International Journal of
Mathematical Modelling & Computations
Vol. 07, No. 03, Summer 2017, 175- 183

Determination of a Source Term in an Inverse Heat Conduction Problem by Radial Basis Functions

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Abstract. In this paper, we propose a technique for determining a source term in an inverse heat conduction problem (IHCP) using Radial Basis Functions (RBFs). Because of being very suitable instruments, the RBFs have been applied for solving Partial Differential Equations (PDEs) by some researchers. In the current study, a stable meshless method will be proposed for solving an (IHCP). The other advantage of the method is that can be applied to the problems with various types of boundary conditions. The results of numerical experiments are presented and compared with analytical solutions. The results demonstrate the reliability and efficiency of the proposed scheme.

Received: 11 January 2017, Revised: 20 August 2017, Accepted: 25 September 2017.

Keywords: Direct and inverse heat conduction problem; heat equation; Radial basis functions.

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1. Introduction

Inverse problems occur in many branches of engineering natural sciences. Such problems are much more difficult to solve than direct ones, because inverse problems are usually ill-posed [12]. On the other hand, methods of solving these kinds of problems are very important for a wide range of problems that cannot be analyzed by simple methods, directlly.
One of the most important inverse problems, appearing in some engineering science is an inverse heat conduction problem (IHCP) that the various kinds of them have been solved by some methods [1–3]. The inverse problem of determining a heat

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source in the heat conduction equation has been studied by researchers as Cannon, DuChateau, Fatullayev and etc [5, 7, 9, 10]. Several numerical methods have been proposed for solving the inverse source problem [4, 6, 8, 11]. We note that, even though these methods are very effective for solving various type of these problems, conditional stability of explicit finite difference procedures and the need to use large amount of CPU time in implicit finite difference schemes limit the applicability of these methods. Also, these methods provide the solution of the problem on the mesh points only and the accuracy of these schemes may be decrese in nonsmooth and nonregular domains. So, to avoid the mesh generation, in the recent years meshless methods have been applied for solving a large numbers of problems by some reseachers. The main advantage of these methods is that mesh generation is not needed. In the circumstances, RBFs method is known as a powerful tool of meshless methods which can be used for solving various kinds of problems such as PDEs [13–16, 22–24].
In the present work we are dealing with the appoximated solution of the following second-order parabolic problem:

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = F(t) \tag{1}$$

namely, the heat source depends on the time variable only. Where α is a known constant coefficient. The layout of the article is as follows. In Section 2, the global radial basis functions are introduced. In Section 3, we will show that how we use the RBFs for solving an IHCP. The results of numerical examples are presented in Section 4. Section 5 is dedicated to a brief conclusion.

2. Global Radial Basis Functions

In this section, we cosider RBF method for interpolation of scattered data. Suppose that x^* and x be a fixed point and an arbitrary point in R^d, respectively. A radial function ϕ^* is defined via : $\phi^* = \phi^*(r)$ where : $r = \| \mathbf{x} - \mathbf{x}^* \|_2$ .i.e. the radial function ϕ^* depends only on the distance between $\mathbf{x}$ and $\mathbf{x}^*$. This property implies that the RBFs ϕ^* are radially symmetric about $\mathbf{x}^*$. Some well-known RBFs are given in the table below:

Infinitely smooth RBFs	$\phi(r)$
Gaussian(GA)	$\exp(-cr^2)$
Multiquadrics (MQ)	$\sqrt{r^2+c^2}$
Inverse multiquadrics (IMQ)	$(\sqrt{r^2+c^2})^{-1}$
Inverse quadric (IQ)	$\phi(r) = (r^2+c^2)^{-1}$

Let $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\}$ be a given set of distinct points in domain Ω in R^d. The main idea of using the RBFs is interpolation with a linear combination of RBFs of the same types as follows :

$$\Phi(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k \phi_k(\mathbf{x}) \tag{2}$$

where $\phi_k(\mathbf{x}) = \phi(\| \mathbf{x} - \mathbf{x}_k \|)$ and λ_k's are unknown coefficients for $k = 1, 2, ..., N$. Assume that we want to interpolate the given values $f_k = f(\mathbf{x}_k), k = 1, 2, ..., N$.

