International Journal of
Mathematical Modelling & Computations
Vol. 07, No. 03, Summer 2017, 231- 237

A Note on the Convergence of Homotopy Analysis Method for Nonlinear Age-Structured Population Models

S. Naghshband*

Department of Mathematics, West Tehran Branch, Islamic Azad University, Tehran, Iran.

Abstract. In this paper, a theorem is proved which presents the series solution obtained from the homotopy analysis method is convergent to the exact solution of nonlinear age-structured population models.

Received: 19 December 2017, Revised: 10 November 2018, Accepted: 12 November 2018.

Keywords: Nonlinear age-structured population models; Homotopy analysis method (HAM); Convergency.

AMS Subject Classification: 35C10, 65M99.

Index to information contained in this paper

1. Introduction

This paper deals with the convergence of the HAM for the following nonlinear age-structured population models [10]

$$\frac{\partial p(x,t)}{\partial t} + \frac{\partial p(x,t)}{\partial x} = -(d_1(x) + d_2(x)P(t))p(x,t), \quad t \geqslant 0, \ 0 \leqslant x < A,$$

$$p(x,0) = p_0(x), \qquad 0 \leqslant x < A, \tag{1}$$

*Corresponding author. Email:s_ naghshband@yahoo.com; naghshband.s@wtiau.ac.ir

$$p(0,t) = \int_a^A (b_1(\xi) - b_2(\xi)P(t))p(\xi,t)d\xi, \qquad t \geqslant 0,$$

$$P(t) = \int_0^A p(x,t)dx, \qquad t \geqslant 0$$

where $A \to +\infty$ and x, t, respectively, denote age and time, $P(t)$ denotes the total population number at time t, $p(x,t)$ is the age-specific density of individuals of age x at time t, i.e.$\int_a^{a+\Delta a} p(x,t)dx$ is the number of individuals that are aged between a and $a+\Delta a$ at time t, $d_1(x)$ is the natural death rate (without considering competition), $d_2(x)$ is the increase of death rate considering competition, $b_1(x)$ is the natural fertility rate (without considering competition), $b_2(x)$ is the decrease of fertility rate considering competition, a denotes the lowest age when an individual can bear, and A is the maximum age that an individual of the population may reach[10]. In [10], Goreishi et al solve the nonlinear age-structured population models numerically by applying the HAM and finally they concluded that the HAM is a valid scheme to solve Eq.(1) as well. This paper tries to prove the convergency of the HAM for nonlinear age-structured population model, during a theorem. Recently, the HAM has been well applied to solve many problems in science and engineering [1–16]. Total discription of this paper is as follows: In section 2, some preliminaries of the HAM and some obtained necessary relations via the HAM of Eq.(1) are given, in section 3 convergence theorem of the HAM is proved and in section 4 a numerical example is considered to certify the convergency of the HAM numerically, as well.

2. Preliminaries

Let the following differential equation:

$$N[w(x,t)] = 0,$$

where N is a nonlinear operator, x and t denote the independent variables and w is an unknown function. Via the HAM, the zeroth-order deformation equations:

$$(1-q)L[\Phi(x,t;q) - w_0(x,t)] = qhH(x,t)N[\Phi(x,t;q)], \tag{2}$$

where $q \in [0,1]$ is the embeding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator and $H(x,t)$ is an auxiliary function. $\Phi(x,t;q)$ is an unknown function and $w_0(x,t)$ is an initial guess of $w(x,t)$. It is clear, if $q = 0$ and $q = 1$ then:

$$\Phi(x,t;0) = w_0(x,t), \quad \Phi(x,t;1) = w(x,t),$$

respectively. Therefore, when q increases from 0 to 1, the solution $\Phi(x,t;q)$ varies from$w_0(x,t)$ to the exact solution $w(x,t)$. By Taylor's theorem, it can be expanded $\Phi(x,t;q)$ in a power series of the embeding parameter q as comes:

$$\Phi(x,t;q) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t)q^m, \tag{3}$$

where

$$w_m(x,t) = \frac{1}{m!}\frac{\partial^m \Phi(x,t;q)}{\partial q^m}|_{q=0}. \tag{4}$$

