

## Perishable Inventory Model with Retrial Demands, Negative Customers and Multiple Working Vacations

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**Abstract.** This paper presents the analysis of a continuous review perishable inventory system wherein the life time of each item follows an exponential distribution. The operating policy is  $(s, S)$  policy where the ordered items are received after a random time which follows exponential distribution. Primary arrival follows Poisson distribution and they may turnout to be positive or negative and then enter into the orbit. The orbiting demands compete their service according to exponential distribution. The server takes multiple working vacations at zero inventory. We assume that the vacation times, service times both during regular busy period and vacation period are exponentially distributed. Matrix analytic method is used for the steady state distribution of the model. Various performance measures and cost analysis are shown with numerical results.

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## 1. Introduction

Many classic inventory models rely on the assumption that the lifetime of inventory items is indefinitely long. However, real-life inventory systems can consist of perishable products, i.e., products with a finite lifetime like drugs and medical products,

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which become unusable after a given date. Some perishable products also deteriorate, i.e., their quality diminishes gradually over time. For example, vegetables, food, meat and fish lose their luster as time elapses.

Clearly, an inventory management approach that ignores product perishability and deterioration is likely to yield sub-optimal outcomes. A review of the literature on fixed time perishable and deteriorating inventory models was given by Nahmias [7] and Raafat [12], respectively. Continuous and discrete perishable inventory models have been studied recently by Baron et al. [1], Lawrence et al. [6] and Nahmias [8].

Vacation queueing models have been investigated extensively in view of their applications in computer systems, communication networks and production managing. In a classical vacation queue, the server completely stops serving customers and may do some additional work. Comprehensive surveys on inventory models with server vacation can be found in Krishnamoorthy et al. [5], Narayanan et al. [9], Sivakumar [16] and Jeganathan [2].

In working vacation (*WV*) systems, instead of completely stopping the service, the server is continuously serving the customers with lower service rate. The classical vacation model is a special case of the *WV* model with zero service rate during vacation. Thus the *WV* model is a generalization of the classical vacation model and it has more analytical complexity than the classical vacation model. Kathiresan et al. [4] presented an inventory system with retrial demands and *WV*. Jeganathan [3] extended the paper [4] with two types of customers, high priority and low priority customers.

An important feature in inventory models is to specify what happens when the arriving customer finds a system out of stock, i.e., the inventory on hand is zero. Two classical situations are considered in the literature:

- (1) Lost-sales case: The blocked customers who arrive during zero stock level are lost.
- (2) Backlog case: The customer arriving while the inventory system is out of stock are backlogged and are satisfied as soon as an appropriate replenishment occurs.

However, in some applications, the demands during the stock-out period go to an orbit of unsatisfied customers and after a random amount of time, retry for service. We can see such situations in various fields such as telecommunication and computer networks. There are numerous studies on retrial inventory models by several researchers. Recently, Jeganathan [3] considered a retrial inventory system with multiple working vacations and two types of customers. The retrial inventory system with negative customers and service interruptions was studied by Vijaya Laxmi and Soujanya [17]. They extended the paper [17] with perishable inventories in Vijaya Laxmi and Soujanya [18].

In all the above models, the authors have considered that the arrival of customers to the service station should join the system until it is full. However in some applications the arriving customers, instead of joining the system, remove some of the waiting customers from the system. This type of customer is called negative customer. Research on inventory models with negative arrivals has been greatly motivated by some practical applications in computers, neural networks and communication networks, etc. Sivakumar and Arivarignan [14, 15] proposed the concept of negative customers in an inventory model with finite and infinite waiting line, respectively. Yadavalli et al. [19] considered a continuous perishable inventory system with multi-server service facility and negative customers. A continuous review inventory system with an orbit of infinite size was proposed by

Rajkumar [13]. He considered ordinary and negative customers and assumed that the ordinary customer will renege from the orbit after a random time.

As a real life application, let us consider a mobile network. It is a known fact that a subscriber who obtains a busy tone, repeats the call until the required connection is established. This situation can be viewed as an example of retrial. And in many situations the data will be transported in the form of packets through a shared common channel. When the packet arrives for a sustained period at a given router or network segment at a rate greater than it is possible to send through, then there is no other option than to drop packets. This can be treated as an example of perishability. In general every switching system of a mobile network will have four important functions. They are event monitoring, call processing, charging, operation & maintenance. If there is no traffic to handle a call, the switching system will perform only the last function operation & maintenance and during this if any call arrives, the switching system realizes the operation & maintenance function at a lower rate and then it handles the traffic. This case can be considered as an example of working vacation. Similarly, if a customer experiences a dropped or discarded connection to the network very frequently, he may opt to go with another network operator and he may also motivate another customer (for example, family member in case of poor signals) to quit the existing network. This case can be taken as an example of negative arrival.

Motivated by such situation, this paper focuses on perishable inventory model with multiple working vacations (*MWV*), negative customers and the backlogged customers retry for service after a random amount of time. The operating policy is  $(s, S)$  policy with exponential replenishment times for the ordered items. Customers arrive according to Poisson process. Whenever the inventory level is zero, the server goes for a *WV*. If the server is busy or at zero inventory, the primary arrival may turnout to be positive or negative and enters into an orbit of infinite size. The service times during busy period, service times during *WV* period, vacation times and life times of inventory are exponentially distributed. Using matrix analytic method, we obtain the steady state distribution of the model. Various performance measures and cost analysis through direct search method are presented.

The rest of the paper is organized as follows. Section 2 and Section 3 present the description and analysis of the model, respectively. Section 4 is devoted to performance measures and cost analysis. Finally, the cost analysis is illustrated by means of numerical examples in Section 5 followed by conclusion in Section 6.

## 2. Description of the model

Let us consider an  $(s, S)$  inventory system in which customers arrive according to a Poisson process with rate  $\lambda$ . Life time of each item has exponential distribution with rate  $\gamma$ . The system starts with  $S$  units of inventory on hand. Each arriving customer is served a single unit of the item. When the inventory level reaches  $s$ , an order is placed for  $Q (= S - s)$  units with an exponential replenishment rate  $\eta$ . When the inventory level reaches to zero, the server leaves for a *WV*, the duration of which is exponentially distributed with rate  $\beta$ . If the server finds an empty stock at the end of a vacation, it takes another vacation immediately, otherwise; it has to serve any arriving customers. The service times during regular period and *WV* follow exponential distribution with parameters  $\mu_b$  and  $\mu_v$ , respectively. If the server is busy or inventory level is zero, any arriving primary customer may turns out to be positive with probability  $p$  or negative with probability  $q (= 1 - p)$  and enter into an orbit of infinite size. These orbiting customers compete for their demands according to an exponential distribution with parameter  $i\theta$  when the

number of customers in the orbit is  $i$ . We also assume that the inter-dependent times between the primary demands, the lead times, retrial demand times and server vacation times are mutually independent random variables.