The unknown coefficients λ_k's are obtained, so that $\Phi(\mathbf{x}_k) = f_k, k = 1, 2, ..., N$, which results in the following linear system of equations :

$$A\Lambda = b \tag{3}$$

where $\Lambda = [\lambda_1, ..., \lambda_N]^T, b = [f_1, ..., f_N]^T$ and $A = a_{ij}$ for $i, j = 1, 2, ..., N$ with entries $a_{ij} = \phi_j(\mathbf{x}_i)$. Generally, the matrix A has been shown to be positive definite (and therfore nonsigular) for distinct interpolation points for GA, IMQ and IQ by Schoenberg's Theorem [20]. Also, using the Micchelli's Theorem [21], we can show that A is a invertible for distinct set of scattered nodes in the case of MQ.

3. Method of solution of an IHCP

3.1 *An inverse problem*

In this section,first, we introducing the mathematical model of an inverse heat conduction problem. Let us consider the problem of finding functions $u = u(x, t)$ and $F = F(t)$ satisfying in one-dimensional heat conduction equation :

$$Lu(x, t) = F(t), \qquad (x, t) \in (0, l) \times (0, T) \tag{4}$$

where $T > 0$, $(0, l) \subset R$ is an open spatial interval and L is a second-order linear parabolic operator as follows :

$$Ł := (\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2}) \tag{5}$$

where $\alpha \in R$, with the following initial and boundary conditions :

$$u(x, 0) = f(x), \qquad x \in [0, l] \tag{6}$$

$$\alpha_1 u(0, t) - \beta_1 u_x(0, t) = p(t), \qquad t \in (0, T] \tag{7}$$

$$\alpha_2 u(l, t) + \beta_2 u_x(l, t) = q(t), \qquad t \in (0, T] \tag{8}$$

where α_i's, β_i's for $i = 1, 2$ are constant coiefficients and the functionsf, p and q are regarded as being known in their domains. To solve the above problem, we consider the extra condition as follows :

$$\int_0^l u(x, t)dx = E(t), \qquad t \in (0, T] \tag{9}$$

where $E(t)$ is a known function in its domain. Existence, uniqeness and stability of a direct problem of the same type, with the various kinds of boundary conditions have been verified [1].

3.2 *Analysis of the method*

Now, we show that how we use the method of RBFs to approximate the solution of the above problem. Since an one-dimensional heat conduction equation depends on both x and t, we consider N scattered nodes in the domain $\Omega = [0, l] \times [0, T]$. So, let :

$$\Xi = \{\mathbf{x}_k = (x_k, t_k)\}_{k=1}^{N} \subset \Omega \tag{10}$$

be a given set of distinct points in domain Ω. Also, we consider two subsets of Ξ as follows :

$$\Xi = \Xi_i \cup \Xi_b \tag{11}$$

$$\Xi_i = \{\mathbf{x}_k\}_{k=1}^{N_i} \tag{12}$$

$$\Xi_b = \{\mathbf{x}_k\}_{k=1}^{N_b} \tag{13}$$

where $N = N_i + N_b$, Ξ_i and Ξ_b are the interior and the boundary points in domain Ω, respectively. Using the radial basis collocation method, the solution of the problem in Ω is looked for as :

$$\widetilde{u}(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k \phi_k(\mathbf{x}) \tag{14}$$

and

$$\widetilde{F}(t) = \sum_{j=1}^{N_i} \mu_j \psi_j(t) \tag{15}$$

where $\phi_k = \phi(\| \mathbf{x} - \mathbf{x}_k \|)$ and $\psi_j = \psi(| t - t_k |)$ for given RBFs ϕ, ψ and N_i is the number of the interior points in domain Ω. To determine the coeifficients in (14), (15) we impose the approximate solutions $\widetilde{u}$ and $\widetilde{F}$ to satisfy the given partial differential equation with the other conditions at any $\mathbf{x} \in \Xi$.