Let the initial guess $w_0(x,t)$, the auxiliary linear operator L, the nonzero auxiliary parameter h and the auxiliary function $H(x,t)$ be properly chosen so that the power series Eq.(3) converges at $q = 1$, then, it can be seen:

$$w(x,t) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t), \tag{5}$$

which must be the solution of the original nonlinear equation. Now, we define the following set of vectors:

$$\vec{w_n} = \{w_0(x,t), w_1(x,t), \ldots, w_n(x,t)\}. \tag{6}$$

By differentiating the zeroth order deformation Eq.(2) m times with respect to the embeding parameter q and then setting $q = 0$ and finally dividing by $m!$, we will have the following mth order deformation equation:

$$L[w_m(x,t) - \chi_m w_{m-1}(x,t)] = hH(x,t)R_m(\vec{w}_{m-1}), \tag{7}$$

where

$$R_m(\vec{w}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\Phi(x,t;q)]}{\partial q^{m-1}}|_{q=0}, \tag{8}$$

and

$$\chi_m = \begin{cases} 0 \; m \leqslant 1, \\ 1 \; m > 1. \end{cases} \tag{9}$$

It should be mentioned that $w_m(x,t)$ for $m \geqslant 1$ is goverend by the linear Eq.(7) with linear boundary conditions that come from the original problem. For more details about the HAM, we refer the reader to [13]. Now, the HAM is applied to solve Eq.(1) .

We consider Eq.(1) as follows:

$$\frac{\partial p(x,t)}{\partial t} + \frac{\partial p(x,t)}{\partial x} + (d_1(x) + d_2(x)P(t))p(x,t) = 0 \tag{10}$$

and

$$L[\Phi(x,t;q)] = \frac{\partial \Phi(x,t;q)}{\partial t}, \quad L(c) = 0, \tag{11}$$

where c is a real constant,

$$\begin{aligned} N[\Phi(x,t;q)] = \frac{\partial \Phi(x,t;q)}{\partial t} + \frac{\partial \Phi(x,t;q)}{\partial x} + d_1(x)\Phi(x,t;q) \\ + d_2(x)\Phi(x,t;q)\int_0^A \Phi(x,t;q)dx, \end{aligned} \tag{12}$$

and $H(x,t) = 1$. The zeroth-order deformation equation is:

$$(1-q)L[\Phi(x,t;q) - p_0] = qhN[\Phi(x,t;q)]. \tag{13}$$

Also, the mth-order deformation equation:

$$L[p_m - \chi_m p_{m-1}] = hR_m(\vec{p}_{m-1}), \tag{14}$$

where

$$R_m(\vec{p}_{m-1}) = \frac{\partial p_{m-1}(x,t)}{\partial t} + \frac{\partial p_{m-1}(x,t)}{\partial x} + d_1(x)p_{m-1}(x,t) + d_2(x)\sum_{i=0}^{m-1} p_i(t)\int_0^A p_{m-1-i}(x,t)dx. \tag{15}$$

So,

$$p_m = \chi_m p_{m-1} + h\int_0^t R_m(\vec{p}_{m-1})dt + c, \quad m \geqslant 1. \tag{16}$$

3. Convergence theorem of the HAM

In this section , we prove the convergence of the series solution obtained from the HAM to the exact solution of the Eq.(1) or Eq.(10).

