta).$$

By taking inverse Laplace transform of equation (19), we get transient state probabilities for $j = 0$ as

$$P_{n,0}(t) = V_n(t) * P_{0,0}(t), \text{for } n = 1, 2, 3, \ldots, \tag{20}$$

where $*$ denotes convolution and $V_n(t)$ is the inverse Laplace transform of $\tilde{V}_n(s)$ which is given by

$$V_n(t) = \frac{n}{b\,t}\left(\sqrt{\frac{\lambda b}{\eta}}\right)^n e^{-(\lambda b+\phi+\eta)t} I_n((2\sqrt{\eta\lambda b})t). \tag{21}$$

Similarly, for $1 \leq j \leq K-1$, from equation (4), we get

$$(s+\lambda+\phi)\tilde{P}_{0,j}(s) - \eta\tilde{P}_{1,j}(s) = \phi\tilde{P}_{0,j-1}(s), \tag{22}$$

and from equations (5) and (6), we obtain

$$\tilde{P}_{n,j}(s) = \tilde{V}_n(s)\ \tilde{P}_{0,j}(s), \text{ for } n \geq 1,\ 1 \leq j \leq K-1. \tag{23}$$

For $n = 1$, equations (22) and (23) give

$$\tilde{P}_{0,j}(s) = (\phi B(s))^j \tilde{P}_{0,0}(s),\ 1 \leq j \leq K-1. \tag{24}$$

where

$$B(s) = \sum_{i=0}^{\infty} \frac{\eta^i}{(s+\lambda+\phi)^{(i+1)}} (\tilde{V}_1(s))^i. \tag{25}$$

Substituting equation (24) in equation (23) and taking inverse Laplace transform, we get

$$P_{n,j}(t) = V_n(t) * (\phi^j\, ILT[(B(s))^j]) * P_{0,0}(t),\ n \geq 1,\ 1 \leq j \leq K-1, \tag{26}$$

where

$$ILT[(B(s))^j] = \sum_{i=0}^{\infty} \binom{-j}{i} (-1)^i e^{-(\lambda+\phi)t} \frac{\eta^i t^{i+j-1}}{(i+j-1)!} * [V_1(t)]^{*i}. \tag{27}$$

Since the transient state probabilities during WV, $P_{n,j}(t)$ for $n \geq 1,\ 0 \leq j \leq K-1$, are dependent on $P_{0,0}(t)$, next we evaluate $P_{0,0}(t)$ as follows:
With $n = 1$, equations (15), (17) and (19) give

$$\begin{aligned} P_{1,K}(t) = \big(\mu\beta^2 - \lambda\bar{b}\big) &\int_0^t P_{0,K}(u)\big[I_0(\alpha(t-u)) \\ &- I_2(\alpha(t-u))\big] e^{-(\lambda b+\mu)(t-u)} du \\ &+ \phi \sum_{j=0}^{K-1} \int_0^t \sum_{m=1}^{\infty} P_{m,j}(u)\beta^{1-m}\big[I_{1-m}(\alpha(t-u)) \\ &- I_{1+m}(\alpha(t-u))\big] e^{-(\lambda b+\mu)(t-u)} du. \end{aligned} \tag{28}$$

Using equation (24), the Laplace transform of equation (16) is obtained as

$$\tilde{P}_{0,K}(s) = \frac{\phi^K (B(s))^{K-1}}{(s+\lambda)} \tilde{P}_{0,0}(s). \tag{29}$$

Taking Laplace transform on equations (28) and using equation (29), and equation (19) together with (17), after some mathematical manipulations, we obtain

$$\begin{aligned} \tilde{P}_{0,0}(s) = &\sum_{l=0}^{\infty}\sum_{r=0}^{l}\sum_{i=0}^{r} \frac{1}{(s+\lambda+\phi)^{l+1}} \binom{l}{r}\binom{r}{i} (\eta\tilde{V}_1(s))^i \\ &\times \left[\mu\lambda(b-\bar{b})\frac{\phi^K B(s)^{K-1}}{s+\lambda} LI\right]^{r-i} \\ &\times \left[\phi\mu \sum_{j=0}^{K-1} (\phi B(s))^j\right]^{l-r} \left[\sum_{m=1}^{\infty} \tilde{V}_m(s)\beta^{1-m} LI_m\right]^{l-r}, \end{aligned} \tag{30}$$