**Notations:**

- $e$  : A column vector of appropriate dimension containing all ones.
- $\mathbf{0}$  : A zero matrix of appropriate dimension.
- $[A]^{ij}$  : The sub-matrix at  $(i, j)^{th}$  position of  $A$ .
- $\phi_N^{(j)}$  : The probability of  $N$  orbital demands at  $j^{th}$  state of the server.
- $\phi_N^{(j)} = \begin{cases} (\Pi_N^{(j, 0)}, \dots, \Pi_N^{(j, S)}), j = 0, 1; \\ (\Pi_N^{(j, 1)}, \dots, \Pi_N^{(j, S)}), j = 2, 3. \end{cases}$
- $\phi_N^{(j,k)}$  :  $\begin{cases} \text{The probability of } N \text{ demands in the orbit when the server is in} \\ j^{th} \text{ state and the inventory level is } k. \end{cases}$
- $\zeta(t) = \begin{cases} 0, \text{ The serve is idle in } WV \text{ period at time } t; \\ 1, \text{ The serve is busy in } WV \text{ period at time } t; \\ 2, \text{ The serve is idle in regular period at time } t; \\ 3, \text{ The serve is busy in regular period at time } t. \end{cases}$

Let  $N(t)$ ,  $\zeta(t)$  and  $L(t)$  be the number of customers in the orbit, the status of the server and the on-hand inventory level at time  $t$ , respectively. Then  $\Omega = \{(N(t), \zeta(t), L(t)), t \geq 0\}$  is a level-dependent quasi-birth-death (*LDQBD*) process on the state space  $E = \{(i, j, k); i \geq 0, j = 0, 1, 0 \leq k \leq S\} \cup \{(i, j, k); i \geq 1, j = 2, 3, 1 \leq k \leq S\}$ . Define the following ordered sets:

$$\begin{aligned} \langle i, j \rangle &= \begin{cases} ((i, j, 0), (i, j, 1), \dots, (i, j, S)), j = 0, 1; \\ ((i, j, 1), (i, j, 2), \dots, (i, j, S)), j = 2, 3. \end{cases} \\ \langle i \rangle &= (\langle i, j \rangle)_{j = 0, 1, 2, 3}. \end{aligned}$$

The state space can be ordered lexicographically ( $\langle 0 \rangle, \langle 1 \rangle, \dots$ ). A typical illustration of the transitions of the Markov chain  $E$  is given in Figure 1 and its infinitesimal generator matrix  $T$  is a block tridiagonal matrix, which is of the following form:

$$T = \begin{matrix} & \langle 0 \rangle & \langle 1 \rangle & \langle 2 \rangle & \langle 3 \rangle & \dots \\ \langle 0 \rangle & \left[ \begin{array}{cccc} A_0 & C & \mathbf{0} & \mathbf{0} & \dots \end{array} \right] \\ \langle 1 \rangle & \left[ \begin{array}{cccc} B_1 & A_1 & C & \mathbf{0} & \dots \end{array} \right] \\ \langle 2 \rangle & \left[ \begin{array}{cccc} \mathbf{0} & B_2 & A_2 & C & \dots \end{array} \right] \\ \langle 3 \rangle & \left[ \begin{array}{cccc} \mathbf{0} & \mathbf{0} & B_3 & A_3 & \dots \end{array} \right] \\ \vdots & \left[ \begin{array}{cccc} \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right] \end{matrix},$$

where the blocks  $C$ ,  $A_i$  ( $i \geq 0$ ) and  $B_i$  ( $i \geq 1$ ) are square matrices, each of order  $(4S + 2)$ , they are give by

$$A_0 = \begin{matrix} & \langle 0, 0 \rangle & \langle 0, 1 \rangle & \langle 0, 2 \rangle & \langle 0, 3 \rangle \\ \langle 0, 0 \rangle & \left[ \begin{array}{cccc} [A_0]^{00} & [A_0]^{01} & [A_0]^{02} & \mathbf{0} \\ [A_0]^{10} & [A_0]^{11} & \mathbf{0} & [A_0]^{13} \\ [A_0]^{20} & \mathbf{0} & [A_0]^{22} & [A_0]^{23} \\ \mathbf{0} & [A_0]^{31} & \mathbf{0} & [A_0]^{33} \end{array} \right] \\ \langle 0, 1 \rangle & & & & \\ \langle 0, 2 \rangle & & & & \\ \langle 0, 3 \rangle & & & & \end{matrix},$$



