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A New Implicit Finite Difference Method for Solving Time Fractional Diffusion Equation

E. Afshari*

Young Researcher Club, khomein Branch, Islamic Azad University, Khomein, Iran.

Abstract. In this paper, a time fractional diffusion equation on a finite domain is considered. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first order time derivative by a fractional derivative of order $0 < \alpha \leqslant 1$ (in the Riemann-Liovill or Caputo sence). In equation that we consider the time fractional derivative is in the Caputo sense. We propose a new finite difference method for solving time fractional diffusion equation. In our method firstly, we transform the Caputo derivative into Riemann-Liovill derivative. The stability and convergence of this method are investigated by a Fourier analysis. We show that this method is unconditionally stable and convergent with the convergence order $O(\tau^2 + h^2)$, where τ and h are time and space steps respectively. Finally, a numerical example is given that confirms our theoretical analysis and the behavior of error is examined to verify the order of convergence.

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1. Introduction

Fractional order partial differential equation models have been proposed in many research field, such as fluid mechanics, biology, plasma physics, finance and so on [1, 2]. In general, the analytical solution of many fractional partial differential

*Corresponding author. Email: elham.afshari@iaukhomein.ac.ir

http://ijm2c.iauctb.ac.ir

equations is not easy to derive, therefore, the numerical solution of these equations have been considered by researchers.

In the standard diffusion equation if the first order time derivative is replaced by a fractional derivative of order $0 < \alpha \leqslant 1$ (in the Riemann-Liovill or Caputo sence), time fractional diffusion equation will obtain.

P. Zhung and F. liu, [5] proposed an implicit difference approximation for solving the time fractional diffusion equation and they showed the implicit difference approximation is unconditionally stable and convergent with order $O(\tau + h^2)$.

In this paper, the following time-fractional diffusion equation is considered.

$$ {}^cD_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 \leqslant x \leqslant L, 0 < t \leqslant T, \tag{1} $$

with initial and boundary conditions

$$ u(x,0) = g(x), \quad 0 < x < L, \tag{2} $$

$$ u(0,t) = u(L,t) = 0, \quad 0 < t \leqslant T. \tag{3} $$

Where $0 < \alpha < 1$.

The fractional derivative ${}^cD_t^\alpha u(x,t)$ in (1) is the Caputo fractional derivative of order α, defined by [3]

$$ {}^cD_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^\alpha}, \quad 0 \leqslant \alpha \leqslant 1. $$

where $\Gamma(.)$ is the gamma function.

In this paper, a new implicit finite difference method for solving equations $(1)-(3)$ is proposed. we show this method is unconditionally stable and convergence with order $O(\tau^2 + h^2)$, where τ and h are time and space step size, respectively.

2. The new implicit method

For numerical scheme, define $t_k = k\tau,\ k = 0,1,...,N$, and $x_i = ih,\ i = 0,1,...,M$, where $\tau = \frac{T}{n}$ and $h = \frac{L}{M}$ are time and space steps, respectively. Let u_i^k be the numerical estimate of the exact value of $u(x,t)$ at the mesh point (x_i, t_k).

Firstly, we define the Riemann-Liouvill fractional derivative and integral and consider several lemmas.

We show The Riemann-Liouvill fractional derivative and integral of order α, $(0 < \alpha \leqslant 1)$ by ${}_0D_t^\alpha f$ and ${}_0D_t^{-\alpha} f$, respectively, and defined by [3]

$$ {}_0D_t^\alpha f(t) = \tfrac{1}{\Gamma(1-\alpha)} \tfrac{d}{dt} \int_0^t \tfrac{f(\xi)}{(t-\xi)^\alpha} d\xi, \quad 0 < \alpha < 1, $$

$$ {}_0D_t^{-\alpha} f(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t \tfrac{f(\xi)}{(t-\xi)^{1-\alpha}} d\xi, \quad \alpha > 0. $$

Lemma 2.1 [7]*Let*

$$ b_j^{(\alpha)} = (j+1)^\alpha - j^\alpha, \qquad j = 0,1,... \tag{4} $$

then $b_j^{(\alpha)}$ *satisfies*

1) $b_0^{(\alpha)} = 1,\ b_j^{(\alpha)} > 0, \quad j = 0,1,...$

2) $b_j^{(\alpha)} > b_{j+1}^{(\alpha)}, \quad j = 0, 1, ...$

3) *there is a positive constant* $c > 0$ *such that* $\tau \leqslant c\, b_j^{(\alpha)} \tau^{\gamma}$.