Remark 1
We assume that there are $N_b < N$ boundary points, $N - N_b$ interior points and Ξ_1, Ξ_2 and Ξ_3 are three subsets of the Ξ_i as follows:

$$\Xi_1 = [0, l] \times \{0\} \tag{16}$$

$$\Xi_2 = \{0\} \times (0, T] \tag{17}$$

$$\Xi_3 = \{l\} \times (0, T] \tag{18}$$

Let ℓ be a boundary operator and ℓu is known for $\mathbf{x} \in \Xi_b$.
So we have :

$$L\widetilde{u}(\mathbf{x}) - \widetilde{F}(t) = 0, \qquad \mathbf{x} \in \Xi_i, \qquad t \in (0, T) \tag{19}$$

$$\ell\widetilde{u}(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Xi_1 \tag{20}$$

$$\ell\widetilde{u}(\mathbf{x}) = p(t), \qquad \mathbf{x} \in \Xi_2 \tag{21}$$

$$\ell\widetilde{u}(\mathbf{x}) = q(t), \qquad \mathbf{x} \in \Xi_3 \tag{22}$$

and

$$\int_0^l \widetilde{u}(x,t)dx = E(t), \qquad t \in (0,T] \tag{23}$$

which results the following linear system of equations :

$$A\Lambda = b \tag{24}$$

where $\Lambda = [\lambda_1, ..., \lambda_N, \mu_1, ..., \mu_{N_1}]^T$, $b = [0,\ f,\ p,\ q,\ E]^T$ and 0 is the zero matrix. Thus the $((N+N_1)\times(N+N_1))$ matrix A can be subdivided into five submatrices as follows:

$$A = \begin{bmatrix} A_1 \\ --- \\ A_2 \\ --- \\ A_3 \\ --- \\ A_4 \end{bmatrix} \tag{25}$$

where

$$A_1 = [L\widetilde{u}(\mathbf{x}) - \widetilde{F}(t)], \quad \mathbf{x} \in \Xi_i, \quad t \in (0,T) \tag{26}$$

$$A_2 = [\ell\widetilde{u}(\mathbf{x})], \qquad \mathbf{x} \in \Xi_1 \tag{27}$$

$$A_3 = [\ell\widetilde{u}(\mathbf{x})], \qquad \mathbf{x} \in \Xi_{2,3} \tag{28}$$

$$A_4 = [\int_0^l \widetilde{u}(x,t)dx], \qquad t \in\ (0,T] \tag{29}$$

By solving this linear system of equations, the unknown values $\lambda_k, k = 1, ..., N$ and $\mu_j, j = 1, ..., N_1$ will be found.

Remark 2
If T be sufficiently small value, we can apply this algorithm to determine the approximate solutions of the problem in the upper levels via replacing Ω to $\Omega_k = [0,l]\times[(k-1)T, kT]$ for $k = 1, 2, 3, ...$

Remark 3
Although, the matrix A is nonsingular in the following examples, usually it is very ill-conditioned, i.e. the condition number of A as $K_i(A) = \| A \|_i \| A^{-1} \|_i, i = 1, 2, ...$

is a very large number for a fixed number of interpolation points. The condition number of A depend on the shape parameter c, on the other hand, it grows with N for fixed values of the shape parameter c. In practice, the shape parameter must be adjusted with the number of interpolating points. Despite various reseach works which are done, the optimal choice of its is still an open problem [17–19].

4. Numerical results

In this section we present some numerical results to test the efficiency and accuracy of the scheme for solving the above inverse heat conduction problem.

Example 4.1
In this example, we consider the problem (4−9) with $\alpha = 2/\pi^2$ and $\Omega = [0,1]\times[0,2]$. Practicing the exact solutions which can be resulte as: $u(x,t) = \exp(-2t)$ $(\cos(\pi x)+1)$, $F(t) = 2\exp(-2t)$ initial, the Dirichlet boundary conditions ($\alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 0$) and the extra condition can be achived.
Using the obtained scheme in the previous section and considering $c = 1$ as the shape parameter in GA, IQ and IMQ-RBFs, we will find $u(x,t)$ and $F(t)$ numerically. The values of the RMS errors and the maximum absolute errors for GA, IQ and IMQ-RBF method are introduced in the below tables. In Table 1 we give the errors for $\mathbf{x} \in \Omega_1 = [0,1] \times [0, 0.5]$ for GA, IQ and IMQ-RBF method. Due to closeness of accuracy of GA, IQ and IMQ-RBF, in Table 2 only the results of GA-RBF will be shown on the other levels. The plots of the error functions for $\mathbf{x} \in \Omega_1$ and Ω_4 are given in Figure 1 and Figure 4.
The maximum absolute errors: $E_1^\infty = \max|u(\mathbf{x}) - \widetilde{u}(\mathbf{x}_k)|$, $E_2^\infty = \max|F(t) - \widetilde{F}(t_j)|$ are obtained.