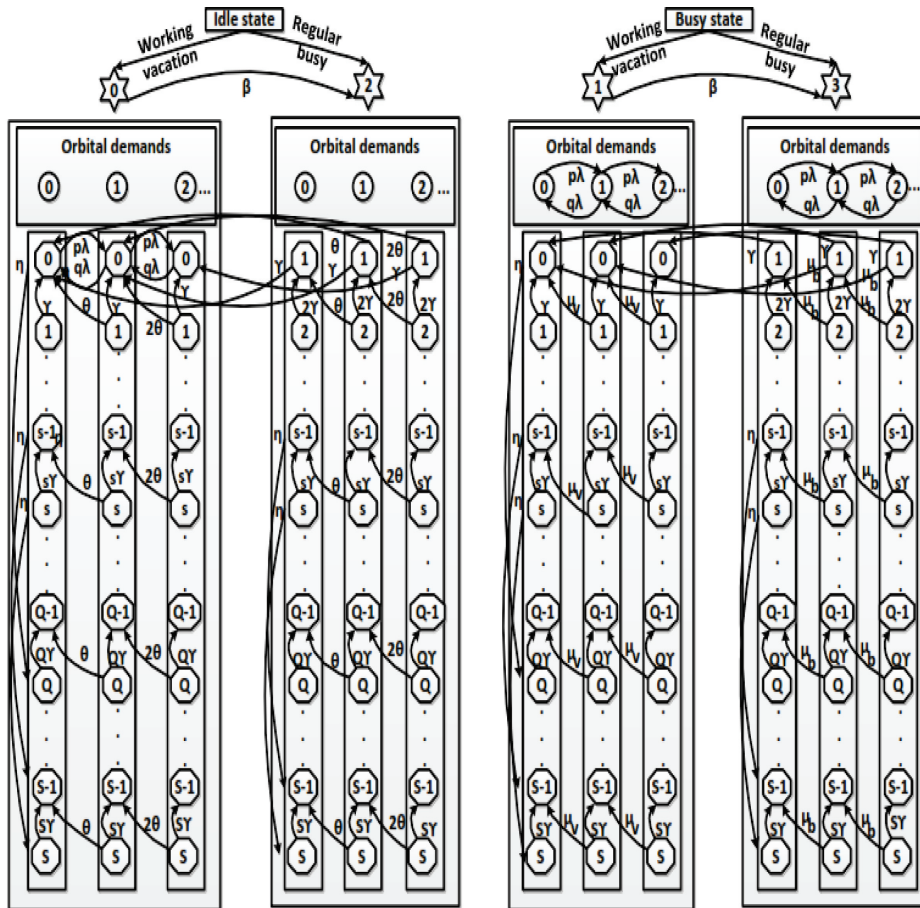


Figure 1. Transition diagram for the Markov chain  $E$ .

$$C = \begin{matrix} & \langle i, 0 \rangle & \langle i, 1 \rangle & \langle i, 2 \rangle & \langle i, 3 \rangle \\ \langle i+1, 0 \rangle & [C]^{00} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \langle i+1, 1 \rangle & \mathbf{0} & [C]^{11} & \mathbf{0} & \mathbf{0} \\ \langle i+1, 2 \rangle & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \langle i+1, 3 \rangle & \mathbf{0} & \mathbf{0} & \mathbf{0} & [C]^{33} \end{matrix},$$

For  $i \geq 1$ ,

$$A_i = \begin{matrix} \langle i, 0 \rangle & \langle i, 1 \rangle & \langle i, 2 \rangle & \langle i, 3 \rangle \\ \langle i, 0 \rangle & [A_i]^{00} & [A_i]^{01} & [A_i]^{02} & \mathbf{0} \\ \langle i, 1 \rangle & [A_i]^{10} & [A_i]^{11} & \mathbf{0} & [A_i]^{13} \\ \langle i, 2 \rangle & \mathbf{0} & \mathbf{0} & [A_i]^{22} & [A_i]^{23} \\ \langle i, 3 \rangle & \mathbf{0} & [A_i]^{31} & \mathbf{0} & [A_i]^{33} \end{matrix},$$

$$B_i = \begin{matrix} \langle i-1, 0 \rangle & \langle i-1, 1 \rangle & \langle i-1, 2 \rangle & \langle i-1, 3 \rangle \\ \langle i, 0 \rangle & [B_i]^{00} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \langle i, 1 \rangle & \mathbf{0} & [B_i]^{11} & \mathbf{0} & \mathbf{0} \\ \langle i, 2 \rangle & [B_i]^{20} & \mathbf{0} & [B_i]^{22} & \mathbf{0} \\ \langle i, 3 \rangle & \mathbf{0} & \mathbf{0} & \mathbf{0} & [B_i]^{33} \end{matrix},$$

$$[[A_0]^{00}]_{kn} = \begin{cases} -(p\lambda + \eta), & n = k, & k = 0; \\ -(\lambda + \eta + \beta + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(\lambda + \beta + k\gamma), & n = k, & k = s + 1, \dots, S; \\ \eta, & n = k + Q, & k = 0, 1, \dots, s; \\ k\gamma, & n = k - 1, & k = 1, 2, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{11}]_{kn} = \begin{cases} -(p\lambda + \eta), & n = k, & k = 0; \\ -(p\lambda + \eta + \beta + \mu_v + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(p\lambda + \beta + \mu_v + k\gamma), & n = k, & k = s + 1, \dots, S; \\ \eta, & n = k + Q, & k = 0, 1, \dots, s; \\ \mu_v + k\gamma, & n = k - 1, & k = 2, 3, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{22}]_{kn} = \begin{cases} -(\lambda + \eta + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(\lambda + k\gamma), & n = k, & k = s + 1, \dots, S; \\ \eta, & n = k + Q, & k = 1, 2, \dots, s; \\ k\gamma, & n = k - 1, & k = 2, 3, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{33}]_{kn} = \begin{cases} -(p\lambda + \eta + \mu_b + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(p\lambda + \mu_b + k\gamma), & n = k, & k = s + 1, \dots, S; \\ \eta, & n = k + Q, & k = 1, 2, \dots, s; \\ \mu_b + k\gamma, & n = k - 1, & k = 2, 3, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{01}]_{kn} = [[A_0]^{23}]_{kn} = \begin{cases} \lambda, & n = k, & k = 1, 2, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{02}]_{kn} = [[A_0]^{13}]_{kn} = \begin{cases} \beta, & n = k, & k = 1, 2, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{10}]_{kn} = \begin{cases} \gamma + \mu_v, & n = k - 1, & k = 1; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{31}]_{kn} = \begin{cases} \gamma + \mu_b, & n = k - 1, & k = 1; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_0]^{20}]_{kn} = \begin{cases} \gamma, & n = k - 1, & k = 1; \\ 0, & \text{otherwise.} \end{cases}, \quad [[C]^{00}]_{kn} = \begin{cases} p\lambda, & n = k, & k = 0; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[C]^{11}]_{kn} = \begin{cases} p\lambda, & n = k, & k = 0, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[C]^{33}]_{kn} = \begin{cases} p\lambda, & n = k, & k = 1, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