LEMMA 2.2 [4]*Suppose* $y(t) \in C^3[0,T]$, *then*

$$
{}_0D_t^{-\alpha} y(t_{k+1}) - {}_0D_t^{-\alpha} y(t_k) =
$$

$$
\frac{\tau^\alpha}{\Gamma(\alpha+1)} \left((a_k^{(\alpha)} - a_{k+1}^{(\alpha)}) y(t_0) + \sum_{j=0}^{k-1} (v_{j+1}^{(\alpha)} - v_j^{(\alpha)}) y(t_{k-j}) + \frac{1}{\Gamma(\alpha+1)} y(t_{k+1}) \right)
$$

$$
+R_1 \tag{5}
$$

where

$$
a_j^{(\alpha)} = (j+1)^\alpha - \frac{1}{\alpha+1}[(j+1)^{\alpha+1} + j^{\alpha+1}], \quad j = 0, 1, ... \tag{6}
$$

$$
a_{-1}^{(\alpha)} = 0, \tag{7}
$$

$$
v_j^{(\alpha)} = \frac{1}{\alpha+1}[(j+1)^{\alpha+1} - 2j^{\alpha+1} + (j-1)^{\alpha+1}], \quad j = 1, 2, ... \tag{8}
$$

$$
v_0^{(\alpha)} = \frac{1}{\alpha+1}, \tag{9}
$$

$$
|R_1| \leqslant C_1 \tau^{\alpha+2} b_k. \tag{10}
$$

The Caputo fractional derivative ${}^cD_t^\alpha u$ and the Riemann-Liovill fractional derivative ${}_0D_t^\alpha u$ are connected with each other by the following relation [3]

$$
({}^cD_t^\alpha u)(x,t) = ({}_0D_t^\alpha u)(x,t) - \frac{u(x,0)}{\Gamma(1-\alpha)} t^{-\alpha}, \quad 0 < \alpha \leqslant 1,
$$

by substituting above equation into (1), we have

$$
({}_0D_t^\alpha u)(x,t) - \frac{u(x,0)}{\Gamma(1-\alpha)} t^{-\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),
$$

therefore

$$
({}_0D_t^\alpha u)(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{u(x,0)}{\Gamma(1-\alpha)} t^{-\alpha} + f(x,t). \tag{11}
$$

By integrating both side of above equation with respect t from t_k to t_{k+1} we have

$$
({}_0D_t^{\alpha-1} u)(x_i, t_{k+1}) - ({}_0D_t^{\alpha-1} u)(x_i, t_k) = \int_{t_k}^{t_{k+1}} \frac{\partial^2 u(x_i,t)}{\partial x^2} dt
$$

$$
+ \frac{u(x_i,t_0)}{(1-\alpha)\Gamma(1-\alpha)} \left(t_{k+1}^{1-\alpha} - t_k^{1-\alpha} \right) + \int_{t_k}^{t_{k+1}} f(x_i,t) dt. \tag{12}
$$

$\int_{t_k}^{t_{k+1}} \frac{\partial^2 u(x_i,t)}{\partial x^2} dt$ can be approximated by the trapezoidal rule. For approximating the second-order space derive, we use of the following symmetric second difference quotient.

$$\frac{\partial^2 u(x_i,t_k)}{\partial x^2} = \frac{\delta_x^2 u(x_i,t_k)}{h^2} + O(h^2), \tag{13}$$

where

$$\delta_x^2 u(x_i,t_k) = u(x_{i+1},t_k) - 2u(x_i,t_k) + u(x_{i-1},t_k),$$

and $({}_0D_t^{\alpha-1}u)(x_i,t_{k+1}) - ({}_0D_t^{\alpha-1}u)(x_i,t_k)$ can be approximated by (5). Thus, we have

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left((a_k^{(1-\alpha)} - a_{k+1}^{(1-\alpha)})u_i^0 + \sum_{j=0}^{k-1}(v_{j+1}^{(1-\alpha)} - v_j^{(1-\alpha)})u_i^{k-j} + \frac{1}{\Gamma(2-\alpha)}u_i^{k+1}\right)$$
$$= \frac{\tau}{2h^2}\left(\delta_x^2 u_i^{k+1} + \delta_x^2 u_i^k\right) + \frac{u_i^0}{(1-\alpha)\Gamma(1-\alpha)}\left(t_{k+1}^{1-\alpha} - t_k^{1-\alpha}\right) + F_i^{k+1},$$

where $F_i^{k+1} = \int_{t_k}^{t^{k+1}} f(x_i,t)dt$.
By noticing $(1-\alpha)\Gamma(1-\alpha) = \Gamma(2-\alpha)$ and $t_{k+1}^{1-\alpha} - t_k^{1-\alpha} = \tau^{1-\alpha}\left((k+1)^{1-\alpha} - k^{1-\alpha}\right)$, we have

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left((a_k^{(1-\alpha)} - a_{k+1}^{(1-\alpha)})u_i^0 + \sum_{j=0}^{k-1}(v_{j+1}^{(1-\alpha)} - v_j^{(1-\alpha)})u_i^{k-j} + \frac{1}{\Gamma(2-\alpha)}u_i^{k+1}\right)$$
$$= \frac{\tau}{2h^2}\left(\delta_x^2 u_i^{k+1} + \delta_x^2 u_i^k\right) + u_i^0\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left((k+1)^{1-\alpha} - k^{1-\alpha}\right) + F_i^{k+1},$$
$$i = 1,2,...,M-1, \quad k = 0,1,...,N-1.$$