where LI is Laplace transform of the expression $[I_0(\alpha t) - I_2(\alpha t)]\, e^{-(\lambda b+\mu)t}$ and LI_m is the Laplace transform of the expression $[I_{1-m}(\alpha t) - I_{1+m}(\alpha t)]\, e^{-(\lambda b+\mu)t}$.
Taking the inverse Laplace transform of equation (30), we get

$$P_{0,0}(t) = \sum_{l=0}^{\infty}\sum_{r=0}^{l}\sum_{i=0}^{r}\binom{l}{r}\binom{r}{i} e^{-(\lambda+\phi)t}\frac{t^l}{l!} * \eta^i [V_1(t)]^{*i}$$
$$* (ILT[B(s)^{K-1}])^{*(r-i)} * \left[e^{-(\lambda b+\mu)t}\left(I_0(\alpha t) - I_2(\alpha t)\right)\right]^{*(r-i)}$$
$$* [\mu\lambda(b-\bar{b})\phi^K]^{r-i} e^{-\lambda t}\frac{t^{r-i-1}}{(r-i-1)!} * [\mu\phi]^{l-r}\left[\sum_{j=0}^{K-1}\phi^j ILT[B(s)^j]\right]^{*(l-r)}$$
$$* \left[\sum_{m=1}^{\infty} V_m(t) * \beta^{1-m}\left(I_{m-1}(\alpha t) - I_{m+1}(\alpha t)\right) e^{-(\lambda b+\mu)t}\right]^{*(l-r)}. \tag{31}$$

where $*$ denotes the convolution, while " $* q$" stands for q-fold convolution. In equation (31), the functions $V_m(t)$, $ILT[B(s)^j]$ and $ILT[B(s)^{K-1}]$ are calculated by using the equations (21) and (27).

4. Performance measures

In this section, we consider the time dependent performance measures of the transient system.

i. The probability that the server is idle during busy period is obtained from equation (29) by taking inverse Laplace transform as

$$P_{0,K}(t) = \phi^K e^{-\lambda t} * \left(\sum_{i=0}^{\infty}\binom{1-K}{i}(-1)^i e^{-(\lambda+\phi)t}\frac{\eta^i t^{i+K-2}}{(i+K-2)!}\right.$$
$$\left. *[V_1(t)]^{*i}\right) * P_{0,0}(t), \qquad \text{for } K > 1, \tag{32}$$

where $V_1(t) = \frac{1}{b\,t}\left(\sqrt{\frac{\lambda b}{\eta}}\right) I_1((2\sqrt{\eta\lambda b})t)e^{-(\lambda b+\phi+\eta)t}$.

Remark 1 From equation (29), for $K = 1$ (SWV), we get $P_{0,1}(t) = \phi e^{-\lambda t} * P_{0,0}(t)$.

ii. The probability that the system is empty during WV is obtained from equation (24) by taking inverse Laplace transform as

$$\sum_{j=0}^{K-1} P_{0,j}(t) = \sum_{j=1}^{K-1}\phi^j\left(\sum_{i=0}^{\infty}\binom{-j}{i}(-1)^i e^{-(\lambda+\phi)t}\frac{\eta^i t^{i+j-1}}{(i+j-1)!}\right.$$
$$\left. *[V_1(t)]^{*i}\right) * P_{0,0}(t) + P_{0,0}(t). \tag{33}$$

iii. Let P_{WV} be the probability that the server is on WV. From equation (26), we obtain

$$\begin{aligned} P_{WV} &= \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} P_{n,j}(t) \\ &= \sum_{n=1}^{\infty} (V_n(t) * P_{0,0}(t)) \\ &\quad + \sum_{j=1}^{K-1} \sum_{n=1}^{\infty} \Big(V_n(t) * (\phi^j \, ILT[B(s)^j]) * P_{0,0}(t) \Big), \end{aligned} \tag{34}$$

where $ILT[B(s)^j]$ are calculated by using equation (27).

iv. The probability that the server is busy is given by

$$\begin{aligned} P_b &= \sum_{n=1}^{\infty} P_{n,K}(t) \\ &= \sum_{n=1}^{\infty} \bigg(\left(\mu\beta^{n+1} - \lambda\bar{b}\beta^{n-1}\right) P_{0,K}(t) * \big[I_{n-1}(\alpha(t)) \\ &\quad - I_{n+1}(\alpha(t))\big] e^{-(\lambda b+\mu)(t)} \\ &\quad + \phi \sum_{j=0}^{K-1} \bigg[\sum_{m=1}^{\infty} P_{m,j}(t) * \beta^{n-m} \big[I_{n-m}(\alpha(t)) \\ &\quad - I_{n+m}(\alpha(t))\big] e^{-(\lambda b+\mu)(t)} \bigg] \bigg). \end{aligned} \tag{35}$$

v. Let $L(t)$ be the number of customers in the system at time t. Then, the average number of customers in the system at time t is given by

$$E[L(t)] = \sum_{n=1}^{\infty} n \left(\sum_{j=0}^{K-1} P_{n,j}(t) + P_{n,K}(t) \right). \tag{36}$$