For  $i \geq 1$ ,

$$[[A_i]^{00}]_{kn} = \begin{cases} -(p\lambda + q\lambda + \eta), & n = k, & k = 0; \\ -(\lambda + \eta + i\theta + \beta + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(\lambda + i\theta + \beta + k\gamma), & n = k, & k = s + 1, \dots, S; \\ k\gamma, & n = k - 1, & k = 1, 2, \dots, S; \\ \eta, & n = k + Q, & k = 0, 1, \dots, s; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_i]^{11}]_{kn} = \begin{cases} -(p\lambda + q\lambda + \eta), & n = k, & k = 0; \\ -(p\lambda + q\lambda + \eta + \beta + \mu_v + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(p\lambda + q\lambda + \beta + \mu_v + k\gamma), & n = k, & k = s + 1, \dots, S; \\ \eta, & n = k + Q, & k = 0, 1, \dots, s; \\ \mu_v + k\gamma, & n = k - 1, & k = 2, 3, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_i]^{22}]_{kn} = \begin{cases} -(\lambda + \eta + i\theta + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(\lambda + i\theta + k\gamma), & n = k, & k = s + 1, \dots, S; \\ k\gamma, & n = k - 1, & k = 2, 3, \dots, S; \\ \eta, & n = k + Q, & k = 1, 2, \dots, s; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_i]^{33}]_{kn} = \begin{cases} -(p\lambda + q\lambda + \eta + \mu_b + k\gamma), & n = k, & k = 1, 2, \dots, s; \\ -(p\lambda + q\lambda + \mu_b + k\gamma), & n = k, & k = s + 1, \dots, S; \\ \eta, & n = k + Q, & k = 1, 2, \dots, s; \\ \mu_b + k\gamma, & n = k - 1, & k = 2, 3, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[A_i]^{01}] = [[A_0]^{01}], \quad [[A_i]^{02}] = [[A_0]^{02}], \quad [[A_i]^{10}] = [[A_0]^{10}], \quad [[A_i]^{13}] = [[A_0]^{13}],$$

$$[[A_i]^{20}] = [[A_0]^{20}], \quad [[A_i]^{23}] = [[A_0]^{23}], \quad [[A_i]^{31}] = [[A_0]^{31}],$$

$$[[B_i]^{00}]_{kn} = \begin{cases} q\lambda, & n = k, & k = 0; \\ i\theta, & n = k - 1, & k = 1, 2, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[B_i]^{11}]_{kn} = \begin{cases} q\lambda, & n = k, & k = 0, 1, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[B_i]^{20}]_{kn} = \begin{cases} i\theta, & n = k - 1, & k = 1; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[B_i]^{22}]_{kn} = \begin{cases} i\theta, & n = k - 1, \quad k = 2, 3, \dots, S; \\ 0, & \text{otherwise.} \end{cases},$$

$$[[B_i]^{33}]_{kn} = \begin{cases} q\lambda, & n = k, \quad k = 1, 2, \dots, S; \\ 0, & \text{otherwise.} \end{cases}.$$

### 3. Steady state analysis

In this section, using matrix analytic method (for matrix analytic method, see Neuts [10]), we perform the steady state analysis. First, we determine the stability condition under which the irreducible Markov chain is positive recurrent which guarantees the existence of steady state solution.

#### 3.1 Stability condition

For investigating the stability condition of the system under study, first we apply Neuts and Rao [11] truncation to the LIQBD. To this end suppose that  $A_i = A_N$  and  $B_i = B_N, \forall i \geq N$ . The generator matrix of the truncated system  $\Omega_N$  will look as under:

$$T_N = \begin{matrix} & \langle 0 \rangle & \langle 1 \rangle & \langle 2 \rangle & \langle 3 \rangle & \dots & \langle N-1 \rangle & \langle N \rangle & \langle N+1 \rangle & \dots \\ \langle 0 \rangle & \left[ \begin{array}{cccccccc} A_0 & C & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ B_1 & A_1 & C & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & B_2 & A_2 & C & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & B_3 & A_3 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle N-1 \rangle & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A_{N-1} & C & \mathbf{0} & \dots \\ \langle N \rangle & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & B_N & A_N & C & \dots \\ \langle N+1 \rangle & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & B_N & A_N & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right] \end{matrix}.$$

Consider the generator matrix  $P_N = B_N + A_N + C$ , which is given by

$$P_N = \begin{matrix} & \langle i, 0 \rangle & \langle i, 1 \rangle & \langle i, 2 \rangle & \langle i, 3 \rangle \\ \langle i, 0 \rangle & \left[ \begin{array}{cccc} [P_N]^{00} & [P_N]^{01} & [P_N]^{02} & \mathbf{0} \\ [P_N]^{10} & [P_N]^{11} & \mathbf{0} & [P_N]^{13} \\ [P_N]^{20} & \mathbf{0} & [P_N]^{22} & [P_N]^{23} \\ \mathbf{0} & [P_N]^{31} & \mathbf{0} & [P_N]^{33} \end{array} \right] \end{matrix},$$

where

$$[P_N]^{00} = [B_N]^{00} + [A_N]^{00} + [C]^{00}, [P_N]^{01} = [A_N]^{01}, [P_N]^{02} = [A_N]^{02},$$

$$[P_N]^{10} = [A_N]^{10}, [P_N]^{11} = [B_N]^{11} + [A_N]^{11} + [C]^{11}, [P_N]^{13} = [A_N]^{13},$$

$$[P_N]^{20} = [B_N]^{20}, [P_N]^{22} = [B_N]^{22} + [A_N]^{22}, [P_N]^{23} = [A_N]^{23},$$

$$[P_N]^{31} = [A_N]^{31}, [P_N]^{33} = [B_N]^{33} + [A_N]^{33} + [C]^{33}.$$

Let  $\phi_N = (\phi_N^{(0)}, \phi_N^{(1)}, \phi_N^{(2)}, \phi_N^{(3)})$  denote the steady state probability vector of  $P_N$ , i.e.,  $\phi_N P_N = 0$  and  $\phi_N e = 1$ . The elements of the vector  $\phi_N$  can be given in the following lemma.