Let $s_1 = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}$ and $s_2 = \frac{\tau}{2h^2}$, then we have

$$s_2\delta_x^2 u_i^{k+1} - \frac{s_1}{2-\alpha}u_i^{k+1} = \tag{14}$$
$$s_1\left((a_k^{(1-\alpha)} - a_{k+1}^{(1-\alpha)})u_i^0 + \sum_{j=0}^{k-1}(v_{j+1}^{(1-\alpha)} - v_j^{(1-\alpha)})u_i^{k-j}\right) - s_2\delta_x^2 u_i^k$$
$$-s_1 u_i^0\left((k+1)^{1-\alpha} - k^{1-\alpha}\right) - F_i^{k+1}, i = 1,2,...,M-1, \quad k = 0,1,...,N-1.$$

The initial and boundary conditions can be discretized by

$$u_i^0 = g_i, \quad i = 1,2,...,M-1, \tag{15}$$
$$u_0^k = u_M^k = 0, \quad k = 1,2,...,N. \tag{16}$$

3. Stability analysis

Firstly, the following lemma is considered

Lemma 3.1 [4]
1) *For $j = 0,1,...,$ the coefficients $a_j^{(1-\alpha)}$ defined by* (6) *are positive and $a_j^{(1-\alpha)} >$*

$a_{j+1}^{(1-\alpha)}$.
2) *For* $j=0,1,...,$ *the coefficients* $v_j^{(1-\alpha)}$ *defined by* (8) *and* (9) *are positive, and for* $j=1,2,...,$ $v_j^{(1-\alpha)} > v_{j+1}^{(1-\alpha)}$.

Now, the stability of the numerical method (14) – (16) is investigated by using Fourier analysis.
Suppose $\tilde{u_i^k}$ be the approximation solution of the implicit method and define

$$\rho_j^k = u_j^k - \tilde{u_i^k},$$

then the following roundoff error equations can be obtained.

$$s_2\delta_x^2\rho_j^{k+1} - \frac{s_1}{2-\alpha}\rho_j^{k+1} = \tag{17}$$

$$s_1\left((a_k^{(1-\alpha)} - a_{k+1}^{(1-\alpha)})\rho_j^0 + \sum_{m=0}^{k-1}(v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)})\rho_j^{k-m}\right) - s_2\delta_x^2\rho_j^k$$

$$-s_1\left((k+1)^{1-\alpha} - k^{1-\alpha}\right)\rho_j^0,\ j=1,2,...,M-1, \quad k=0,1,...,N-1.$$

$$\rho_0^k = \rho_M^k = 0,\ k=1,2,...,N. \tag{18}$$

Now, we define the grid functions

$$\rho^k = \begin{cases} \rho_j^k \quad when \quad x_j - \frac{h}{2} < x \leqslant x_j + \frac{h}{2}\ j=1,2,...,M-1, \\ 0 \quad when \quad 0 \leqslant x \leqslant \frac{h}{2}\ or\ L - \frac{h}{2} < x \leqslant L, \end{cases} \tag{19}$$

$\rho^k(x)$ can be expanded in Fourier series as

$$\rho^k(x) = \sum_{l=-\infty}^{+\infty} d_k(l)e^{i2\pi lx/L} \quad k=1,2,..,N,$$

where

$$d_k(l) = \frac{1}{L}\int_0^L \rho^k(x)e^{-i2\pi lx/L}dx,\ i=\sqrt{-1}.$$

Let $\rho^k = [\rho_1^k, \rho_2^k, ..., \rho_{M-1}^k]$, and introduce the following norm

$$\|\rho^k\|_2^2 = \left(\sum_{j=1}^{m-1} h|\rho_j^k|^2\right)^{1/2} = \left[\int_0^L |\rho^k(x)|^2dx\right]^{1/2}.$$

Based on the Parseval equality

$$\int_0^L |\rho^k(x)|^2dx = \sum_{l=-\partial}^{+\infty} |d_k(l)|^2,$$

Thus

$$\|\rho^k\|_2^2 = \sum_{l=-\partial}^{+\infty} |d_k(l)|^2. \tag{20}$$

Now, with using above analysis,we assume that the solution of equations (17) and (18) have the following form

$$\rho_j^k = d_k e^{i\sigma jh},$$

where $\sigma = \frac{2\pi l}{L}$. Substituting into (17), we have

$$\begin{aligned}
&s_2 d_{k+1}\delta_x^2 e^{ij\sigma h} - \frac{s_1}{2-\alpha} d_{k+1} e^{ij\sigma h} = \\
&s_1 \left((a_k^{(1-\alpha)} - a_{k-1}^{(1-\alpha)}) d_0 e^{ij\sigma h} + \sum_{m=0}^{k-1} (v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)}) d_{k-m} e^{ij\sigma h} \right) \\
&-s_2 d_k \delta_x^2 e^{ij\sigma h} - s_1 d_0 e^{ij\sigma h} \left((k+1)^{1-\alpha} - k^{1-\alpha} \right) \\
&j = 1, ..., M-1 \quad k = 0, ..., N-1,
\end{aligned}$$