From (36), we get

$$E[L'(t)] = \sum_{n=1}^{\infty} n \left(\sum_{j=0}^{K-1} P'_{n,j}(t) + P'_{n,K}(t) \right).$$

From equations (2), (3), (5), (6), (8) and (9), after considerable mathematical manipulations, the above equation will lead to the following differential equation

$$\begin{aligned} \frac{dE[L(t)]}{dt} =& \lambda b - \lambda\bar{b} \left(\sum_{j=0}^{K=1} P_{0,j}(t) + P_{0,K}(t) \right) - \eta \sum_{n=1}^{\infty} \sum_{j=0}^{K=1} P_{n,j}(t) \\ & - \mu \sum_{n=1}^{\infty} P_{n,K}(t). \end{aligned}$$

Therefore, integration of the above equation yields

$$\begin{aligned} E[L(t)] =& \lambda bt - \lambda\bar{b}\left(\sum_{j=0}^{K=1}\int_0^t P_{0,j}(u)du + \int_0^t P_{0,K}(u)du\right) \\ & - \eta\sum_{n=1}^{\infty}\left(\sum_{j=0}^{K=1}\int_0^t P_{n,j}(u)du\right) - \mu\sum_{n=1}^{\infty}\int_0^t P_{n,K}(u)du. \end{aligned} \tag{37}$$

vi. The variance of the number of the customers in the system at time t is given by

$$Var[L(t)] = E[L^2(t)] - E[L(t)]^2, \tag{38}$$

where

$$E[L^2(t)] = \sum_{n=1}^{\infty} n^2\left(\sum_{j=0}^{K-1} P_{n,j}(t) + P_{n,K}(t)\right) \text{ and } E[L^2(0)] = 0.$$

We have $E'[L^2(t)] = \sum_{n=1}^{\infty} n^2\left(\sum_{j=0}^{K-1} P'_{n,j}(t) + P'_{n,K}(t)\right)$ and from equations (2),(3), (5), (6), (8) and (9), after considerable mathematical manipulations, the above equation will lead to the following differential equation

$$\begin{aligned} \frac{dE[L^2(t)]}{dt} =& \lambda b - \lambda\bar{b}\sum_{j=0}^{K} P_{0,j}(t) + 2\lambda b E[L(t)] \\ & - \eta\sum_{n=1}^{\infty}(2n-1)\sum_{j=0}^{K=1} P_{n,j}(t) - \mu\sum_{n=1}^{\infty}(2n-1)P_{n,K}(t). \end{aligned}$$

On integration, we have

$$\begin{aligned} E[L^2(t)] = & \lambda bt - \lambda\bar{b}\sum_{j=0}^{K}\int_0^t P_{0,j}(u)du + 2\lambda b\int_0^t E[L(u)]du \\ & -\eta\sum_{n=1}^{\infty}(2n-1)\left(\sum_{j=0}^{K=1}\int_0^t P_{n,j}(u)du\right) \\ & -\mu\sum_{n=1}^{\infty}(2n-1)\int_0^t P_{n,K}(u)du. \end{aligned} \tag{39}$$

5. Conclusions

In this paper, we have studied the transient solutions of an $M/M/1$ queueing system with variant WVs and balking. We have obtained the explicit analytical expressions for the transient state system size probabilities in terms of modified Bessel function of the first kind by using probability generating function, continued fractions and Laplace transform. We have also derived the closed-form expressions for some performance measures. The numerical results for this model are left for further investigation.

