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ABS-Type Methods for Solving m Linear Equations in $\frac{m}{k}$ Steps for $k = 1, 2, \cdots, m$

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Abstract. The ABS methods, introduced by Abaffy, Broyden and Spedicato, are direct iteration methods for solving a linear system where the i-th iteration satisfies the first i equations, therefore a system of m equations is solved in at most m steps. In this paper, we introduce a class of ABS-type methods for solving a full row rank linear equations, where the i-th iteration solves the first $3i$ equations. We also extended this method for k steps. So, termination is achieved in at most $\left[\frac{m+(k-1)}{k}\right]$ steps. Morever in our new method in each iteration, we have the the general solution of each iteration.

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Keywords: ABS methods; Rank k update; Linear system; General solution of a system; General solution of an iteration.

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1. Introduction

The ABS methods, introduce by Abaffy, Broyden and Spedicato [1], are a general class of algorithms for solving linear and nonlinear algebraic systems. There are many papers on this topic. Some authhors used this method for solving linear system of equations [2, 4, 5, 7, 8, 10, 11], and there are some works on Diophantine linear systems [3, 8]. Solving some linear inequality systems considered by this

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method [6, 7]. ABS algorithms are also used in optimization [6, 7, 12, 13]. Recently this method has been used for solving fuzzy linear system of equations [9].

The basic algorithm works on a system of the form

$$Ax = b, \tag{1}$$

where $A = \begin{bmatrix} a_1 \cdots a_m \end{bmatrix}^T$, $a_i \in \mathbb{R}^n$, $1 \leqslant i \leqslant m$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. The basic ABS methods determine the solution of (1) or signify lack of its existence in at most m iterations. In [4, 5], the authores introduced a class of ABS-type methods for solving full row rank linear equations, where the i-th iteration solves the first $2i$ equations.

Here we suggest an approach, based on ABS methods, which in any iteration, k new equations, if compatible, are satisfied and we use two new updates for the Abaffian matrix, one of them leads to a solution of the i-th iterate and the other leads to the general solution of that iterate.

Section 2 provides an overview of the ABS methods. There we discuss a new rank three update and present a new algorithm for solving compatible systems. We also state and prove some results about the algorithm in this section. In Section 3, we describe the main parameters for solving full row rank linear systems by $\frac{m}{k}$ $(k = 1, \cdots, m)$ steps. In Section 4, we discuss on computational and numerical results.

2. ABS Type Methods for Solving m Linear Equations in $\frac{m}{3}$ Steps.

The basic ABS algorithm starts with an initial vector $x_0 \in \mathbb{R}^n$ and a nonsingular matrix $H_0 \in \mathbb{R}^{n\times n}$ (Spedicato's parameter). Given that x_i is a solution of the first i equations, the ABS algorithm computes x_i as the solution of the first $i+1$ equations performing the following steps (See [2, 4, 5]):

(1) Determine z_i (Broyden's parameter) so that $z_i^T H_i a_i \neq 0$ and set

$$P_i = H_i^T z_i,$$

(2) Update the solution by

$$x_{i+1} = x_i + \alpha_i P_i,$$

where the step size α_i is given by

$$\alpha_i = \frac{b_i - a_i^T x_i}{a_i^T P_i}.$$

(3) Update the Abaffian matrix H_i by

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i},$$

It is easily observed that the ABS methods satisfy a new equation at each iteration. So, at most m iterations are needed to determine a solution or signify the lack of it.

We now discuss an approach to satisfy three equations at a time. We consider the system (1) and we assume that rank(A)=m, where $m = 3l$.

Remark 2.1 if $m = 3q + 1$ or $m = 3q + 2$, we can consider the augmented systems.

$$\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}$$

respectively, that contains the same solution x as (1).

Alternatively, we can use a rank one or a rank two update at the final iteration. We shall see that if a solution exists, it will be found in at most $\frac{m}{3}$ steps.
Let

$$A^{3i} = \begin{bmatrix} a_1 \cdots a_{3i} \end{bmatrix}^T,$$

$$b^{3i} = \begin{bmatrix} b_1 \cdots b_{3i} \end{bmatrix}^T,$$

and

$$r_j(x) = a_j^T x - b_j, \qquad j = 1, \cdots, m.$$

Assume that we are at the i-th step and x_i satisfies $A^{3i}x = b^{3i}$. We determine $H_i \in \mathbb{R}^{n\times n}$, $z_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$ so that

$$x_i = x_{i-1} - \lambda_i H_i^T z_i, \tag{2}$$

be a solution of the first $3i$ equations of the system (1). That is

$$A^{3i} x_i = b^{3i}, \tag{3}$$

or equivalently we have

$$r_j(x_i) = 0, \qquad j = 1, \cdots, 3i.$$

Thus for $j = 3i-2$, $j = 3i-1$ and $j = 3i$, we have:

$$\begin{cases} a_{3i-2}^T \left(x_{i-1} - \lambda_i H_i^T z_i\right) - b_{3i-2} = 0, \\ a_{3i-1}^T \left(x_{i-1} - \lambda_i H_i^T z_i\right) - b_{3i-1} = 0, \\ a_{3i}^T \left(x_{i-1} - \lambda_i H_i^T z_i\right) - b_{3i} = 0, \end{cases}$$

or equivalently we have

$$\begin{cases} \lambda_i (H_i a_{3i-2})^T z_i = r_{3i-2}(x_{i-1}), \\ \lambda_i (H_i a_{3i-1})^T z_i = r_{3i-1}(x_{i-1}), \\ \lambda_i (H_i a_{3i})^T z_i = r_{3i}(x_{i-1}). \end{cases} \tag{4}$$

Suppose that $r_{3i-2}(x_{i-1}) \neq 0$, $r_{3i-1}(x_{i-1}) \neq 0$ and $r_{3i}(x_{i-1}) \neq 0$. Then λ_i must be nonzero and (4) is compatible if and only if we have:

$$\lambda_i = \frac{\overline{r}_{3i-2}(x_{i-1})}{(H_i a_{3i-2})^T z_i} = \frac{\overline{r}_{3i-1}(x_{i-1})}{(H_i a_{3i-1})^T z_i} = \frac{\overline{r}_{3i}(x_{i-1})}{(H_i a_{3i})^T z_i} \tag{5}$$

There are several ways to satisfy (5); for example:

(1) Choose an appropriate update for H_i so that

$$H_i a_{3i} = H_i a_{3i-1} = H_i a_{3i-2} \neq 0$$

(2) Choose a vector z_i from an orthogonal space to the vectors: $H_i(a_{3i} - a_{3i-1})$ and $H_i(a_{3i} - a_{3i-2})$; or equivalently , we choose a vector z_i from an orthogonal space to the vector $H_i a_{3i}$. So that

$$z_i^T H_i a_{3i} \neq 0 \tag{6}$$

Since three new equations are considered in each step, we use a rank two update as (7) and a rank one update as (8), where (7) help us to compute a solution of each iterate and (8) leads to the general solution of each iteration.
Here we present H_i and $H_{l_{i-1}}$ such that they satisfy the following properties:

$$\begin{cases} H_i a_j = 0, \;\; j = 1, \cdots, 3i-3 \\ H_i a_j \neq 0, j = 3i-2, \cdots, 3i \end{cases} \tag{7}$$

and

$$H_{l_{i-1}} a_j = 0, \quad j = 1, \cdots, 3i-3 \tag{8}$$

Now, we let

$$c_j = \begin{cases} a_{3i} - a_j, \, j \neq 3i \\ \\ a_{3i}, \qquad j = 3i \end{cases} \tag{9}$$

Using (9), the systems (7) and (8) are written as (10) and (11), respectively:

$$H_i c_j = 0, \qquad j = 1, \cdots, 3i - 1, \tag{10}$$

$$H_{l_{i-1}} c_j = 0, \qquad j = 1, \cdots, 3i - 3. \tag{11}$$

We compute H_i from $H_{l_{i-1}}$, such that the relations (10)and (11) hold. We proceed inductively. Suppose that the matrix H_i satisfies (10) and the matrix $H_{l_{i-1}}$ satisfies (11). We define

$$H_{i+1} = H_{l_i} + g_i d_i^T + e_i f_i^T.$$

where $g_i, d_i, e_i, f_i \in \mathbb{R}^n$. We need to have

$$\begin{cases} H_{i+1} c_j = 0, \, j = 1, \cdots, 3i + 2, \\ \\ H_{l_i} c_j = 0, \quad j = 1, \cdots, 3i, \end{cases}$$

or equivalently,

$$\begin{cases} \left(H_{l_i} + g_i d_i^T + e_i f_i^T\right) c_j = 0, \, j = 1, \cdots, 3i + 2, \\ \\ H_{l_i} c_j = 0, \qquad\qquad\qquad j = 1, \cdots, 3i. \end{cases}$$