therefore

$$\begin{aligned}
&d_{k+1}\left(-4s_2 \sin^2 \frac{\sigma h}{2} - \frac{s_1}{2-\alpha} \right) = \\
&s_1 \left((a_k^{(1-\alpha)} - a_{k-1}^{(1-\alpha)}) d_0 + \sum_{m=0}^{k-1} (v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)}) d_{k-m} \right) \\
&+4s_2 d_k \sin^2 \frac{\sigma h}{2} - s_1 d_0 \left((k+1)^{1-\alpha} - k^{1-\alpha} \right), \quad k = 0, ..., N-1.
\end{aligned}$$

Let $\mu = 4s_2 \sin^2 \frac{\sigma h}{2}$, then we have

$$\begin{aligned}
d_{k+1} = &\frac{s_1 \left((a_k^{(1-\alpha)} - a_{k-1}^{(1-\alpha)}) d_0 + \sum_{m=0}^{k-1} (v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)}) d_{k-m} \right)}{-\mu - \frac{s_1}{2-\alpha}} \\
&+ \frac{\mu d_k - s_1 d_0 \left((k+1)^{1-\alpha} - k^{1-\alpha} \right)}{-\mu - \frac{s_1}{2-\alpha}}, \quad k = 0, 1, ..., N-1.
\end{aligned}$$

By noticing (4) we have

$$\begin{aligned}
&d_{k+1} = \frac{s_1 \left((a_k^{(1-\alpha)} - a_{k-1}^{(1-\alpha)}) d_0 + \sum_{m=0}^{k-1} (v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)}) d_{k-m} \right) + \mu d_k - s_1 d_0 b_k^{(1-\alpha)}}{-\mu - \frac{s_1}{2-\alpha}} \\
&k = 0, 1, ..., N-1.
\end{aligned} \tag{21}$$

Proposition 3.2 *Suppose that d_{k+1} $k = 0, ..., N-1$ is the solution of (21) and $2^{2-\alpha} \leqslant 3$, then*

$$|d_{k+1}| \leqslant |d_0|,$$

Proof We proof this proposition by using mathematical induction.
For $k=0$ we have

$$d_1 = \frac{s_1(a_0-1)+\mu}{-\mu-\frac{s_1}{2-\alpha}}d_0 = \frac{-\frac{s_1}{2-\alpha}+\mu}{-\mu-\frac{s_1}{2-\alpha}}d_0,$$

noticing that $\mu>0$ and $\frac{s_1}{2-\alpha}>0$,we have

$$|d_1| = \frac{|-\frac{s_1}{2-\alpha}+\mu|}{\mu+\frac{s_1}{2-\alpha}}|d_0| \leqslant \frac{\mu+\frac{s_1}{2-\alpha}}{\mu+\frac{s_1}{2-\alpha}}|d_0| = |d_0|.$$

Now, suppose that

$$|d_n| \leqslant |d_0|, \quad n=1,2,...,k. \tag{22}$$

Using $(21),(23)$ and lemmas (2.1) and (3.1), we have

$$\begin{aligned}
|d_{k+1}| &\leqslant \frac{s_1\left((a_{k-1}^{(1-\alpha)}-a_k^{(1-\alpha)})|d_0| + \sum_{m=0}^{k-1}(|v_{m+1}^{(1-\alpha)}-v_m^{(1-\alpha)}|)|d_{k-m}|\right)}{\mu+\frac{s_1}{2-\alpha}} \\
&+ \frac{\mu|d_k| + s_1|d_0|b_k^{(1-\alpha)}}{\mu+\frac{s_1}{2-\alpha}} \\
&\leqslant \frac{s_1\left((a_{k-1}^{(1-\alpha)}-a_k^{(1-\alpha)}) + \sum_{m=0}^{k-1}|v_{m+1}^{(1-\alpha)}-v_m^{(1-\alpha)}|\right)+\mu+s_1 b_k^{(1-\alpha)}}{\mu+\frac{s_1}{2-\alpha}}|d_0| \\
&\leqslant \frac{s_1\left(a_{k-1}^{(1-\alpha)}-a_k^{(1-\alpha)}+v_0^{(1-\alpha)}-v_k^{(1-\alpha)}\right)+\mu+s_1 b_k^{(1-\alpha)}}{\mu+\frac{s_1}{2-\alpha}}|d_0| \\
&= \frac{s_1\left(v_0^{(1-\alpha)}-b_k^{(1-\alpha)}\right)+\mu+s_1 b_k^{(1-\alpha)}}{\mu+\frac{s_1}{2-\alpha}}|d_0| = |d_0|
\end{aligned}$$

THEOREM 3.3 *The new implicit finite difference method* $(13)-(16)$ *is unconditionally stable.*

Proof Apply proposition (3.2) and noticing (20), we have

$$\|\rho^{k+1}\|_2 \leqslant \|\rho^0\|_2, \quad k=0,1,...,N-1,$$

which implies that the new implicit scheme $(14)-(16)$ is unconditionally stable.