Table 1. The RMS and absolute errors GA, IQ and IMQ-RBFs for N=100, $\mathbf{x} \in \Omega_1$.

error	GA	IQ	IMQ
RMS error of u	5.5163e-005	5.4514e-005	4.7659e-005
RMS error of F	4.0323e-005	1.8952e-004	1.3555e-004
E_1^∞	5.6379e-004	5.2078e-004	4.3279e-004
E_2^∞	9.9058e-005	5.7306e-004	3.1860e-004

Table 2. The RMS and the maximume absolute errors GA-RBF for $\mathbf{x} \in \Omega_k = [0,1] \times [(k-1)\times 0.5, k \times 0.5], k = 2,3,4$.

Ω_k	RMS error of u	RMS error of F	E_1^∞	E_2^∞
Ω_2	2.3333e-005	5.7405e-004	5.7405e-004	1.7795e-003
Ω_3	1.2625e-005	1.4476e-004	1.3727e-004	4.4654e-004
Ω_4	1.1857e-005	1.9755e-004	1.4207e-004	5.3676e-004

Example 4.2
As another example, let us consider the problem $(4-9)$ as follows : $\alpha = 1/\pi^2$, $\Omega = [0,1] \times [0,2]$ and $\alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 1$. By these assumptions, the analytical solutions are given as: $u(x,t) = \exp(-t)(\sin(\pi x) - t)$, $F(t) = \exp(-t)(1 - t)$. Applying the mentioned method, we will find the estimated solutions. Similar to previous example, in Tables 3 and 4 we give the RMS errors and the maximume absolute errors for GA, IQ and IMQ-RBF method for $\mathbf{x} \in \Omega_1$ and $\mathbf{x} \in \Omega_k, k =$

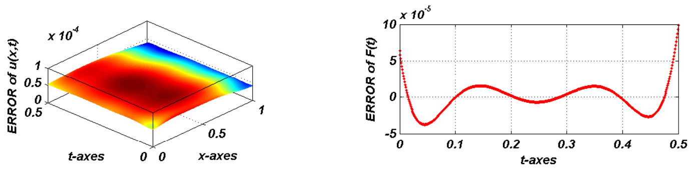

Figure 1. The errors of the approximated solutions for $\mathbf{x} \in \Omega_1$.

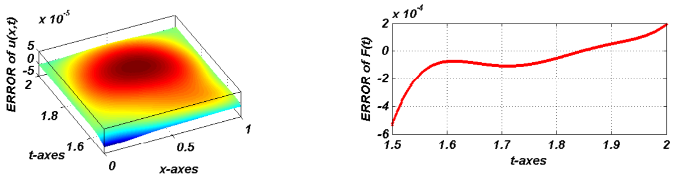

Figure 2. The errors of the approximated solutions for $\mathbf{x} \in \Omega_4$.

$2, 3, 4$, respectively. Also, the plots of the estimated solutions on the levels Ω_1 and Ω_4 are given in Figures 3 and 4.

Table 3. The RMS and absolute errors GA, IQ and IMQ-RBFs for N=100, $\mathbf{x} \in \Omega_1$.

error	GA	IQ	IMQ
RMS error of u	1.4110e-004	1.9003e-004	1.3194e-004
RMS error of F	4.2432e-005	1.4615e-004	5.3797e-004
E_1^∞	1.4220e-003	1.9561e-003	1.5455e-003
E_2^∞	9.8357e-005	3.9446e-004	8.7500e-004

Table 4. The RMS and the maximume absolute errors GA-RBF for $\mathbf{x} \in \Omega_k = [0, 1] \times [(k-1) \times 0.5, k \times 0.5], k = 2, 3, 4$.

Ω_k	RMS error of u	RMS error of F	E_1^∞	E_2^∞
Ω_2	6.9110e-005	1.4483e-003	1.7629e-003	4.4924e-003
Ω_3	4.8475e-005	3.7855e-004	7.6842e-004	1.1736e-003
Ω_4	3.7282e-005	4.8028e-004	7.1946e-004	1.4907e-003