So, We must define $g_i, d_i, e_i, f_i \in \mathbb{R}^n$ in such a way that

$$H_{l_i} c_j + \left(d_i^T c_j\right) g_i + \left(f_i^T c_j\right) e_i = 0, \qquad j = 1, \cdots, 3i + 2 \tag{12}$$

and

$$H_{l_i} c_j = 0, \qquad j = 1, \cdots, 3i. \tag{13}$$

By defining

$$d_i = H_{l_i}^T \overline{w}_{i+1}, \quad f_i = H_{l_i}^T \tilde{w}_{i+1}, \tag{14}$$

for some $\overline{w}_{i+1}, \tilde{w}_{i+1} \in \mathbb{R}^n$, the conditions (12) and (13) are satisfied for $j \leqslant 3i - 1$ and $j \leqslant 3i - 3$, by the induction hypothesis, respectively. Letting $j = 3i + 1$ and $j = 3i + 2$ in (12), we get:

$$\begin{cases} (d_i^T c_{3i+1}) g_i + (f_i^T c_{3i+1}) e_i = -H_{l_i} c_{3i+1}, \\ \\ (d_i^T c_{3i+2}) g_i + (f_i^T c_{3i+2}) e_i = -H_{l_i} c_{3i+2}. \end{cases} \tag{15}$$

We consider that the choices

$$e_i = -H_{l_i} c_{3i+2}, \qquad g_i = -H_{l_i} c_{3i+1}, \tag{16}$$

with

$$\begin{cases} d_i^T c_{3i+1} = 1, \\ d_i^T c_{3i+2} = 0, \end{cases} \qquad \begin{cases} f_i^T c_{3i+1} = 0, \\ f_i^T c_{3i+2} = 1, \end{cases} \tag{17}$$

satisfy (15). In order to have (17), $\overline{w}_{i+1}$ and $\tilde{w}_{i+1}$ may be defined as below:

$$\begin{cases} \overline{w}_{i+1}^T H_{l_i} c_{3i+1} = 1, \\ \overline{w}_{i+1}^T H_{l_i} c_{3i+2} = 0, \end{cases} \qquad \begin{cases} \tilde{w}_{i+1}^T H_{l_i} c_{3i+1} = 0, \\ \tilde{w}_{i+1}^T H_{l_i} c_{3i+2} = 1, \end{cases} \tag{18}$$

It is apparent that the system (18) has a solution if and only if the vectors $H_{l_i} c_{3i+1}$ and $H_{l_i} c_{3i+2}$ are linearly independent. Due to Theorem 2.3 to be seen later, if a_i are linearly independent, then $H_i a_j$ for i, $1 \leqslant i \leqslant l$ and j, $3i-2 \leqslant j \leqslant m$ are nonzero and linearly independent. So $H_{l_i} c_{3i+1}$ and $H_{l_i} c_{3i+2}$ will also be linearly independent and (18) will have a solution for all i and hence H_{i+1} and x_{i+1} are well defined for all i.
Therefore the updating formula for H_i turns out to be

$$H_{i+1} = H_{l_i} - H_{l_i} c_{3i+1} \overline{w}_{i+1}^T H_{l_i} - H_{l_i} c_{3i+2} \tilde{w}_{i+1}^T H_{l_i} \tag{19}$$

where $\overline{w}_{i+1}$ and $\tilde{w}_{i+1}$ can be any vectors satisfying (18). Now to satisfy (2.18) to complete the induction, H_1 and H_{l_1} should be chosen so that

$$H_1 a_1 = H_1 a_2 = H_1 a_3 \quad \textit{or} \quad H_1 c_j = 0, j = 1, 2 \tag{20}$$

and

$$H_{l_1} c_j = 0, j = 1, 2, 3 \tag{21}$$

Let H_0 be an arbitrary nonsingular matrix. We obtain H_1 from H_0 by using a rank two update and H_{l_1} from H_1 by using a rank one update. Let $H_1 = H_0 - uv^T - pq^T$. Where $u, v, p, q \in \mathbb{R}^n$ are chosen so that (20) is satisfied; that is

$$H_0 c_j - (v^T c_j) u - (q^T c_j) p = 0, j = 1, 2$$

This equation is satisfied if we set

$$u = H_0 c_1 \quad v = H_0^T \overline{w}_1 \quad p = H_0 c_2 \quad q = H_0^T \tilde{w}_1$$

for some $\overline{w}_1, \tilde{w}_1 \in \mathbb{R}^n$ which in turn satisfies the condition

$$\begin{cases} \overline{w}_1^T H_0 c_1 = 1, \\ \overline{w}_1^T H_0 c_2 = 0, \end{cases} \qquad \begin{cases} \tilde{w}_1^T H_0 c_1 = 0, \\ \tilde{w}_1^T H_0 c_2 = 1, \end{cases} \tag{22}$$

It is easily seen that (22) can be held with a proper choice of $\overline{w}_1$ and $\tilde{w}_1$ whenever a_1, a_2 and a_3 are linearly independent. So we have

$$H_1 = H_0 - H_0 c_1 \overline{w}_1^T H_0 - H_0 c_2 \tilde{w}_1^T H_0, \tag{23}$$

where $\overline{w}_1$ and $\tilde{w}_1$ are arbitrary vectors satisfying (22), also the matrix H_{l_1} can be computed by a rank one update as:

$$H_{l_1} = H_1 - H_1 a_3 w_1^T H_1 = H_1 - H_1 c_3 w_1^T H_1, \tag{24}$$

where $w_1 \in \mathbb{R}^n$ is an arbitrary vector satisfying

$$w_1^T H_1 a_3 = 1 \qquad \big(w_1^T H_1 c_3 = 1\big) \tag{25}$$

Since a_i being linearly independent, $H_1 a_3 (H_1 c_3)$ is a nonzero vector and hence (24) is well defined with a proper choice of $w_1 \in \mathbb{R}^n$.

Remark 2.2 For the matrices H_i are generated by (19) and (23), we have:

$$x_i = x_{i-1} - \lambda_i H_i^T z_i \qquad i = 1, \cdots, l$$

is a solution of the first $3i$ equations of the system.

To compute the general solution of each iteration, we need a matrix H_{l_i} with the following properties:

$$H_{l_i} a_j = 0, \qquad j = 1, \cdots, 3i$$

It can be easily verified that the matrix H_{l_i} can be computed by a rank one update as:

$$H_{l_i} = H_i - H_i a_{3i} w_i^T H_i = H_i - H_i c_{3i} w_i^T H_i, \qquad i = 1, \cdots, l \tag{26}$$

where $w_i \in \mathbb{R}^n$ is an arbitrary vector satisfying

$$w_i^T H_i a_{3i} = 1 \quad \big(w_i^T H_i c_{3i} = 1\big) \tag{27}$$

Hence the general solution of the i-th iterate is given by

$$x_{l_i} = x_i - H_{l_i}^T s$$

where $s \in \mathbb{R}^n$ is arbitrary.
Therefore, we proved Theorem 2.1.