4. Convergence analysis

Let $u(x_i,t_k)$ be the exact solution of equation (1). By integration both sides of (11), with respect t, from t_k to t_{k+1}, we have

$$\begin{aligned}
({}_0D_t^{\alpha-1}u)(x_i,t_{k+1}) - ({}_0D_t^{\alpha-1}u)(x_i,t_k) &= \int_{t_k}^{t_{k+1}} \frac{\partial^2 u(x_i,t)}{\partial x^2}dt \\
&+\frac{u(x_i,t_0)}{(1-\alpha)\Gamma(1-\alpha)}\left(t_{k+1}^{1-\alpha}-t_k^{1-\alpha}\right) + \int_{t_k}^{t_{k+1}} f(x_i,t)dt.
\end{aligned}$$

By using (13) and lemma (2.2) and using the trapezoidal rule for approximation $\int_{t_k}^{t_{k+1}} \frac{\partial^2 u(x_i,t)}{\partial x^2} dt$, we have

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \left((a_k^{(1-\alpha)} - a_{k+1}^{(1-\alpha)}) u(x_i,t_0) + \sum_{j=0}^{k-1} (v_{j+1}^{(1-\alpha)} - v_j^{(1-\alpha)}) u(x_i,t_{k-j}) + \frac{1}{\Gamma(2-\alpha)} u(x_i,t_{k+1}) \right)$$
$$+R_1 = \frac{\tau}{2h^2} \left(\delta_x^2 u(x_i,t_{k+1}) + \delta_x^2 u(x_i,t_k) + 2R_2 \right) + R_3 + \frac{u(x_i,t_0)}{(1-\alpha)\Gamma(1-\alpha)} \left(t_{k+1}^{1-\alpha} - t_k^{1-\alpha} \right) + F_i^{k+1},$$

where $F_i^{k+1} = \int_{t_k}^{t^{k+1}} f(x_i,t)dt$.
Therefore

$$\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \left((a_k^{(1-\alpha)} - a_{k+1}^{(1-\alpha)}) u(x_i,t_0) + \sum_{j=0}^{k-1} (v_{j+1}^{(1-\alpha)} - v_j^{(1-\alpha)}) u(x_i,t_{k-j}) + \frac{1}{\Gamma(2-\alpha)} u(x_i,t_{k+1}) \right)$$
$$= \frac{\tau}{2h^2} \left(\delta_x^2 u(x_i,t_{k+1}) + \delta_x^2 u(x_i,t_k) + \right) + \frac{u(x_i,t_0)}{(1-\alpha)\Gamma(1-\alpha)} \left(t_{k+1}^{1-\alpha} - t_k^{1-\alpha} \right)$$
$$+F_i^{k+1} + R_i^{k+1}, \tag{23}$$

where $R_i^{k+1} = R_1 + \tau R_2 + R_3$.
For approximation $\int_{t_k}^{t_{k+1}} \frac{\partial^2 u(x_i,t)}{\partial x^2} dt$,we use the trapezoidal rule,therefore we have $R_3 = o(\tau^3)$, thus exist the positive constant C_3 such that $|R_3| \leqslant C_3\tau^3$, now by using lemma (2.1),we have

$$|R_3| \leqslant C_3' \tau^{1-\alpha} b_k^{1-\alpha} \tau^2, \tag{24}$$

therefore, by using (24) and lemma (2.2), we have

$$|R_i^{k+1}| \leqslant C_1 \tau^{3-\alpha} b_k^{(1-\alpha)} + C_2 \tau h^2 + C_3' \tau^{1-\alpha} b_k^{(1-\alpha)} \tau^2.$$

by using Lemma (2.1), we can obtain

$$\begin{aligned} |R_i^{k+1}| &\leqslant C_1 \tau^{3-\alpha} b_k^{(1-\alpha)} + C_2' b_k^{(1-\alpha)} \tau^{1-\alpha} h^2 + C_3' \tau^{1-\alpha} b_k^{(1-\alpha)} \tau^2 \\ &= \tau^{1-\alpha} b_k^{(1-\alpha)} \left(C_1 \tau^2 + C_2' h^2 + C_3' \tau^2 \right) \\ &\leqslant C_4 \tau^{1-\alpha} b_k^{(1-\alpha)} (\tau^2 + h^2), \end{aligned} \tag{25}$$

where $C_4 = Max\{C_1 + C_3', C_2'\}$.
Define $e_i^k = u(x_i,t_k) - u_i^k,\ i = 1,2,...,M-1,\ k = 1,...,N$. For boundary and initial conditions, we have

$$e_0^k = e_M^k = 0, \quad k = 1,2,...,N,$$
$$e_j^0 = 0, \quad j = 1,...,M-1.$$

By subtracting (14) from (23), and noticing that $e_j^0 = 0$, we have

$$s_2\delta_x^2 e_j^{k+1} - \frac{s_1}{2-\alpha} e_j^{k+1} = s_1 \sum_{m=0}^{k-1} \left(v_{m+1}^{1-\alpha} - v_m^{1-\alpha}\right) e_j^{k-m} - s_2\delta_x^2 e_j^k + R_j^{k+1} \qquad (26)$$

$$j = 1, ..., M-1,\ k = 0, ..., N-1,$$

where $\delta_x^2 e_j^k = e_{j+1}^k - 2e_j^k + e_{j-1}^k$.
And for $k = 0$, we have

$$s_2\delta_x^2 e_j^1 - \frac{s_1}{2-\alpha} e_j^1 = R_j^1, \quad j = 1, ..., M-1. \qquad (27)$$