LEMMA 3.1 *The vector  $\phi_N$  corresponding to the generator  $P_N$  is given by*

$$\phi_N^{(j)} = -\phi_N^{(0)} \vartheta_N^{(j)}; j = 1, 2, 3,$$

where,

$$\vartheta_N^{(1)} = ([P_N]^{00} - [P_N]^{02}([P_N]^{22})^{-1}[P_N]^{20})([P_N]^{10})^{-1},$$

$$\vartheta_N^{(2)} = [P_N]^{02}([P_N]^{22})^{-1},$$

$$\vartheta_N^{(3)} = ([P_N]^{01} - ([P_N]^{00} - [P_N]^{02}([P_N]^{22})^{-1}[P_N]^{20})([P_N]^{10})^{-1}[P_N]^{11})([P_N]^{31})^{-1}$$

and  $\phi_N^{(0)}$  can be obtained by solving

$$\left. \begin{aligned} \phi_N^{(0)} \left[ [P_N]^{00} \left( ([P_N]^{10})^{-1} \left( [P_N]^{11} ([P_N]^{31})^{-1} [P_N]^{33} - [P_N]^{13} \right) \right) \right. \\ \left. - [P_N]^{01} ([P_N]^{31})^{-1} [P_N]^{33} + [P_N]^{02} ([P_N]^{22})^{-1} \left( [P_N]^{20} \right. \right. \\ \left. \left. ([P_N]^{10})^{-1} \left( [P_N]^{13} - [P_N]^{11} ([P_N]^{31})^{-1} [P_N]^{33} \right) - [P_N]^{23} \right) \right] \end{aligned} \right\} = 0 \quad (1)$$

and

$$\left. \begin{aligned} \phi_N^{(0)} \left[ 1 - [P_N]^{00} ([P_N]^{10})^{-1} \left( 1 - [P_N]^{11} ([P_N]^{31})^{-1} \right) - [P_N]^{01} \right. \\ \left. ([P_N]^{31})^{-1} - [P_N]^{02} ([P_N]^{22})^{-1} \left( 1 + [P_N]^{20} ([P_N]^{10})^{-1} \right) \right. \\ \left. \left( 1 + [P_N]^{11} ([P_N]^{31})^{-1} \right) \right] \end{aligned} \right\} = 1. \quad (2)$$

*Proof* The equation  $\phi_N P_N = 0$  yields the following set of equations.

$$\phi_N^{(0)} [P_N]^{00} + \phi_N^{(1)} [P_N]^{10} + \phi_N^{(2)} [P_N]^{20} = 0, \quad (3)$$

$$\phi_N^{(0)} [P_N]^{01} + \phi_N^{(1)} [P_N]^{11} + \phi_N^{(3)} [P_N]^{31} = 0, \quad (4)$$

$$\phi_N^{(0)} [P_N]^{02} + \phi_N^{(2)} [P_N]^{22} = 0, \quad (5)$$

$$\phi_N^{(1)} [P_N]^{13} + \phi_N^{(2)} [P_N]^{23} + \phi_N^{(3)} [P_N]^{33} = 0. \quad (6)$$

The equations (3) to (5) can be recursively solved to get  $\phi_N^{(j)}, j = 1, 2, 3$ , as stated in Lemma 3.1. Now, substituting the values of  $\vartheta_N^{(j)}$  in (6) and in the normalizing condition  $\phi_N e = 1$  we get the constraints (1) and (2) to compute  $\phi_N^{(0)}$ . ■

The following result gives the stability condition.

LEMMA 3.2 *The stability condition of the system under study is given by*

$$(p\lambda - q\lambda) < (p\lambda - q\lambda + \theta) (\phi_N^{(0)} - \phi_N^{(0,0)} + \phi_N^{(2)}).$$

*Proof* From the well known result of Neuts [10] on the positive recurrence of  $P_N$ , we have  $\phi_N C e < \phi_N B_N e$ . Simplification of this yields the stated result. ■

### 3.2 Computation of steady state vector

We find the steady state vector of  $\Omega$ , by approximating it with the steady state vector of the truncated system,  $\Omega_N$  with generator matrix  $P_N$ . Let  $\Pi^{(N)} = (\Pi_0, \Pi_1, \Pi_2, \dots)$ , be the steady state vector of  $\Omega_N$  where each  $\Pi_i$  is a row vector consisting of  $4S + 2$  elements represented as  $\Pi_i = (\Pi(i, 0, 0), \dots, \Pi(i, 0, S), \Pi(i, 1, 0), \dots, \Pi(i, 1, S), \Pi(i, 2, 1), \dots, \Pi(i, 2, S), \Pi(i, 3, 1), \dots, \Pi(i, 3, S))$ . From the known results of Matrix Analytic Methods (see Neuts [10]), it follows that,  $\Pi_{N+r} = \Pi_{N-1}(R_N)^{r+1}$ , for  $r \geq 0$ , where  $R_N$  is the minimal non-negative solution of the matrix quadratic equation

$$(R_N)^2 B_N + R_N A_N + C = 0 \text{ and } \Pi_{N-i} = \Pi_{N-i-1} R_{N-i}, \text{ for } 1 \leq i \leq N-1,$$

where  $R_{N-i} = -C(A_{N-i} + R_{N-i+1}B_{N-i+1})^{-1}$ . Now, for computing  $\Pi_0$ , we have the equation  $\Pi_0(A_0 + R_1B_1) = 0$ . First we take  $\Pi_0$  as the steady state vector of the generator matrix  $A_0 + R_1B_1$ . Then  $\Pi_i$ , for  $1 \leq i \leq N-1$ , can be found using recursive formulae;  $\Pi_i = \Pi_{i-1}R_i$ . The steady state probability distribution of the truncated system is then obtained by dividing each  $\Pi_i$ , with the normalizing constant

$$(\Pi_0 + \Pi_1 + \Pi_2 + \dots)e = [\Pi_0 + \Pi_1 + \dots + \Pi_{N-2} + \Pi_{N-1}(I - R_n)^{-1}]e.$$

## 4. System performance measures

In this section, we derive some performance measures of the system. Using these measures we can construct the total expected cost per unit time.

- (i) **Expected inventory level** : Since  $\Pi(i, j, k)$  is the steady state probability vector for  $k^{th}$  inventory level with each component specifying a particular combination of number of customers in the orbit and the state of the server, the expected inventory level ( $E_{IL}$ ) is given by

$$E_{IL} = \sum_{i=1}^{\infty} \sum_{j=0}^3 \sum_{k=1}^S k \Pi(i, j, k).$$

- (ii) **Expected reorder rate** : Let  $E_{OR}$  denote the expected reorder rate in the steady state. A reorder is triggered when the inventory level drops to  $s$  from the level  $s + 1$ , due to any one of the following events:

- if the service of any one of the primary arrival is completed.
- if an inventory is perishable.
- if anyone of the customers in the orbit is selected.