Theorem 2.1 Given $m = 3l$ arbitrary linearly independent vectors $a_1, \cdots, a_m \in \mathbb{R}^n$ and an arbitrary nonsingular matrix $H_0 \in \mathbb{R}^{n \times n}$. Let H_1 be generated by (23) with $\overline{w}_1$ and $\tilde{w}_1$ satisfying (22) and the sequence of matrices H_i, $i = 2, \cdots, l$ be generated by

$$H_i = H_{l_{i-1}} - H_{l_{i-1}} c_{3i-2} \overline{w}_i^T H_{l_{i-1}} - H_{l_{i-1}} c_{3i-1} \tilde{w}_i^T H_{l_{i-1}} \tag{28}$$

with $\overline{w}_i$ and $\tilde{w}_i \in \mathbb{R}^n$ satisfying

$$\begin{cases} \overline{w}_i^T H_{l_{i-1}} c_{3i-2} = 1 \\ \overline{w}_i^T H_{l_{i-1}} c_{3i-1} = 0 \end{cases} \qquad \begin{cases} \tilde{w}_i^T H_{l_{i-1}} c_{3i-2} = 0 \\ \tilde{w}_i^T H_{l_{i-1}} c_{3i-1} = 1 \end{cases} \tag{29}$$

also let the sequence of matrices $H_{l_1}, \cdots, H_{l_l}$ be generated by (26) with $w_i \in \mathbb{R}^n$ satisfying (27), Then the following properties $((i)-(iii))$ hold for $i = 1, \cdots, l$

$(i) H_{l_i} a_j = 0, \qquad j = 1, \cdots, 3i,$

$(ii) H_i a_{3i-2} = H_i a_{3i-1} = H_i a_{3i} \neq 0$

$(iii) H_{l_i} c_j = 0, \qquad j = 1, \cdots, 3i$

also, for $i = 2, \cdots, l$, we have

$(iv) H_i c_j = 0, \; j = 1, \cdots, 3i-3$ and for $i = 1$, we have $H_i c_j = 0 \;\; j = 1, 2$

Remark 2.3 Before we present the algorithm, we need to explain the definition of λ_i based on the values of the residues of three new equations being considered. We saw that λ_i must be nonzero when the corresponding residues $(r_{3i-2}(x_{i-1}) = \alpha_i, r_{3i-1}(x_{i-1}) = \beta_i \; and \; r_{3i}(x_{i-1}) = \gamma_i)$ are nonzero. We use the following strategy for the definition of λ_i:

(i) If $\alpha_i \beta_i \gamma_i \neq 0$, then we let

$$\begin{cases} a_{3i-2} = \beta_i \gamma_i a_{3i-2} \\ b_{3i-2} = \beta_i \gamma_i b_{3i-2} \end{cases} \qquad \begin{cases} a_{3i-1} = \alpha_i \gamma_i a_{3i-1} \\ b_{3i-1} = \alpha_i \gamma_i b_{3i-1} \end{cases} \qquad \begin{cases} a_{3i} = \alpha_i \beta_i a_{3i} \\ b_{3i} = \alpha_i \beta_i b_{3i} \end{cases}$$

and we have:

$$\lambda_i = \frac{\alpha_i \beta_i \gamma_i}{z_i^T H_i a_{3i-2}} = \frac{\alpha_i \beta_i \gamma_i}{z_i^T H_i a_{3i-1}} = \frac{\alpha_i \beta_i \gamma_i}{z_i^T H_i a_{3i}}$$

(ii) If $(\alpha_i = 0, \beta_i = 0 \;\; and \;\; \gamma_i \neq 0)$ then we let:

$$\begin{cases} a_{3i-2} = a_{3i-2} + a_{3i} \\ b_{3i-2} = b_{3i-2} + b_{3i} \end{cases} \qquad \begin{cases} a_{3i-1} = a_{3i-1} + a_{3i} \\ b_{3i-1} = b_{3i-1} + b_{3i} \end{cases} \qquad \begin{cases} a_{3i} = a_{3i} \\ b_{3i} = b_{3i} \end{cases}$$

and we have

$$\lambda_i = \frac{\gamma_i}{z_i^T H_i a_{3i-2}} = \frac{\gamma_i}{z_i^T H_i a_{3i-1}} = \frac{\gamma_i}{z_i^T H_i a_{3i}}$$

(iii)If ($\alpha_i \neq 0, \beta_i = 0$ *and* $\gamma_i \neq 0$) then we let

$$\begin{cases} a_{3i-2} = \gamma_i a_{3i-2} \\ b_{3i-2} = \gamma_i b_{3i-2} \end{cases} \qquad \begin{cases} a_{3i} = \alpha_i a_{3i} \\ b_{3i} = \alpha_i b_{3i} \end{cases} \qquad \begin{cases} a_{3i-1} = a_{3i-1} + a_{3i} \\ b_{3i-1} = b_{3i-1} + b_{3i} \end{cases}$$

and we have:

$$\lambda_i = \frac{\alpha_i \gamma_i}{z_i^T H_i a_{3i-2}} = \frac{\alpha_i \gamma_i}{z_i^T H_i a_{3i-1}} = \frac{\alpha_i \gamma_i}{z_i^T H_i a_{3i}}$$

(iv)If ($\alpha_i = 0, \beta_i \neq 0$ *and* $\gamma_i \neq 0$), then we let

$$\begin{cases} a_{3i-1} = \gamma_i a_{3i-1} \\ b_{3i-1} = \gamma_i b_{3i-1} \end{cases} \qquad \begin{cases} a_{3i} = \beta_i a_{3i} \\ b_{3i} = \beta_i b_{3i} \end{cases} \qquad \begin{cases} a_{3i-2} = a_{3i-2} + a_{3i} \\ b_{3i-2} = b_{3i-2} + b_{3i} \end{cases}$$

and we have:

$$\lambda_i = \frac{\beta_i \gamma_i}{z_i^T H_i a_{3i-2}} = \frac{\beta_i \gamma_i}{z_i^T H_i a_{3i-1}} = \frac{\beta_i \gamma_i}{z_i^T H_i a_{3i}}$$

Since we define c_j based on the $3i$-th equation, in each step, γ_i must be nonzero for all $i = 1, \cdots, l$. So if we have $\gamma_i = 0$ and α_i or β_i (at least one of them) be nonzero, by proper interchanging a_{3i} and b_{3i} by a_{3i-2} and b_{3i-2} or a_{3i-1} and b_{3i-1}, we have one of the before cases.
But if α_i, β_i and γ_i, all of them, be zero, then λ_i will be zero and x_i will be set to x_{i-1} so x_{l_i} will be set to $x_{l_{i-1}}$, as expected.

Now, we can present the steps of the new algorithm for solving full row rank (and hence compatible) systems.

Algorithm 1.(Assume that $A_{m\times n}$ has full row rank and $m = 3l$.)

(1) Let $x_0 \in \mathbb{R}^n$ be an arbitrary vector and choose $H_0 \in \mathbb{R}^{n\times n}$ (an arbitrary nonsingular matrix). Set $i = 1$.
(2) (a) Compute $\alpha_1 = r_1(x_0)$, $\beta_1 = r_2(x_0)$ and $\gamma_1 = r_3(x_0)$
(b) If $\alpha_1\beta_1\gamma_1 \neq 0$, then we let

$$\begin{cases} a_1 = \beta_1\gamma_1 a_1 \\ b_1 = \beta_1\gamma_1 b_1 \end{cases} \qquad \begin{cases} a_2 = \alpha_1\gamma_1 a_2 \\ b_2 = \alpha_1\gamma_1 b_2 \end{cases} \qquad \begin{cases} a_3 = \alpha_1\beta_1 a_3 \\ b_3 = \alpha_1\beta_1 b_3 \end{cases}$$

(c) If ($\alpha_1 = 0, \beta_1 = 0$ *and* $\gamma_1 \neq 0$), then we let

$$\begin{cases} a_1 = a_1 + a_3 \\ b_1 = b_1 + b_3 \end{cases} \qquad \begin{cases} a_2 = a_2 + a_3 \\ b_2 = b_2 + b_3 \end{cases} \qquad \begin{cases} a_3 = a_3 \\ b_3 = b_3 \end{cases}$$

(d) If ($\alpha_1 \neq 0, \beta_1 = 0$ *and* $\gamma_1 \neq 0$), then we let

$$\begin{cases} a_1 = \gamma_1 a_1 \\ b_1 = \gamma_1 b_1 \end{cases} \qquad \begin{cases} a_3 = \alpha_1 a_3 \\ b_3 = \alpha_1 b_3 \end{cases} \qquad \begin{cases} a_2 = a_2 + a_3 \\ b_2 = b_2 + b_3 \end{cases}$$