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References

[1] S. I. Ammar, Transient analysis of an $M/M/1$ queue with impatient behavior and multiple vacations, Applied Mathematics and Computation, **260** (2015) 97–105.
[2] A. D. Banik, The infinite-buffer single server queue with a variant of multiple vacation policy and batch Markovian arrival process, Applied Mathematical Modelling, **33 (7)** (2009) 3025-3039.
[3] B. T. Doshi, Queueing systems with vacations: A survey, Queueing Systems, **1 (1)** (1986) 29-66.
[4] J. D. Griffiths, G. M. Leonenko and J. E. Williams, The transient solution to $M/E_k/1$ queue, Operations Research Letters, **34 (3)** (2006) 349–354.
[5] K. Kalidass and K. Ramanath, Time dependent analysis of $M/M/1$ queue with server vacations and a waiting server, Proceedings of the 6th International Conference on Queueing Theory and Network Applications, Korea University, South Korea, (2011) 77–83.
[6] K. Kalidass, J. Gnanaraj, S. Gopinath and R. Kasturi, Transient analysis of an $M/M/1$ queue with a repairable server and multiple vacations, International Journal of Mathematics in Operational Research, **6 (2)** (2014) 193–216.
[7] J. C. Ke, K. B. Huang and W. L. Pearn, Randomized policy of a Poisson input queue with J vacations, Journal of System Science and System Engineering, **19 (1)** (2010) 50-71.
[8] B. Krishna Kumar and D. Arivudainambi, Transient solution of an $M/M/1$ queue with catastrophes, Computers & Mathematics with applications, **40(10)** (2000) 1233–1240.
[9] G. M. Leonenko, A new formula for the transient solution of the Erlang queueing model, Statistics & Probability Letters, **79 (3)** (2009) 400–406.
[10] W. Liu, X. Xu and N. Tian, Stochastic decompositions in the $M/M/1$ queue with working vacations, Operations Research Letters, **35 (5)** (2007) 595-600.
[11] P. R. Parthasarathy and R. B. Lenin, On the exact transient solution of finite birth and death processes with specific quadratic rates, Mathematical Scientist, **22** (1997) 92–105.
[12] P. R. Parthasarathy and R. B. Lenin, On the numerical solution of transient probabilities of quadratic birth and death processes, Journal of Difference Equations and Applications, **4 (4)** (1998) 365–379.
[13] P. R. Parthasarathy and N. Selvaraju, Transient analysis of a queue where potential customers are discouraged by queue length, Mathematical Problems in Engineering, **7 (5)** (2001) 433–454.
[14] P. R. Parthasarathy and R. Sudhesh, Transient solution of a multi server Poisson queue with N-policy, Computers & Mathematics with Applications, **55 (3)** (2008) 550–562.
[15] L. D. Servi and S. G. Finn, $M/M/1$ queues with working vacations ($M/M/1/WV$), Performance Evaluation, **50 (1)** (2002) 41-52.
[16] R. Sudhesh, Transient analysis of a queue with system disasters and customer impatience, Queueing systems, **66 (1)** (2010) 95–105.
[17] R. Sudhesh and L. Francis Raj, Computational analysis of stationary and transient distribution of single server queue with working vacation, Global Trends in Computing and Communication Systems, (2012) 480–489.
[18] H. Takagi, Queueing Analysis: A Foundation of Performance Evaluation, Volume 1: Vacation and Priority Systems. Part 1. Elsevier Science Publishers, Amsterdam, (1991).
[19] A. M. K. Tarabia and A. H. El-Baz, Exact transient solutions of nonempty Markovian queues, Computers & Mathematics with Applications, **52** (2006) 985–996.
[20] N. Tian and Z. G. Zhang, Vacation Queueing Models: Theory and Applications, Springer Science & Business Media, (2006).
[21] P. Vijaya Laxmi and K. Jyothsna, Performance analysis of variant working vacation queue with balking and reneging, International Journal of Mathematics in Operational Research, **6 (4)** (2014) 505-521.
[22] T. Y. Wang, J. C. Ke and F. M. Chang, On the discrete-time $Geo/G/1$ queue with randomized vacations and at most J vacations, Applied Mathematical Modelling, **35 (5)** (2011) 2297-2308.
[23] D. Wu and H. Takagi, $M/G/1$ queue with multiple working vacations, Performance Evaluation, **63 (7)** (2006) 654-681.
[24] D. Y. Yang and C. H. Wu, Cost-minimization analysis of a working vacation queue with N-policy and server breakdowns, Computers & Industrial Engineering, **82** (2015) 151-158.
[25] D. Yue, W. Yue, Z. Saffer and X. Chen, Analysis of an $M/M/1$ queueing system with impatient customers and a variant of multiple vacation policy, Journal of Industrial and Management Optimization, **10 (1)** (2014) 89-112.
[26] Z. G. Zhang and N. Tian, Discrete time $Geo/G/1$ queue with multiple adaptive vacations, Queueing Systems, **38 (4)** (2001) 419-429.
[27] M. Zhang, and Z. Hou, Steady state analysis of the $GI/M/1/N$ queue with a variant of multiple working vacations, Computers & Industrial Engineering, **61 (4)** (2011) 1296-1301.