Now, by using a Fourier analysis, we analyze the convergence of new implicit method.
Firstly,we define grid functions

$$e^k(x) = \begin{cases} e_j^k & when \quad x_j - \frac{h}{2} < x \leqslant x_j + \frac{h}{2}\ j = 1, 2, ..., M-1, \\ 0 & when \quad 0 \leqslant x \leqslant \frac{h}{2}\ or\ L - \frac{h}{2} < x \leqslant L, \end{cases}$$

and

$$R^k(x) = \begin{cases} R_j^k & when \quad x_j - \frac{h}{2} < x \leqslant x_j + \frac{h}{2}\ j = 1, 2, ..., M-1, \\ 0 & when \quad 0 \leqslant x \leqslant \frac{h}{2}\ or\ L - \frac{h}{2} < x \leqslant L, \end{cases}$$

Therefore $e^k(x)$ and $R^k(x)$ can be expanded in following Fourier series

$$e^k(x) = \sum_{l=-\infty}^{+\infty} \xi_k(l) e^{i2\pi lx/L} \quad k = 1, 2, .., N,$$

$$R^k(x) = \sum_{l=-\infty}^{+\infty} \eta_k(l) e^{i2\pi lx/L} \quad k = 1, 2, .., N,$$

where

$$\xi_k(l) = \frac{1}{L}\int_0^L e^k(x) e^{-i2\pi lx/L} dx,$$

and

$$\eta_k(l) = \frac{1}{L}\int_0^L R^k(x) e^{-i2\pi lx/L} dx, .$$

Let

$$e^k = [e_1^k, e_2^k, ..., e_{M-1}^k]^T,\ k = 0, 1, ..., N,$$
$$R^k = [R_1^k, R_2^k, ..., R_{M-1}^k]^T,\ k = 0, 1, ..., N.$$

We introduce the following norms

$$\|e^k\|_2 = \left(\sum_{j=1}^{m-1} h|e_j^k|^2\right)^{1/2} = \left[\int_0^L |e^k(x)|^2 dx\right]^{1/2}, k = 0, 1, ..., N. \tag{28}$$

and

$$\|R^k\|_2 = \left(\sum_{j=1}^{m-1} h|R_j^k|^2\right)^{1/2} = \left[\int_0^L |R^k(x)|^2 dx\right]^{1/2}, k = 0, 1, ..., N. \tag{29}$$

By using Parseval's equality, we have

$$\int_0^L |e^k(x)|^2 dx = \sum_{l=-\partial}^{+\infty} |\xi_k(l)|^2, \quad k = 0, ..., N, \tag{30}$$

$$\int_0^L |R^k(x)|^2 dx = \sum_{l=-\partial}^{+\infty} |\eta_k(l)|^2, \quad k = 0, ..., N. \tag{31}$$

Thus

$$\|e^k\|_2^2 = \sum_{l=-\infty}^{+\infty} |\xi_k(l)|^2, \quad k = 0, 1, ..., N, \tag{32}$$

$$\|R^k\|_2^2 = \sum_{l=-\infty}^{+\infty} |\eta_k(l)|^2, \quad k = 0, 1, ..., N. \tag{33}$$

Now, from above analysis,we can suppose that

$$e_j^k = \xi_k e^{i\sigma jh}, \tag{34}$$

$$R_j^k = \eta_k e^{i\sigma jh}, \tag{35}$$

where $\sigma = \frac{2\pi l}{L}$. By substituting (34) and (35) in (26) and (25) , we have

$$-4s_2\xi_{k+1}\sin^2\frac{\sigma h}{2}e^{i\sigma jh} - \frac{s_1}{2-\alpha}\xi_{k+1}e^{i\sigma jh} =$$
$$s_1\sum_{m=0}^{k-1}\left(v_{m+1}^{1-\alpha} - v_m^{1-\alpha}\right)\xi_{k-m}e^{i\sigma jh} + 4s_2\xi_k\sin^2\frac{\sigma h}{2}e^{i\sigma jh} + \eta_{k+1}e^{i\sigma jh}$$
$$j = 1, ..., M-1, \quad k = 1, ..., N-1,$$

and for $k = 0$

$$-4s_2\xi_1\sin^2\frac{\sigma h}{2}e^{i\sigma jh} - \frac{s_1}{2-\alpha}\xi_1 e^{i\sigma jh} = \eta_1 e^{i\sigma jh} \quad j = 1, ..., M-1$$