(e) If ($\alpha_1 = 0, \beta_1 \neq 0$ *and* $\gamma_1 \neq 0$), then we let

$$\begin{cases} a_2 = \gamma_1 a_2 \\ b_2 = \gamma_1 b_2 \end{cases} \qquad \begin{cases} a_3 = \beta_1 a_3 \\ b_3 = \beta_1 b_3 \end{cases} \qquad \begin{cases} a_1 = a_1 + a_3 \\ b_1 = b_1 + b_3 \end{cases}$$

(f) If $\gamma_1 = 0$ and $\alpha_1 \neq 0$, then we let

$$\begin{cases} a_1 = a_3 \\ b_1 = b_3 \end{cases} \qquad \begin{cases} a_2 = a_2 \\ b_2 = b_2 \end{cases} \qquad \begin{cases} a_3 = a_1 \\ b_3 = b_1 \end{cases}$$

and go to (2).
(g) If $\gamma_1 = 0, \alpha_1 = 0$ and $\beta_1 \neq 0$, then we let

$$\begin{cases} a_1 = a_1 \\ b_1 = b_1 \end{cases} \qquad \begin{cases} a_2 = a_3 \\ b_2 = b_3 \end{cases} \qquad \begin{cases} a_3 = a_2 \\ b_3 = b_2 \end{cases}$$

and go to (2).
(3) (a) Let $c_1 = a_3 - a_1$ and $c_2 = a_3 - a_2$
(b) Select $\overline{w}_1, \tilde{w}_1 \in \mathbb{R}^n$ so that

$$\begin{cases} \overline{w}_1^T H_0 c_1 = 1 \\ \overline{w}_1^T H_0 c_2 = 0 \end{cases} \qquad \begin{cases} \tilde{w}_1^T H_0 c_1 = 0 \\ \tilde{w}_1^T H_0 c_2 = 1 \end{cases}$$

and compute

$$H_1 = H_0 - H_0 c_1 \overline{w}_1^T H_0 - H_0 c_2 \tilde{w}_1^T H_0$$

(c) Select $w_1 \in \mathbb{R}^n$ so that $w_1^T H_1 a_3 = 1$ and compute

$$H_{l_1} = H_1 - H_1 a_3 w_1^T H_1$$

(d) Select $z_1 \in \mathbb{R}^n$ so that $z_1^T H_1 a_3 \neq 0$, and compute

$\lambda_1 = \frac{\alpha_1 \beta_1 \gamma_1}{z_1^T H_1 a_3}$, if $\alpha_1 \beta_1 \gamma_1 \neq 0$

$\lambda_1 = \frac{\gamma_1}{z_1^T H_1 a_3}$, if ($\alpha_1 = 0, \beta_1 = 0$ and $\gamma_1 \neq 0$)

$\lambda_1 = \frac{\alpha_1 \gamma_1}{z_1^T H_1 a_3}$, if ($\alpha_1 \neq 0, \beta_1 = 0$ and $\gamma_1 \neq 0$)

$\lambda_1 = \frac{\beta_1\gamma_1}{z_1^T H_1 a_3}$, if ($\alpha_1 = 0, \beta_1 \neq 0$ and $\gamma_1 \neq 0$)

(4) (a) Let $x_1 = x_0 - \lambda_1 H_1^T z_1$
(b) Let $x_{l_1} = x_1 - H_{l_1}^T s$, where $s \in \mathbb{R}^n$ is arbitrary.

(5) While $i < \frac{m}{3}$ do step (6)-(9)

(6) (a) Compute $\alpha_i = r_{3i-2}(x_{i-1}), \beta_i = r_{3i-1}(x_{i-1})$ and $\gamma_i = r_{3i}(x_{i-1})$
(b) If $\alpha_i\beta_i\gamma_i \neq 0$, then we let

$$\begin{cases} a_{3i-2} = \beta_i\gamma_i a_{3i-2} \\ b_{3i-2} = \beta_i\gamma_i b_{3i-2} \end{cases} \quad \begin{cases} a_{3i-1} = \alpha_i\gamma_i a_{3i-1} \\ b_{3i-1} = \alpha_i\gamma_i b_{3i-1} \end{cases} \quad \begin{cases} a_{3i} = \alpha_i\beta_i a_{3i} \\ b_{3i} = \alpha_i\beta_i b_{3i} \end{cases}$$

(c) If ($\alpha_i = 0, \beta_i = 0$ *and* $\gamma_i \neq 0$), then we let

$$\begin{cases} a_{3i-2} = a_{3i-2} + a_{3i} \\ b_{3i-2} = b_{3i-2} + b_{3i} \end{cases} \quad \begin{cases} a_{3i-1} = a_{3i-1} + a_{3i} \\ b_{3i-1} = b_{3i-1} + b_{3i} \end{cases} \quad \begin{cases} a_{3i} = a_{3i} \\ b_{3i} = b_{3i} \end{cases}$$

(d) If ($\alpha_i \neq 0, \beta_i = 0$ *and* $\gamma_i \neq 0$), then we let

$$\begin{cases} a_{3i-2} = \gamma_i a_{3i-2} \\ b_{3i-2} = \gamma_i b_{3i-2} \end{cases} \quad \begin{cases} a_{3i} = \alpha_i a_{3i} \\ b_{3i} = \alpha_i b_{3i} \end{cases} \quad \begin{cases} a_{3i-1} = a_{3i-1} + a_{3i} \\ b_{3i-1} = b_{3i-1} + b_{3i} \end{cases}$$

(e) If ($\alpha_i = 0, \beta_i \neq 0$ *and* $\gamma_i \neq 0$), then we let

$$\begin{cases} a_{3i-1} = \gamma_i a_{3i-1} \\ b_{3i-1} = \gamma_i b_{3i-1} \end{cases} \quad \begin{cases} a_{3i} = \beta_i a_{3i} \\ b_{3i} = \beta_i b_{3i} \end{cases} \quad \begin{cases} a_{3i-2} = a_{3i-2} + a_{3i} \\ b_{3i-2} = b_{3i-2} + b_{3i} \end{cases}$$

(f) If $\gamma_i = 0$ and $\alpha_i \neq 0$, then we let

$$\begin{cases} a_{3i-2} = a_{3i} \\ b_{3i-2} = b_{3i} \end{cases} \quad \begin{cases} a_{3i-1} = a_{3i-1} \\ b_{3i-1} = b_{3i-1} \end{cases} \quad \begin{cases} a_{3i} = a_{3i-2} \\ b_{3i} = b_{3i-2} \end{cases}$$

and go to (6).
(g) If $\gamma_i = 0$, $\alpha_i = 0$ and $\beta_i \neq 0$, then we let

$$\begin{cases} a_{3i-2} = a_{3i-2} \\ b_{3i-2} = b_{3i-2} \end{cases} \quad \begin{cases} a_{3i-1} = a_{3i} \\ b_{3i-1} = b_{3i} \end{cases} \quad \begin{cases} a_{3i} = a_{3i-1} \\ b_{3i} = b_{3i-1} \end{cases}$$

and go to (6).

(7) (a) Let $c_{3i-2} = a_{3i} - a_{3i-2}$ and $c_{3i-1} = a_{3i} - a_{3i-1}$
(b) Select $\overline{w}_i, \tilde{w}_i \in \mathbb{R}^n$ so that

$$\begin{cases} \overline{w}_i^T H_{l_{i-1}} c_{3i-2} = 1 \\ \overline{w}_i^T H_{l_{i-1}} c_{3i-1} = 0 \end{cases} \quad \begin{cases} \tilde{w}_i^T H_{l_{i-1}} c_{3i-2} = 0 \\ \tilde{w}_i^T H_{l_{i-1}} c_{3i-1} = 1 \end{cases}$$

and compute

$$H_i = H_{l_{i-1}} - H_{l_{i-1}} c_{3i-2} \overline{w}_i^T H_{l_{i-1}} - H_{l_{i-1}} c_{3i-1} \tilde{w}_i^T H_{l_{i-1}}$$

(c) Select $w_i \in \mathbb{R}^n$ so that $w_i^T H_i a_{3i} = 1$ and compute

$$H_{l_i} = H_i - H_i a_{3i} w_i^T H_i$$

(d) Select $z_i \in \mathbb{R}^n$ so that $z_i^T H_i a_{3i} \neq 0$, and compute

$\lambda_i = \frac{\alpha_i \beta_i \gamma_i}{z_i^T H_i a_{3i}}$, if $\alpha_i \beta_i \gamma_i \neq 0$

$\lambda_i = \frac{\gamma_i}{z_i^T H_i a_{3i}}$ if $(\alpha_i = 0, \beta_i = 0 \ \ and \ \ \gamma_i \neq 0)$

$\lambda_i = \frac{\alpha_i \gamma_i}{z_i^T H_i a_{3i}}$ if $(\alpha_i \neq 0, \beta_i = 0 \ \ and \ \ \gamma_i \neq 0)$

$\lambda_i = \frac{\beta_i \gamma_i}{z_i^T H_i a_{3i}}$ if $(\alpha_i = 0, \beta_i \neq 0 \ \ and \ \ \gamma_i \neq 0)$

(8) (a) Let $x_i = x_{i-1} - \lambda_i H_i^T z_i$.
(b) Let $x_{l_i} = x_i - H_{l_i}^T s$, where $s \in \mathbb{R}^n$ is arbitrary.
Set $i = i + 1$
Endwhile

(9) Stop (x_l is a solution and x_{l_l} is the general solution of the system.)