This leads to

$$E_{OR} = \left\{ \begin{aligned} &\sum_{i=0}^{\infty} \left[ \mu_v \pi(i, 1, s + 1) + \mu_b \pi(i, 3, s + 1) \right] + \sum_{i=1}^{\infty} \theta \left[ \pi(i, 0, s + 1) \right. \\ &\left. + \pi(i, 2, s + 1) \right] + \sum_{i=1}^{\infty} \sum_{j=0}^3 (s + 1) \gamma \Pi(i, j, s + 1). \end{aligned} \right.$$

- (iii) **Expected replenishment rate** : The expected replenishment rate ( $E_{RR}$ ) is given by

$$E_{RR} = \sum_{i=0}^{\infty} \eta \left[ \sum_{j=0}^1 \sum_{k=0}^s \Pi(i, j, k) + \sum_{j=2}^3 \sum_{k=1}^s \Pi(i, j, k) \right].$$

- (iv) **Expected failure rate** : The expected failure rate ( $E_{FR}$ ) is given by

$$E_{FR} = \sum_{i=0}^{\infty} \sum_{j=0}^3 \sum_{k=1}^S k \gamma \Pi(i, j, k).$$

- (v) **Mean rate of arrivals of negative customers** : Let  $E_{NC}$  denote the mean arrival rate of negative customers. This is given by

$$E_{NC} = \sum_{i=1}^{\infty} q \lambda \Pi(i, 0, 0) + \sum_{i=1}^{\infty} \sum_{k=0}^S q \lambda \Pi(i, 1, k) + \sum_{i=1}^{\infty} \sum_{k=1}^S q \lambda \Pi(i, 3, k).$$

- (vi) **Expected number of customers in the orbit** : Since  $\Pi_i$  denote the steady state probability when the number of customers in the orbit is  $i$ , the expected number of customers in the orbit ( $E_{CO}$ ) is given by

$$E_{CO} = \sum_{i=1}^{\infty} \left[ \sum_{j=0}^1 \sum_{k=0}^S i \Pi(i, j, k) + \sum_{j=2}^3 \sum_{k=1}^S i \Pi(i, j, k) \right].$$

- (vii) **Successful rate of retrials** : Let  $S_{RR}$  denote the successful rate of retrials in the steady state. The orbiting customers receive their demands only when the server is idle and there is an inventory. Hence  $S_{RR}$  is given by

$$S_{RR} = \sum_{i=1}^{\infty} \sum_{k=1}^S \theta \left[ \Pi(i, 0, k) + \Pi(i, 2, k) \right].$$

- (viii) **Fraction of time the server is on vacation** : The fraction of time the server is on vacation ( $F_{SV}$ ) is given by

$$F_{SV} = \sum_{i=0}^{\infty} \sum_{j=0}^1 \sum_{k=0}^S \Pi(i, j, k).$$

#### 4.1 Cost analysis

We develop a total expected cost function per unit time with an objective to determine the optimum values of  $s$  and  $S$  so that the total cost is minimized. Let

$C_S$  = Setup cost per order,

$C_H$  = Holding cost per unit item per unit time,

$C_N$  = Loss per unit time due to arrival of a negative customer,

$C_O$  = Waiting cost of a customer in the orbit per unit time,

$C_V$  = Fixed cost per unit time when the server is on vacation, and

$C_F$  = Failure cost per unit item unit time.

Based on the definitions of each cost element listed above and various performance measures of the model, the total expected cost function per unit time is defined as

$$TC(s, S) = C_S E_{OR} + C_H E_{IL} + C_N E_{NC} + C_O E_{CO} + C_V F_{SV} + C_F E_{FR}.$$

Substituting  $E_{OR}$ ,  $E_{IL}$ ,  $E_{NC}$ ,  $E_{CO}$ ,  $F_{SV}$  and  $E_{FR}$  in the above equation, we get

$$TC(s, S) = \left\{ \begin{array}{l} C_S \left[ \sum_{i=0}^{\infty} [\mu_v \Pi(i, 1, s + 1) + \mu_b \Pi(i, 3, s + 1)] + \sum_{i=1}^{\infty} \theta [\Pi(i, 0, s + 1) + \right. \\ \left. \Pi(i, 2, s + 1)] \right] + C_H \sum_{i=1}^{\infty} \sum_{j=0}^3 \sum_{k=1}^S k \Pi(i, j, k) + C_N \left[ \sum_{i=1}^{\infty} q \lambda \right. \\ \left. \Pi(i, 0, 0) + \sum_{i=1}^{\infty} \sum_{k=0}^S q \lambda \Pi(i, 1, k) + \sum_{i=1}^{\infty} \sum_{k=1}^S q \lambda \Pi(i, 3, k) \right] + \\ C_O \sum_{i=1}^{\infty} \left[ \sum_{j=0}^1 \sum_{k=0}^S i \Pi(i, j, k) + \sum_{j=2}^3 \sum_{k=1}^S i \Pi(i, j, k) \right] + \\ C_V \sum_{i=0}^{\infty} \sum_{j=0}^1 \sum_{k=0}^S \Pi(i, j, k) + C_F \sum_{i=0}^{\infty} \sum_{j=0}^3 \sum_{k=1}^S k \gamma \Pi(i, j, k). \end{array} \right.$$

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically. Here, we present the following numerical examples to demonstrate the computability of the results derived in our work and to illustrate the effect of the parameters on the main performance characteristics.

### 5. Numerical analysis

To demonstrate the applicability of the formulae obtained in the previous sections, we present some numerical results in the form of tables and graphs. The various parameters of the model are chosen as  $S = 10$ ,  $s = 3$ ,  $p = 0.7$ ,  $q = 1 - p$ ,  $\lambda = 5.0$ ,  $\mu_b = 15.0$ ,  $\mu_v = 10.0$ ,  $\theta = 6.0$ ,  $\eta = 2.8$ ,  $\beta = 1.3$ ,  $\gamma = 3.4$  and the cost values are  $C_S = 0.1$ ,  $C_H = 1.0$ ,  $C_N = 5.0$ ,  $C_O = 1.5$ ,  $C_V = 3.8$  and  $C_F = 1.8$  unless they are considered as variables in some graphs and tables.

Figure 2 shows the effect of perishable rate ( $\gamma$ ) on the expected reorder rate ( $E_{OR}$ ), expected failure rate ( $E_{FR}$ ) and the expected inventory level ( $E_{IL}$ ). As depicted in the figure, it is evident that the expected inventory level decreases and the expected failure rate increases as the perishable rate increases. It can also be observed that expected reorder rate increases with the increase of perishable rate. This is to say that, the inventory reaches the reordering point frequently due to perishability.