above equation can be rewritten as follow

$$\xi_{k+1} = \frac{s_1 \sum_{m=0}^{k-1} \left(v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)} \right) \xi_{k-m} + \mu \xi_k}{-\mu - \frac{s_1}{2-\alpha}} + \frac{\eta_{k+1}}{-\mu - \frac{s_1}{2-\alpha}} \quad k = 1, ..., N-1, \tag{36}$$

and

$$\xi_1 = \frac{\eta_1}{-\mu - \frac{s_1}{2-\alpha}}, \tag{37}$$

where $\mu = 4s_2 \sin^2 \frac{\sigma h}{2} > 0$.

PROPOSITION 4.1 *Let* ξ_k, $(k = 1, ..., N)$ *be the solution of equations* (36) *and* (37). *If* $2^{2-\alpha} \leqslant 3$ *and* $\mu + \frac{s_1}{2-\alpha} > 1$, *then there is a positive constant* C, *such that*

$$|\xi_{k+1}| \leqslant C(k+1)|\eta_1|, \quad k = 0, ..., N-1.$$

Proof The series in the right hand side of (4.11) is convergence, therefore, a positive constant C_k exist, such that

$$|\eta_k| \equiv |\eta_k(l)| \leqslant C_k |\eta_1|, \quad k = 1, ..., N,$$

therefore

$$|\eta_k| \leqslant C|\eta_1|, \quad k = 1, ..., N, \tag{38}$$

where

$$C = Max\{C_k\}, \quad 1 \leqslant k \leqslant N.$$

Now, we can complete the proof using mathematical induction.
For $k = 0$, we have

$$\xi_1 = \frac{\eta_1}{-\mu - \frac{s_1}{2-\alpha}}.$$

We know $\mu + \frac{s_1}{2-\alpha} \geqslant 1$, thus according to (38) we have

$$|\xi_1| \leqslant |\eta_1| \leqslant C|\eta_1|$$

Now, suppose that

$$|\xi_n| \leqslant Cn|\eta_1|, \quad n = 1, 2, ..., k, \tag{39}$$

by noticing $\mu > 0$, $0 < \alpha \leqslant 1$ and $\mu + \frac{s_1}{2-\alpha} \geqslant 1$ and using (38), (39) and lemma

(3.1), from (36), we have

$$\begin{aligned}
|\xi_{k+1}| &\leqslant \frac{s_1 \sum_{m=0}^{k-1} |v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)}||\xi_{k-m}| + \mu|\xi_k|}{\mu + \frac{s_1}{2-\alpha}} + \frac{|\eta_{k+1}|}{\mu + \frac{s_1}{2-\alpha}} \\
&\leqslant \frac{s_1 \sum_{m=0}^{k-1} C|v_{m+1}^{(1-\alpha)} - v_m^{(1-\alpha)}|(k-m)|\eta_1| + \mu C|\eta_1|}{\mu + \frac{s_1}{2-\alpha}} + \frac{C|\eta_1|}{\mu + \frac{s_1}{2-\alpha}} \\
&\leqslant \left(\frac{s_1(v_0 - v_k) + \mu}{\mu + \frac{s_1}{2-\alpha}} k + \frac{1}{\mu + \frac{s_1}{2-\alpha}} \right) C|\eta_1| \\
&\leqslant \left(\frac{s_1 v_0 + \mu}{\mu + \frac{s_1}{2-\alpha}} k + \frac{1}{\mu + \frac{s_1}{2-\alpha}} \right) C|\eta_1| \\
&\leqslant \left(\frac{\frac{s_1}{2-\alpha} + \mu}{\mu + \frac{s_1}{2-\alpha}} k + \frac{1}{\mu + \frac{s_1}{2-\alpha}} \right) C|\eta_1| \\
&\leqslant C(k+1)|\eta_1|
\end{aligned}$$

Theorem 4.2 *If* $2^{2-\alpha \leqslant 3}$ *and* $\mu + \frac{s_1}{2-\alpha} \geqslant 1$, *then the new implicit method* (14)$-$(16) *is convergence with order* $O(\tau^2 + h^2)$.

Proof Using (25) and (29), we can obtain

$$\|R^k\|_2 \leqslant C_4 \tau^{1-\alpha} b_k^{(1-\alpha)} (\tau^2 + h^2)\sqrt{L} = C' \tau^{1-\alpha} (k+1)^{-\alpha} \frac{b_k^{(1-\alpha)}}{(k+1)^{-\alpha}} (\tau^2 + h^2),$$

$$\begin{aligned}
\lim_{k \to +\infty} \frac{b_{k+1}^{(1-\alpha)}}{(k+1)^{-\alpha}} &= \lim_{k \to +\infty} \frac{(k+2)^{1-\alpha} - (k+1)^{1-\alpha}}{(k+1)^{-\alpha}} \\
&= \lim_{k \to +\infty} \frac{(k+1)^{1-\alpha} \left[\left(1 + \frac{1}{k+1}\right)^{1-\alpha} - 1 \right]}{(k+1)^{-\alpha}} \\
&= \lim_{k \to +\infty} (k+1) \left[\left(1 + \frac{1}{k+1}\right)^{1-\alpha} - 1 \right] = 1 - \alpha.
\end{aligned}$$