Theorem 2.2 Assume that $a_1, \cdots, a_m$ are linearly independent vectors in $\mathbb{R}^n$. Let $H_0 \in \mathbb{R}^{n \times n}$ be an arbitrary nonsingular matrix, and for $i = 1, \cdots, l$, the sequence of matrices H_i be generated by:

$$H_i = H_{i-1} - H_{i-1} c_{3i-2} \overline{w}_i^T H_{i-1} - H_{i-1} c_{3i-1} \tilde{w}_i^T H_{i-1} \tag{30}$$

for some $\overline{w}_i, \tilde{w}_i \in \mathbb{R}^n$ satisfying the below conditions:

$$\begin{cases} \overline{w}_i^T H_{i-1} c_{3i-2} = 1 \\ \\ \overline{w}_i^T H_{i-1} c_{3i-1} = 0 \end{cases} \qquad \begin{cases} \tilde{w}_i^T H_{i-1} c_{3i-2} = 0 \\ \\ \tilde{w}_i^T H_{i-1} c_{3i-1} = 1 \end{cases} \tag{31}$$

then for any i, $1 \leqslant i \leqslant l$, and j, $3i - 2 \leqslant j \leqslant m$, the vectors $H_i a_j$ are nonzero and linearly independent (or equivalently $H_i a_{3i-2}$ and $H_i a_j$, $3i + 1 \leqslant j \leqslant m$, are nonzero and linearly independent).

Proof We proceed by induction. For $i = 1$, the theorem is true, since if $\sum\limits_{j=3}^{m} \alpha_j H_1 a_j = 0$, then

$$\sum_{j=3}^{m} \alpha_j \left(H_0 - H_0 c_1 \overline{w}_1^T H_0 - H_0 c_2 \tilde{w}_1^T H_0 \right) a_j = 0.$$

$$\beta H_0 a_1 + \beta' H_0 a_2 - (\beta + \beta' - \alpha_3) H_0 a_3 + \sum_{j=4}^{m} \alpha_j H_0 a_j = 0,$$

where $\beta = \sum\limits_{j=3}^{m} \beta_j$ and $\beta' = \sum\limits_{j=3}^{m} \beta'_j$.

Now, since $a_1, \cdots, a_m$ are linearly independent and H_0 is nonsingular, then $H_0 a_j$, for $1 \leqslant j \leqslant m$, are linearly independent.
Hence $\beta = \beta' = \alpha_3 = \cdots = \alpha_m = 0$.
Therefore the vectors $H_1 a_j$, for $3 \leqslant j \leqslant m$, are linearly independent. Now we assume that the theorem is true up to $1 \leqslant k \leqslant l$, and then we prove it to be true for $k+1$. From (30), for $3k+1 \leqslant j \leqslant m$, we have,

$$H_{k+1}a_j = H_k a_j - (\overline{w}_k^T H_k a_j) H_k c_{3k+1} - (\tilde{w}_k^T H_k a_j) H_k c_{3k+2} \tag{32}$$

We need to show that the relation

$$\sum_{j=3k+1}^{m} \alpha_j H_{k+1} a_j = 0, \tag{33}$$

implies that $\alpha_j = 0$, for $3k+1 \leqslant j \leqslant m$. Using (32)we can write (33) as follows:

$$\sum_{j=3k+1}^{m} \alpha_j H_k a_j - \left(\sum_{j=3k+1}^{m} \alpha_j \overline{w}_k^T H_k a_j \right) H_k c_{3k+1} - \left(\sum_{j=3k+1}^{m} \alpha_j \tilde{w}_k^T H_k a_j \right) H_k c_{3k+2} = 0.$$

By taking $\beta_1 = \sum\limits_{j=3k+1}^{m} \alpha_j \overline{w}_k^T H_k a_j$ and $\beta_2 = \sum\limits_{j=3k+1}^{m} \alpha_j \tilde{w}_k^T H_k a_j$, we have

$$\sum_{j=3k+1}^{m} \alpha_j H_k a_j - \beta_1 H_k (a_{3k+3} - a_{3k+1}) - \beta_2 H_k (a_{3k+3} - a_{3k+2}) = 0.$$

or equivalently

$$\begin{aligned} \sum_{j=3k+3}^{m} \alpha_j H_k a_j + (\alpha_{3k+1} + \beta_1) H_k a_{3k+1} \\ + (\alpha_{3k+2} \beta_2) H_k a_{3k+2} \\ - (\beta_1 + \beta_2) H_k a_{3k+3} = 0. \end{aligned}$$

By the induction hypothesis, the vectors $H_k a_j$, for $3k-2 \leqslant j \leqslant m$, are linearly independent, we have

$$\beta_1 = \beta_2 = \alpha_{3k+1} = \alpha_{3k+2} = \alpha_{3k+3} = \cdots = \alpha_m = 0.$$

Hence, the vectors $H_{k+1} a_j$, for $3k+1 \leqslant j \leqslant m$ are linearly independent (the statement in the parenthesis in Theorem 2.2 is now simply verified by the fact that $H_i a_{3i-2} = H_i a_{3i-1} = H_i a_{3i}$). ■

Theorem 2.3 Assume that $a_1, \cdots, a_m$ are linearly independent vectors in $\mathbb{R}^n$. Let $H_0 \in \mathbb{R}^{n \times n}$ be an arbitrary nonsingular matrix, H_1 be defined as (23), and for $i = 2, \cdots, l$, the sequence of matrices H_i be generated by (2.29) then for i, $1 \leqslant i \leqslant l$, and j, $3i-2 \leqslant j \leqslant m$, the vectors $H_i a_j$ are nonzero and linearly independent (or equivalently, $H_i a_{3i-2}$ and $H_i a_j$, $3i+1 \leqslant j \leqslant m$, are nonzero and linearly independent).

Proof The proof is same as the proof of Theorem 2.2. ■

Corollary 2.4 For all i, $i = 1, \cdots, \frac{m}{3}$, if the vectors $a_1, a_2, \cdots, a_{3i}$ are linearly independent, then $H_i a_{3i-2} = H_i a_{3i-1} = H_i a_{3i} \neq 0$, and there exists $z_i \in \mathbb{R}^n$ such that $z_i^T H_i a_{3i} \neq 0$.

Considering the theorems 2.2 and 2.3, the following corollary is now immediately at hand.

Corollary 2.5 If $a_1, \cdots, a_m$ are linearly independent, then the system (18) has solution for every i, $1 \leqslant i \leqslant l-1$, and $H_{i+1}, x_{i+1}, H_{l_{i+1}}$ and $x_{l_{i+1}}$, are well defined. Similarly the system (2.23) has solution for $i = 0$, and H_1, x_1, H_{l_1} and x_{l_1}, are well defined.

The proof of the following lemma is obvious.

Lemma 2.6 The vectors $a_1, \cdots, a_m$ are linearly independent if and only if the vectors $c_1, \cdots, c_m$ are linearly independent.

We can now easily prove the following theorem, using Lemma 2.6.