The effect of arrival rate ( $\lambda$ ) and vacation rate ( $\beta$ ) on the expected number of negative customers ( $E_{NC}$ ) is presented in Figure 3. We know that  $E_{NC}$  increases with  $\lambda$  and  $\beta$ . In case of vacation rate, if it increases, the server spends more time in busy period and hence according to our assumptions in this model, some arriving customer are turn out to be negative and enter into an orbit. This implies the rise in  $E_{NC}$  as shown in Figure 3. Figure 4 presents the effect of  $\lambda$  and  $\theta$  on  $E_{CO}$ . In



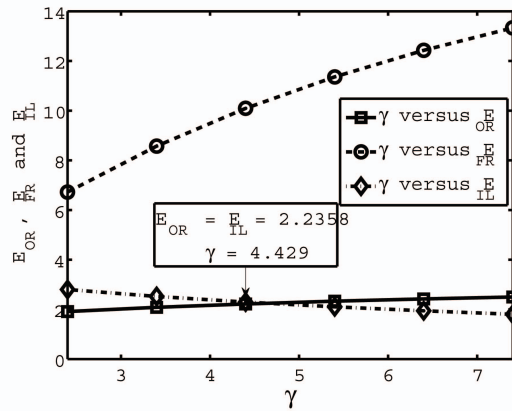


Figure 2. Effect of  $\gamma$  on  $E_{OR}$ ,  $E_{FR}$  and  $E_{IL}$ .

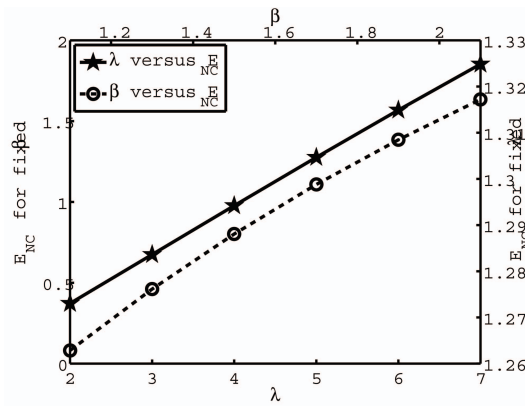


Figure 3. Effect of  $\lambda$  and  $\beta$  on  $E_{NC}$ .

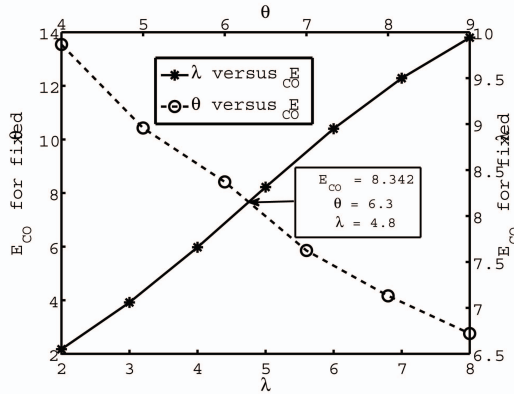


Figure 4. Effect of  $\lambda$  and  $\theta$  on  $E_{CO}$ .

real life, we can expect an increase in the expected number of orbital customers with the arrival rate and a decrease in it with the service rate. From Figure 4, we can clearly observe the increase in  $E_{CO}$  with  $\lambda$  and the decrease with  $\theta$ . The effect of  $\theta$  and  $\beta$  on  $S_{RR}$  is plotted in Figure 5. According to the assumptions we consider in this paper, the successful retrial rate increases with the retrial rate and decreases with the vacation rate as shown in Figure 5.

Figure 6 depicts the effect of service rates  $\mu_b$  and  $\mu_v$  on  $E_{IL}$ ,  $E_{NC}$ ,  $S_{RR}$  and  $E_{CO}$ . From the figures it can be observed that as the service rates increases, the expected inventory level, expected number of customers in the orbit and expected number of negative customers decreases and successful retrial rate increases as it

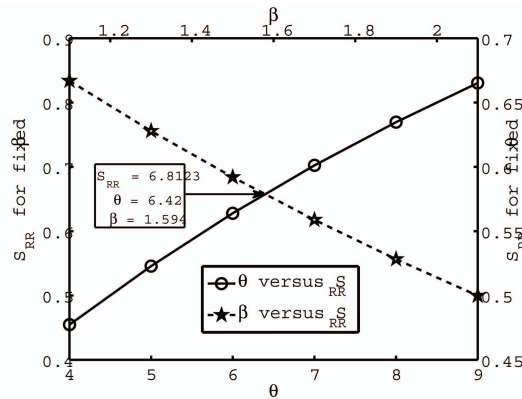


Figure 5. Effect of  $\theta$  and  $\beta$  on  $S_{RR}$ .