Example 4.3
In this example, we apply the new method to find the solution of the problem $(4-9)$ with $\alpha = 8/\pi^2$ and $\Omega = [0, 1] \times [0, 2]$. With the known initial, Neumann boundary conditions $(\alpha_1 = 0, \alpha_2 = 0, \beta_1 = -1, \beta_2 = 1,)$ and extra condition, we will find the exact solutions of this problem as:
$u(x, t) = \exp(-2t)(\sin((\pi x/2)) + \cos(t)),\ F(t) = \exp(-2t)(2\cos(t) + \sin(t))$.
The values of RMS errors and the maximum absolute errors for GA, IQ and IMQ-RBFs for $\mathbf{x} \in \Omega_1 = [0, 1] \times [0, 0.5]$ and $\mathbf{x} \in \Omega_k, k = 2, 3, 4$ are introduced in Tables 5 and 6, as well. For GA-RBFs, the plots of the estimated solutions on the levels Ω_1 and Ω_4 are given in Figure 5 and 6, respectively.

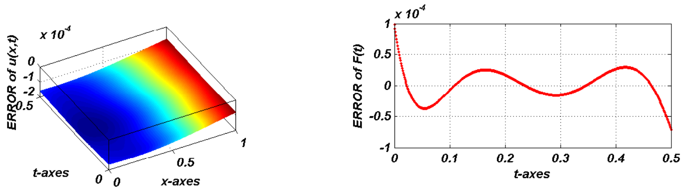

Figure 3. The errors of the approximated solutions for $\mathbf{x} \in \Omega_1$.

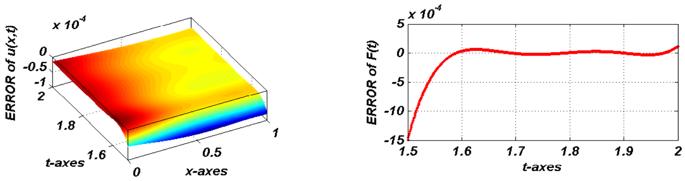

Figure 4. The errors of the approximated solutions for $\mathbf{x} \in \Omega_4$.

Table 5. The RMS and absolute errors GA, IQ and IMQ-RBFs for N=100, $\mathbf{x} \in \Omega_1$.

error	GA	IQ	IMQ
RMS error of u	9.7797e-006	2.2820e-004	2.7616e-004
RMS error of F	1.7974e-005	3.7984e-003	3.4079e-003
E_1^∞	1.0247e-004	2.0437e-003	2.3844e-003
E_2^∞	4.7279e-005	4.3207e-003	3.9366e-003

Table 6. The RMS and the maximume absolute errors GA-RBF for $\mathbf{x} \in \Omega_k = [0,1] \times [(k-1) \times 0.5, k \times 0.5], k = 2, 3, 4$.

Ω_k	RMS error of u	RMS error of F	E_1^∞	E_2^∞
Ω_2	3.9047e-006	1.0436e-004	1.0206e-004	3.2842e-004
Ω_3	4.1055e-006	3.8108e-005	6.7689e-005	1.1961e-004
Ω_4	1.2688e-006	3.7304e-005	3.5714e-005	1.1673e-004

5. Conclusion

In this work, we discussed on an inverse heat conduction problem. we proposed a numerical scheme to solve this problem using collocation points and approximating directly the solutions using the radial basis functions. The numerical result given in the previous section demonstrate the good accuracy of the scheme proposed in this study. Clearly, the method proposed in this work can be extended to solve the varios kinds of the inverse heat conduction problems.

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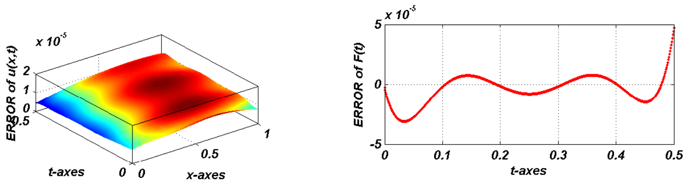

Figure 5. The errors of the approximated solutions for $\mathbf{x} \in \Omega_1$.

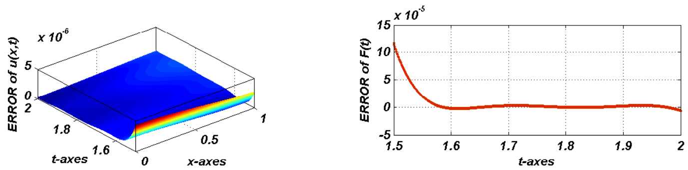

Figure 6. The errors of the approximated solutions for $\mathbf{x} \in \Omega_4$.

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