Theorem 2.4 For the matrices H_i generated by (2.24) and (2.29) at Algorithm 1 and the matrices H_{l_i} given by (26), we have:

$$\begin{aligned}
\dim R(H_i) &= n - 3i + 1, \quad 1 \leqslant i \leqslant l \\
\dim N(H_i) &= 3i - 1, \qquad\quad 1 \leqslant i \leqslant l \\
\dim R(H_{l_i}) &= n - 3i, \qquad\quad 1 \leqslant i \leqslant l \\
\dim N(H_{l_i}) &= 3i, \qquad\qquad\;\; 1 \leqslant i \leqslant l. \\
\dim R(H_{l_l}) &= n - m, \\
\dim N(H_{l_l}) &= m.
\end{aligned}$$

An interesting question of concern arise when

$$H_i a_{3i-2} = H_i a_{3i-1} = H_i a_{3i} = 0$$

Theorem 2.8 below shows this to be equivalent to the vectors $a_1, \cdots, a_{3i}$ being linearly dependent.

Theorem 2.5 Assume that $a_1, \cdots, a_{3i}$ are linearly independent. Assume H_1 can be defined as H_0 according to (2.24) and H_i can be defined according to (2.29) for $i = 2, \cdots, l$, that is the systems (2.30) and (2.23) has a solution for the cases of $H_{l_{i-1}}$ and H_0 respectively, then $H_i a_{3i-2} (= H_i a_{3i-1} = H_i a_{3i}) = 0$, if and only if $a_1, \cdots, a_{3i}$ are linearly dependent.

Proof By corollary 2.4., if $H_i a_{3i-2} (= H_i a_{3i-1} = H_i a_{3i}) = 0$, then the vectors $a_1, \cdots, a_{3i}$ are linearly dependent.
To prove the converse, for $i = 1$, let $a_3 = \alpha a_1$, $\alpha_1 \neq 1$ or $a_3 = \beta a_2$, $\beta \neq 1$.
(for $\alpha = \beta = 1$, it is easily verified that $a_1 = a_2 = a_3$, which can not allow the definition of H_1). If $a_3 = \alpha a_1, \alpha \neq 1$, we have:

$$0 = H_1 c_1 = H_1(a_3 - a_1) = H_1(\alpha a_1 - a_1) = (\alpha - 1) H_1 a_1$$

This implies that $H_1a_1 = H_1a_2 = H_1a_3 = 0$.
If $a_3 = \beta a_2$, $\beta \neq 1$, similar to the previous case we have:

$$0 = H_1c_2 = H_1(a_3 - a_2) = H_1(\beta a_2 - a_2) = (\beta - 1)H_1a_2$$

and $H_1a_1 = H_1a_2 = H_1a_3 = 0$.
For $i > 1$, since $a_1, \cdots, a_{3i}$ are linearly independent, then the dependence of $a_1, \cdots, a_{3i}$ can happen in any one of the nonexclusive ways:
(i) a_{3i-2} or a_{3i-1} or a_{3i} is linearly dependent on $a_1, \cdots, a_{3i-3}$,
or
(ii) a_{3i-2}, a_{3i-1} and a_{3i} are linearly dependent.
In case (i), let's assume, without loss of generality, that

$$a_{3i} = \sum_{j=1}^{3i-3} \alpha_j a_j.$$

Then, using the fact that $H_ia_j = 0$, $j = 1, \cdots, 3i-3$, We have

$$H_ia_{3i} = H_i\left(\sum_{j=1}^{3i-3} \alpha_j a_j\right) = \sum_{j=1}^{3i-3} \alpha_j H_i a_j = 0.$$

In case (ii), let's assume, without loss of generality, that

$$a_{3i} = \alpha a_{3i-2}, \qquad \alpha \neq 1.$$

(For $\alpha = 1$, we have $a_{3i} = a_{3i-2}$, which implies that $c_{3i-2} = 0$ and hence H_i cannot be defined $H_{l_{i-1}}$, contradicting the assumption of the theorem.)
Then, Using the fact that $H_ic_{3i-2} = 0$, we have:

$$0 = H_ia_{3i} - H_ia_{3i-2} = H_i(\alpha a_{3i-2}) - H_ia_{3i-2} = (\alpha - 1)H_ia_{3i-2},$$

which shows $H_ia_{3i-2} = 0$. ■

Remark 2.7 To reduce the compution time, the setting of x_0 and H_0 as the zero vector and the identity matrix I_n, respectively, are proper, in step (1) of Algorithm 1. Also, in this algorithm, we can select the $w_i \in \mathbb{R}^{n-3i+1}$, $z_i \in \mathbb{R}^{n-3i+3}$ considering (2.28) and (2.7) respectively, for $i = 2, \cdots, l$, by the following strategy: (The cases $w_1 \in \mathbb{R}^{n-2}$ and $z_1 \in \mathbb{R}^n$ will be discussed)
We let $d_i = H_ia_{3i}$, and

$$(z)_i = \begin{cases} \frac{1}{d_{j_M}}, & i = j_M \\ 0, & i \neq j_M \end{cases}$$

$$(w)_i = \begin{cases} \frac{1}{d'_{j_M}}, & i = j_M \\ 0, & i \neq j_M \end{cases}$$

where

$$|d_{j_M}| = \max\{|d_i| : i \in \{1, \cdots, n\}\backslash\{\ell_1, \cdots, \ell_{3i-3}\} \text{ such that } z_i^T d = 1\}$$
$$\left|d'_{j_M}\right| = \max\{|d_i| : i \in \{1, \cdots, n\}\backslash\{\ell_1, \cdots, \ell_{3i-1}\} \text{ such that } w_i^T d = 1\}$$

The rows $\ell_1 \cdots \ell_{3i-3}$ of $H_{l_{i-1}}$ and the rows $\ell_1 \cdots \ell_{3i-1}$ of H_i are all the zero vectors. For $i = 1$, we define $d = H_0 a_3$, where

$$(z)_1 = \begin{cases} \frac{1}{d_{j_M}}, & i = j_M \\ 0, & i \neq j_M \end{cases}$$

$$(w)_1 = \begin{cases} \frac{1}{d'_{j_M}}, & i = j_M \\ 0, & i \neq j_M \end{cases}$$

$$|d_{j_M}| = \max\{|d_i| : i \in \{1, \cdots, n\} \text{ such that } z_1^T d = 1\}$$
$$\left|d'_{j_M}\right| = \max\{|d_i| : i \in \{1, \cdots, n\}\backslash\{\ell_1, \ell_2\} \text{ such that } w_1^T d = 1\}$$

The rows ℓ_1 and ℓ_2 of H_1 are zero as Theorem 2.7.

Remark 2.8 If $H_i a_{3i-2} = 0$, it is clear that neither x_i nor H_i can be defined, so x_{l_i} and H_{l_i} can not be defined too. In this case one should identify the case and propose alternative steps to define x_i and H_i and resultly x_{l_i} and H_{l_i}. (of course one can always make use of the regular rank one or rank two ABS steps as alternatives). We also not that H_i fails to be defined if and only if the systems (2.30) and (22) lacks a solution, that is the vectors $H_i c_{3i-2}$ and $H_i c_{3i-1}$ are linearly dependent. A similar argument, is given in the proof for Theorem 2.8., shows that this can happen if and only if $a_1, \cdots, a_{3i}$ are linearly dependent.

In this section we discussed ABS type methods for solving m linear equations by $\frac{m}{3}$ steps, extentively. Now, we are ready to present the main formula (parameters) for solving full row rank (and hence compatible) systems by $\frac{m}{k}$ steps for $k = 1, \cdots, m$.