Therefore, there is a constant C'', so that

$$\|R^k\|_2 \leqslant C'' \tau ((k+1)\tau)^{-\alpha} (\tau^2 + h^2),$$

because $(k+1)\tau \leqslant T$ is finite, we have

$$\|R^k\|_2 \leqslant \tau \tilde{C} (\tau^2 + h^2), \tag{40}$$

where $\tilde{C} = C'' T^{-\alpha}$.
Using (32), (33), (40) and proposition (4.1), we have

$$\|e^{k+1}\|_2 \leqslant C(k+1)\|R_1\|_2 \leqslant C(k+1)\tau \tilde{C} (\tau^2 + h^2)$$

because $(k+1)\tau \leqslant T$ is finite, we have

$$\|e^{k+1}\|_2 \leqslant \overline{C}(\tau^2 + h^2)$$

where $\overline{C} = C\tilde{C}T$.

Comment. For analyses of stability and convergence of our method, we suppose $2^{2-\alpha} \leqslant 3$, in order to keep the monotonicity of $v_k^{1-\alpha}$, $(k = 0, 1, ...)$. When $2^{2-\alpha} > 3$, we can not prove the stability and convergence of the method. However, numerous numerical experiment demonstrate that are no symptoms of numerical instability.

5. Numerical example

In this section we consider a numerical example that confirms our theoretical analysis.
consider the following problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\partial} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 \leqslant x \leqslant 1 \tag{41}$$

subject to

$$u(x,0) = 0, \quad 0 < x < 1, \tag{42}$$

$$u(0,t) = u(L,t) = 0, \quad 0 < t \leqslant 1, \tag{43}$$

where

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x).$$

The exact of the problem is $u(x,t) = t^2 \sin(2\pi x)$.
We define the maximum absolute error

$$E = Max_{\,0<i<M} |u_i^N - u(x_i, 1)|.$$

Figure 1 shows numerical and exact solution of $(41)-(43)$ when $\alpha = 0.8$ and $t = 1$. The following table shows the maximum absolute error between the exact solution and numerical solution for different $\tau = h$ when $\alpha = 0.3$, 0.6, 0.8 and $t = 1$.

and the following table shows the convergence order for different $\tau = h$ when $\alpha = 0.3$, 0.6, 0.8 and $t = 1$.

6. Conclusion

In this paper a new implicit finite difference method for solving time fractional diffusion equation was considered. The stability and convergency of this method were investigated and was shown that the method is unconditionally stable and convergence with order $O(\tau^2 + h^2)$.

Table 1. The maximum absolute error between the exact solution and numerical solution for different $\tau = h$ when $\alpha = 0.3$, 0.6, 0.8 and $t = 1$.

$\tau = h$	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.8$
$\frac{1}{4}$	2.2305E-1	2.1737E-1	2.1268E-1
$\frac{1}{8}$	5.0620E-2	4.8870E-2	4.7944E-2
$\frac{1}{16}$	1.2228E-2	1.1793E-2	1.1707E-2
$\frac{1}{32}$	$2.9892E-3$	2.9283E-3	2.9118E-3
$\frac{1}{64}$	$7.3834E-4$	7.3113E-4	7.2703E-4

Table 2. The convergence order for different $\tau = h$ when $\alpha = 0.3$, 0.6, 0.8 and $t = 1$.

$\tau = h$	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.8$
$\frac{1}{4}$	—	—	—
$\frac{1}{8}$	4.4064	4.4479	4.4360
$\frac{1}{16}$	4.1397	4.1439	4.0953
$\frac{1}{32}$	4.0907	4.0273	4.0205
$\frac{1}{64}$	4.0485	4.0052	4.0051

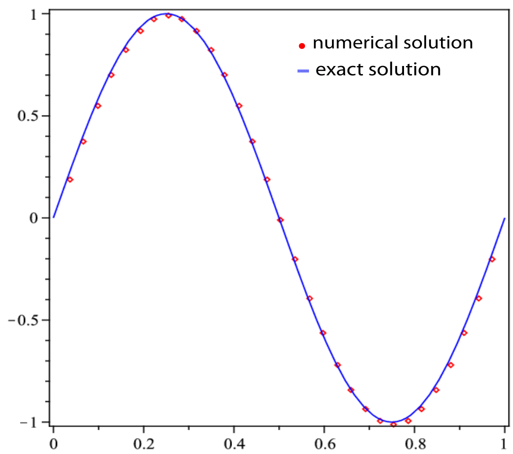

Figure 1. numerical and exact solution of $(41) - (43)$ when $\alpha = 0.8$ and $t = 1$.

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