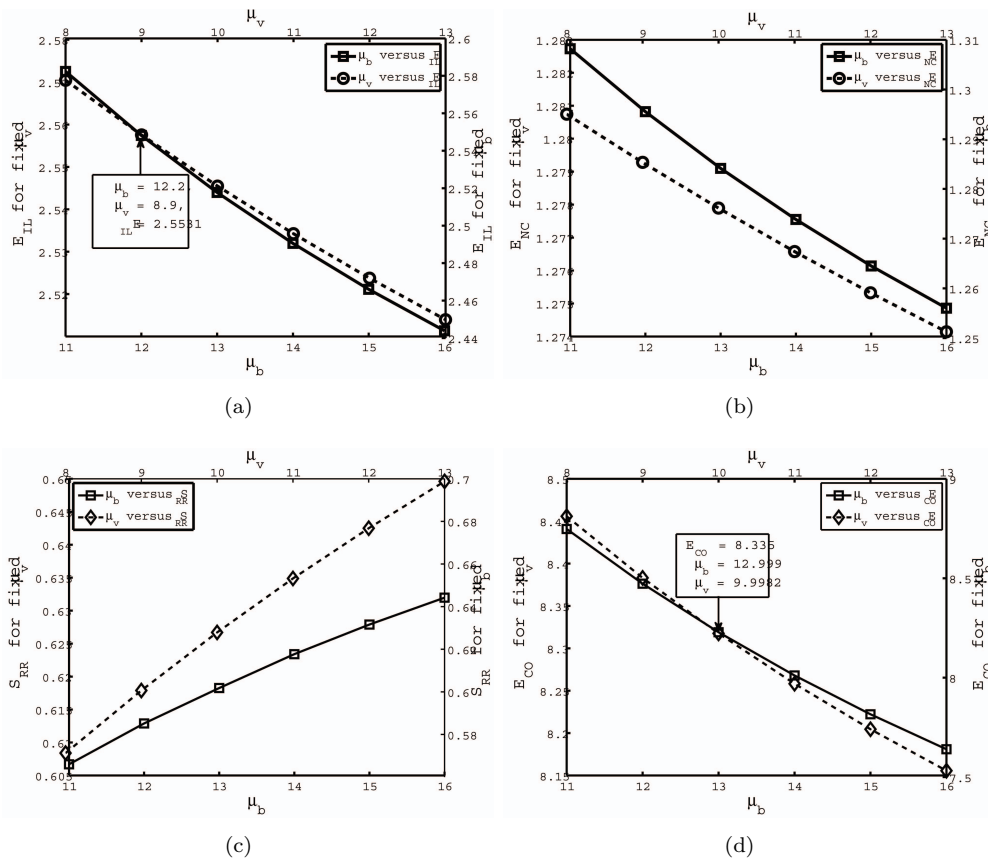


Figure 6. Effect of  $\mu_b$  and  $\mu_v$  on (a)  $E_{IL}$ , (b)  $E_{NC}$ , (c)  $S_{RR}$  and (d)  $E_{CO}$ .

should be. Moreover, in figures 6(b), 6(c) and 6(d), the effect of  $\mu_v$  on  $E_{NC}$ ,  $S_{RR}$  and  $E_{CO}$  is greater than the effect of  $\mu_b$ . For the given parameters this highlights that the working vacation policy can decrease the waiting time of a customer in the queue and enhance the system efficiency. The optimal values of  $s$  and  $S$  that minimize the total expected cost function  $TC(s, S)$  for different pairs of cost values are shown in Table 1. One can see the following observations:

- (1) The total expected cost increases with the increase of cost values according to the definition of cost function. The optimal cost is more sensitive to  $C_S$  than to  $C_H$ ,  $C_O$ ,  $C_N$ ,  $C_V$  and  $C_F$ .
- (2) As  $C_H$  increases, the optimal values of  $s$  and  $S$  decrease monotonically.

Table 1. The optimal values  $s^*$ ,  $S^*$  and  $TC(s^*, S^*)$  for different cost values

$C_H \backslash C_S$	0.1	0.2	0.3	0.4
2.0	(2, 4) 6.31484	(2, 4) 6.34397	(2, 4) 6.37310	(2, 4) 6.40223
3.0	(2, 4) 6.56011	(2, 4) 6.58924	(2, 4) 6.61837	(2, 4) 6.64750
4.0	(1, 3) 6.85567	(1, 3) 6.92608	(1, 3) 6.99648	(1, 3) 7.06689
5.0	(1, 3) 7.06305	(1, 3) 7.13345	(1, 3) 7.20386	(1, 3) 7.27426

$C_N \backslash C_S$	0.1	0.2	0.3	0.4
4.0	(2, 4) 5.95798	(2, 4) 5.98711	(2, 4) 6.06124	(2, 4) 6.04537
5.0	(2, 4) 6.31484	(2, 4) 6.34397	(2, 4) 6.37310	(2, 4) 6.40223
6.0	(1, 4) 6.66192	(2, 4) 6.70084	(2, 4) 6.72997	(2, 4) 6.75910
7.0	(1, 4) 6.98525	(1, 4) 7.04928	(2, 4) 7.08683	(2, 4) 7.11596

$C_O \backslash C_S$	0.1	0.2	0.3	0.4
1.0	(2, 4) 6.00548	(2, 4) 6.03461	(2, 4) 6.06374	(2, 4) 6.09287
1.5	(2, 4) 6.31484	(2, 4) 6.34397	(2, 4) 6.37310	(2, 4) 6.40230
2.0	(1, 4) 6.59228	(2, 4) 6.65334	(2, 4) 6.68247	(2, 4) 6.71159
2.5	(1, 4) 6.84595	(1, 4) 6.90999	(1, 4) 6.97403	(1, 4) 7.03807

$C_V \backslash C_S$	0.1	0.2	0.3	0.4
3.4	(2, 4) 6.14834	(2, 4) 6.17747	(2, 4) 6.20660	(2, 4) 6.23573
3.6	(2, 4) 6.23159	(2, 4) 6.26072	(2, 4) 6.28985	(2, 4) 6.31898
3.8	(2, 4) 6.31484	(2, 4) 6.34397	(2, 4) 6.37310	(2, 4) 6.40223
4.0	(2, 4) 6.39809	(2, 4) 6.42722	(2, 4) 6.45635	(2, 4) 6.48548

$C_F \backslash C_S$	0.1	0.2	0.3	0.4
1.2	(3, 6) 5.63782	(3, 6) 5.66988	(3, 6) 5.70194	(3, 6) 5.73399
1.4	(1, 4) 5.93725	(1, 4) 6.00129	(2, 4) 6.01256	(2, 4) 6.06867
1.6	(1, 4) 6.13792	(1, 4) 6.17719	(2, 4) 6.20632	(2, 4) 6.23545
1.8	(1, 3) 6.31484	(1, 4) 6.34397	(2, 4) 6.37310	(2, 4) 6.40223

This is as expected since the holding cost increases, we resort to maintain low stock in the inventory.

- (3) The optimal values of  $S$  and  $s$  are decreasing monotonically with  $C_F$ .
- (4) Also, we notice that the optimal value of  $s$  decreases when the waiting cost,  $C_O$  increases as expected in practical case.

## 6. Conclusions

This paper presents a perishable inventory model with negative customers, retrial demands and multiple working vacations. The steady state distributions are obtained using matrix analytic method. Various system performance measures are derived and the total expected cost is calculated. By assuming a suitable cost structure on the inventory system, we have presented numerical results to obtain the optimum values of  $s$  and  $S$  that minimize the total expected cost function. The method of analysis used in this paper can be applied to a perishable inventory model with service interruptions. Further, the present paper can be extended by considering the impatient customers. One can also implement these concepts with *MAP* arrivals and *MSP*. These topics are left for future investigation.

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