3. ABS Type Methods for Solving Full Row Rank Systems by $\frac{m}{k}$ Steps

Assume that $a_1, \cdots, a_m$ are linearly independent vectors in $\mathbb{R}^n$.
Let $x_0 \in \mathbb{R}^n$ be an arbitrary vector and $H_0 \in \mathbb{R}^{n\times n}$ be an arbitrary nonsingular matrix.
At first we explain how to compute the stepsize, based on the values of the residues of the k new equations being considered. If we call the stepsize of the i-th iterate by α_i, we use the following strategy for the definition of α_i.
Without loss of the generality assume that the residues values of the k-th equation

be nonzero in each steps.
(i) If $r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki}(x_{i-1}) \neq 0$ then we let

$$a_{ki-(k-1)} = r_{ki-(k-2)}(x_{i-1})r_{ki-(k-3)}(x_{i-1})\cdots r_{ki}(x_{i-1})a_{ki-(k-1)}$$

$$b_{ki-(k-1)} = r_{ki-(k-2)}(x_{i-1})r_{ki-(k-3)}(x_{i-1})\cdots r_{ki}(x_{i-1})b_{ki-(k-1)}$$

$$a_{ki-(k-2)} = r_{ki-(k-1)}(x_{i-1})r_{ki-(k-3)}(x_{i-1})\cdots r_{ki}(x_{i-1})a_{ki-(k-2)}$$

$$b_{ki-(k-2)} = r_{ki-(k-1)}(x_{i-1})r_{ki-(k-3)}(x_{i-1})\cdots r_{ki}(x_{i-1})b_{ki-(k-2)}$$

$$\vdots$$

$$a_{ki-1} = r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki-2}(x_{i-1})r_{ki}(x_{i-1})a_{ki-1}$$

$$b_{ki-1} = r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki-2}(x_{i-1})r_{ki}(x_{i-1})b_{ki-1}$$

$$a_{ki} = r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki-2}(x_{i-1})r_{ki-1}(x_{i-1})a_{ki}$$

$$b_{ki} = r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki-2}(x_{i-1})r_{ki-1}(x_{i-1})b_{ki}$$

and the stepsize is given by:

$$\alpha_i = \frac{r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki-1}(x_{i-1})r_{ki}(x_{i-1})}{z_i^T H_i a_{k_i}} \tag{34}$$

z_i and H_i, will be defined, too.
(ii) If $r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki}(x_{i-1}) = 0$, and all of the k residues values are zero α_i will be zero and x_i will be set to x_{i-1} as expected so, x_{l_i} will be set to $x_{l_{i-1}}$. Now we assume, without loss of generality, that the first p residues values are zero and the others are nonzero. At first we must compute which equations that their residues values of them are nonzero, then we compute the other equations based on the k-th equation. We emphasize again that the k-th equation has nonzero residual value.
(iii) If $r_{ki-(k-1)}(x_{i-1})r_{ki-(k-2)}(x_{i-1})\cdots r_{ki}(x_{i-1}) = 0$ and some of the k residues values be nonzero, without loss of the generality, assume that the k-th equation has nonzero residual value. At first we must compute which equations that their residues values of them are nonzero based on the k-th equation, in each iterate. (Similar to Remark 2.3) and, the step size is given by:

$$\alpha_i = \frac{\text{The product of nonzero residues values}}{z_i^T H_i a_{k_i}}$$

Now, by defining c_j as the next relation

$$c_j = \begin{cases} a_{ki} - a_j, \; j \neq ki \\ \\ a_{ki}, \qquad\quad j = ki \end{cases} \tag{35}$$

we have:

$$H_1 = H_0 - H_0 c_1 \overline{w}_1^T H_0 - H_0 c_2 \tilde{w}_2^T H_0 - \cdots - H_0 c_{k-1} \hat{w}_{k-1}^T H_0 \tag{36}$$

with $\overline{w}_1, \tilde{w}_2, \cdots, \hat{w}_{k-1} \in \mathbb{R}^n$ satisfying the following conditions:

$$\begin{aligned} \overline{w}_1^T H_0 c_1 = 1, \tilde{w}_2^T H_0 c_1 = 0, \cdots, \hat{w}_{k-1}^T H_0 c_1 = 0, \\ \overline{w}_1^T H_0 c_2 = 0, \tilde{w}_2^T H_0 c_2 = 1, \cdots, \hat{w}_{k-1}^T H_0 c_2 = 0, \\ \vdots \qquad\qquad\qquad\qquad \\ \overline{w}_1^T H_0 c_{k-1} = 0, \tilde{w}_2^T H_0 c_{k-1} = 0, \cdots, \hat{w}_{k-1}^T H_0 c_{k-1} = 1, \end{aligned} \tag{37}$$

and for all $i = 2, \cdots, l$, we let

$$\begin{aligned} H_i = H_{l_{i-1}} - H_{l_{i-1}} c_{ki-(k-1)} \overline{w}_{ki-(k-1)}^T H_{l_{i-1}} - H_{l_{i-1}} c_{ki-(k-2)} \tilde{w}_{ki-(k-2)}^T H_{l_{i-1}} \\ - \cdots - H_{l_{i-1}} c_{ki-1} \hat{w}_{ki-1}^T H_{l_{i-1}} \end{aligned} \tag{38}$$

with $\overline{w}_{ki-(k-1)}, \tilde{w}_{ki-(k-2)}, \cdots, \hat{w}_{ki-1} \in \mathbb{R}^n$ satisfying the following conditions:

$$\begin{aligned} \overline{w}_{ki-(k-1)}^T H_{l_{i-1}} c_{ki-(k-1)} = 1, \tilde{w}_{ki-(k-2)}^T H_{l_{i-1}} c_{ki-(k-1)} = 0, \cdots, \hat{w}_{ki-1}^T H_{l_{i-1}} c_{ki-(k-1)} = 0, \\ \overline{w}_{ki-(k-1)}^T H_{l_{i-1}} c_{ki-(k-2)} = 0, \tilde{w}_{ki-(k-2)}^T H_{l_{i-1}} c_{ki-(k-2)} = 1, \cdots, \hat{w}_{ki-1}^T H_{l_{i-1}} c_{ki-(k-2)} = 0, \\ \vdots \qquad\qquad\qquad\qquad \\ \overline{w}_{ki-(k-1)}^T H_{l_{i-1}} c_{ki-1} = 0, \tilde{w}_{ki-(k-2)}^T H_{l_{i-1}} c_{ki-1} = 0, \cdots, \hat{w}_{ki-1}^T H_{l_{i-1}} c_{ki-1} = 1, \end{aligned} \tag{39}$$

and H_{l_i} for all $i = 1, \cdots, l$ is written as:

$$H_{l_i} = H_i - H_i a_{ki} w_{ki}^T H_i = H_i - H_i c_{ki} w_{ki}^T H_i \tag{40}$$

where $w_i \in \mathbb{R}^n$ is an arbitrary vector satisfying

$$w_i^T H_i a_{ki} = w_i^T H_i c_{ki} = 1 \tag{41}$$

It is easily proved that if a_i being linearly independent, $H_i a_{ki}$ is a nonzero vector and hence (40) is well defined with a proper choice of w_i. Hence if x_i be a solution of the first ki equations, for all i, $i = 1, \cdots, l$, we have:

$$x_i = x_{i-1} - \alpha_i H_i^T z_i \tag{42}$$

where $z_i \in \mathbb{R}^n$ is selected so that

$$z_i^T H_i a_{ki} \neq 0 \tag{43}$$

and the general solution of the first ki equations is given by (44), for all i, $i = 1, \cdots, l$

$$x_{l_i} = x_i - H_i^T s \tag{44}$$

where $s \in \mathbb{R}^n$ is arbitrary.

Remark 3.1 To reduce the computation time and space we can select x_0 as the zero vector and H_0 as the identity matrix. Also we can select $z_i \in \mathbb{R}^{n-(ki-k)}$ for all $i = 1, \cdots, l$ such that, z_i has only one nonzero component and $z_i^T H_i a_{ki} = 1$. The other ways to reduce the compution time and space are selecting $\overline{w}_i, \tilde{w}_i, \cdots, \hat{w}_i \in \mathbb{R}^{n-(ki-k)}$ for all $i = 2, \cdots, l$ as (39) and $\overline{w}_1, \tilde{w}_1, \cdots, \hat{w}_1 \in \mathbb{R}^n$ as (37) and $w_i \in \mathbb{R}^{n-ki+1}$ for all $i = 1, \cdots, n$ as (41), by extenting Remark 2.4.

Remark 3.2 All of the theorem in section 2 are proveable (extensive-able) for k steps.

Remark 3.3 We know that for $k = m$ we have one step. In this case if we have x_0 the zero vector and p of the residues values are nonzero and the others are zero, by proper interchenching rows of A and b, we have two systems:
(i) A system with rank p
(ii) A system with rank $m - p$
Now it is sufficient to compute (i) because the solution of (ii) is the zero vector.

Theorem 3.1 For the matrices H_i in (38) and the matrices H_{l_i} generated by (40), we have:

$$\begin{aligned}
&\dim R(H_i) = n - ki + 1 && 1 \leqslant i \leqslant l \\
&\dim N(H_i) = ki - 1 && 1 \leqslant i \leqslant l \\
&\dim R(H_{l_i}) = n - ki && 1 \leqslant i \leqslant l \\
&\dim N(H_{l_i}) = ki && 1 \leqslant i \leqslant l \\
&\dim R(H_{l_l}) = n - m && \\
&\dim N(H_{l_l}) = m &&
\end{aligned}$$

4. Computational and Numerical Results

Assume that $A_{m \times n} (m = kl)$ is a full row rank matrix. We can compute the number of multiplications as follows, at the worst case. The major work is shown in Table 1; by considering Remark 3.1. Notice that we need $\frac{(k-1)^3}{3} + (k-1)^2 - \frac{(k-1)}{3}$ multiplications only for computing $\overline{w}_{ki-(k-1)}^T, \cdots, \hat{w}_{ki-1}^T$, in each step.

Table 1. Numbers of multiplications required for solving m linear equations in $\frac{m}{k}$ steps, for the main parameters.

	The number of multiplications in each step
residues values	kn
$a_{ki-(k-1)}, \cdots, a_{ki}$	$kn(k-2)$, $k \geqslant 2$
$b_{ki-(k-1)}, \cdots, b_{ki}$	k
H_i	$(k-1)(2n^2 - 2nki + 3nk - n)$
H_{l_i}	$2n^2 - 2nki + 3n$
stepsize	1
x_i	$n+1$

Hence the total number of multiplications for the l iteration is:

$$N = \sum_{i=1}^{l} [kn + kn(k-2) + k + (k-1)(2n^2 - 2nki + 3nk - n)$$
$$+ (2n^2 - 2nki + 3n) + 1 + n + 1 + O(m)$$

$$N = 2n^2 m - nm^2 + 3knm - 5nm + \frac{5nm}{k} + O(m) \tag{45}$$

We saw [4] that the algorithm of Huang, when implemented with care, requires $\frac{3}{2}mn^2 + O(mn)$ multiplications, and the algorithm by $\frac{m}{2}$ step as [4, 5], requires

$$3mn^2 - \frac{7}{4}m^2 n + \frac{1}{6}m^3 + O(m^2) + O(n^2).$$

comparing this with our results, we see that our new class of algorithm for $\frac{m}{k}$ steps, when m gets close to n requires less work than the previous methods.
In fact for the square systems ($m = n$), the leading terms is $\frac{3}{2}n^3$ for Huang's method and $\frac{17}{12}n^3$ for the method in [4, 5]. But the leading terms for our new method amount to n^3 as apposed to the two previous methods. Of course, when m and n are not too large, the lower order terms of the computation time will also affect the efficiency.

5. Numerical Example

Example 5.1

$$A = \begin{bmatrix} \frac{3}{10} & -\frac{2}{10} & \frac{1}{10} & \frac{2}{10} & \frac{4}{10} \\ -\frac{2}{10} & \frac{1}{10} & 0 & \frac{1}{10} & \frac{5}{10} \\ \frac{1}{10} & 0 & -\frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ -\frac{2}{10} & \frac{1}{10} & \frac{3}{10} & \frac{2}{10} & 0 \\ -\frac{7}{10} & \frac{5}{10} & -\frac{1}{10} & 0 & \frac{3}{10} \end{bmatrix}, \qquad b = \begin{bmatrix} 8 \\ 5 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

$A_{5\times 5}$ has full row rank. For solving this system we use two steps. At the first step we have $m = 3l$ and at last we have $m = 2l$.

By considering

$$x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad H_0 = \begin{bmatrix} 3 & 2 & 1 & 2 & 4 \\ -2 & 1 & 0 & 1 & -5 \\ 1 & 0 & -1 & 3 & 1 \\ 2 & 1 & 3 & 2 & 0 \\ -1 & 5 & 1 & 0 & -3 \end{bmatrix}$$

We have

$$\overline{w}_1^T = \left[\frac{107.52}{399.36}, \frac{24}{416}, 0, 0, 0\right]$$

$$\tilde{w}_1^T = \left[\frac{-9}{41.6}, \frac{-1}{41.6}, 0, 0, 0\right]$$

and the matrix H_1 is as follows

$$H_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{675.84}{399.36} & -\frac{460.8}{399.36} & -\frac{491.52}{399.36} & \frac{737.28}{399.36} & \frac{1413.12}{399.36} \\ \frac{6919.68}{399.36} & \frac{5107.2}{399.36} & \frac{3417.6}{399.36} & \frac{5506.56}{399.36} & \frac{7534.08}{399.36} \\ -\frac{6796.8}{399.36} & -\frac{3648}{399.36} & -\frac{2127.36}{399.36} & -\frac{5644.8}{399.36} & -\frac{8348.16}{399.36} \end{bmatrix}$$

and z_1 is as follows

$$z_1^T = \left[0, 0, 0, 0, -\frac{399.36}{119808}\right]$$

The step size is $\frac{r_1(x_0)r_2(x_0)r_3(x_0)}{z_1^T H_1 a_3} = -160$ and we have

$$x_1 = x_0 - (stepsize) \times H_1^T z_1 = \begin{bmatrix} \frac{354}{39} \\ \frac{190}{39} \\ \frac{110.8}{39} \\ \frac{294}{39} \\ \frac{434.8}{39} \end{bmatrix}$$

We see that x_1 satisfies the first three equations.

Now, we let $w_1^T = \left[0, 0, 0, 0, -\frac{399.36}{119808}\right] \Longrightarrow \left(w_1^T H_1 a_3 = 1\right)$ thus we have

$$H_{l_1} = H_1 - H_1 a_3 w_1^T H_1$$

$$H_{l_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{0.134}{0.13} & -\frac{0.34}{0.13} & -\frac{0.2708}{0.13} & -\frac{0.054}{0.13} & \frac{0.0252}{0.13} \\ \frac{0.217}{0.13} & \frac{0.57}{0.13} & \frac{0.4754}{0.13} & \frac{0.102}{0.13} & -\frac{0.0476}{0.13} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the general solution of the first three equation is: $x_{l_1} = x_1 - H_{l_1}^T s$, where $s \in \mathbb{R}^n$ is arbitrary.

i=2:

$$\begin{cases} r_4(x_1) = -\frac{289.4}{97.5} \\ r_5(x_1) = -\frac{83.6}{97.5} \end{cases} \Longrightarrow \begin{cases} a_4 = -\frac{83.6}{97.5} a_4 \\ b_4 = -\frac{83.6}{97.5} b_4 \end{cases}, \quad \begin{cases} a_5 = -\frac{289.4}{97.5} a_5 \\ b_5 = -\frac{289.4}{97.5} b_5 \end{cases}$$

By similar situation it can be shown that

$$H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{0.01845}{0.5863} & \frac{0.1025}{0.5863} & \frac{0.14309}{0.5863} & \frac{0.0492}{0.5863} & -\frac{0.02296}{0.5863} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

consequently we have

$x_2 = x_1 - (stepsize) \times H_2^T z_2 = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$ which is the exact solution of the system

and we have

$$w_2^T = \left[0, 0, -\frac{571.6425}{49.597372}, 0, 0\right] \Longrightarrow w_2^T H_2 a_5 = 1$$

$$H_{l_2} = H_2 - H_2 a_5 w_2^T H_2 = \begin{bmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{bmatrix}$$

The general solution of the system is:
$x_{l_2} = x_2 - H_{l_2}^T s$, when $s \in \mathbb{R}^n$ is an arbitrary vector.

Remark 5.2 If we let $H_i a_{ki} = d_i$ and

$$|d_{j_M}| = \max\{|d_i| : i \in \{1, \cdots, n\} \backslash \{\ell_1, \cdots, \ell_{ki-1}\}\}$$

for reducing the time of compution we can let

$$z_i^T = w_i^T = \left[0, 0, \cdots, 0, \frac{1}{d_{j_M}}, 0, 0, \cdots, 0\right]$$

originally we can select z_i as intersection space with w_i.

6. Conclusion

In this work we proposed an ABS method for solving a system of linear equations. We showed that this method has less complexity than the previous methods.

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