THEOREM 3.1 *If the series solution*

$$p(x,t) = p_0(x,t) + p_1(x,t) + \dots,$$

obtained from the HAM and the series $\sum_{m=0}^{\infty} \frac{\partial p_m(x,t)}{\partial t}$, $\sum_{m=0}^{\infty} \frac{\partial p_m(x,t)}{\partial x}$, $\sum_{m=0}^{\infty} \int_0^A p_m(x,t)dx$ *are convergent, then* $\sum_{m=0}^{\infty} p_m(x,t)$ *converges to the exact solution of the Eq.(1).*

Proof .We consider

$$p(x,t) = \sum_{m=0}^{\infty} p_m(x,t)$$

then, in this case, we will have,

$$\lim_{m\to\infty} p_m(x,t) = 0 \tag{17}$$

so

$$\sum_{m=1}^{n} [p_m(x,t) - \chi_m p_{m-1}(x,t)] = p_n(x,t) \tag{18}$$

then we can write :

$$\sum_{m=1}^{\infty} L[p_m(x,t) - \chi_m p_{m-1}(x,t)] = L(\sum_{m=1}^{\infty}(p_m(x,t) - \chi_m p_{m-1}(x,t)) = 0. \quad (19)$$

It can be written

$$\sum_{m=1}^{\infty} L[p_m(x,t) - \chi_m p_{m-1}] = hH(x,t)\sum_{m=1}^{\infty} R_m(p_{m-1}). \quad (20)$$

Moreover, we know $h, H(x,t) \neq 0$ then

$$\sum_{m=1}^{\infty}[R_m(p_{m-1})] = 0 \quad (21)$$

According to the Eq.(15), it can be seen:

$$\begin{aligned}
&\textstyle\sum_{m=1}^{\infty}[R_m(p_{m-1})] \\
&= \sum_{m=1}^{\infty}\frac{\partial p_{m-1}(x,t)}{\partial t} + \sum_{m=1}^{\infty}\frac{\partial p_{m-1}(x,t)}{\partial x} + d_1(x)\sum_{m=1}^{\infty} p_{m-1}(x,t) \\
&\qquad + d_2(x)\sum_{m=1}^{\infty}\sum_{i=0}^{m-1} p_i(t)\int_0^A p_{m-1-i}(x,t)dx \\
&= \sum_{m=0}^{\infty}\frac{\partial p_m(x,t)}{\partial t} + \sum_{m=0}^{\infty}\frac{\partial p_m(x,t)}{\partial x} + d_1(x)\sum_{m=0}^{\infty} p_m(x,t) \\
&\qquad + d_2(x)\sum_{i=0}^{\infty}\sum_{m=i+1}^{\infty} p_i(t)\int_0^A p_{m-1-i}(x,t)dx \\
&= \sum_{m=0}^{\infty}\frac{\partial p_m(x,t)}{\partial t} + \sum_{m=0}^{\infty}\frac{\partial p_m(x,t)}{\partial x} + d_1(x)\sum_{m=0}^{\infty} p_m(x,t) \\
&\qquad + d_2(x)\sum_{i=0}^{\infty} p_i(t)\sum_{m=i+1}^{\infty}\int_0^A p_{m-1-i}(x,t)dx \\
&= \sum_{m=0}^{\infty}\frac{\partial p_m(x,t)}{\partial t} + \sum_{m=0}^{\infty}\frac{\partial p_m(x,t)}{\partial x} + d_1(x)\sum_{m=0}^{\infty} p_m(x,t) \\
&\qquad + d_2(x)\sum_{i=0}^{\infty} p_i(t)\sum_{m=0}^{\infty}\int_0^A p_m(x,t)dx \\
&= \frac{\partial \sum_{m=0}^{\infty} p_m(x,t)}{\partial t} + \frac{\partial \sum_{m=0}^{\infty} p_m(x,t)}{\partial x} + d_1(x)\sum_{m=0}^{\infty} p_m(x,t) \\
&\qquad + d_2(x)\sum_{i=0}^{\infty} p_i(t)\int_0^A \sum_{m=0}^{\infty} p_m(x,t)dx.
\end{aligned} \quad (22)$$

So, by considering Eqs.(15) and (22), it can be seen easily that $\sum_{m=0}^{\infty} p_m(x,t)$ is the exact solution of Eq.(1) or Eq.(10).

■

4. Numerical example

With regard to the main aim of this paper which is to prove the convergence theorem, this example is given, only to confirm the convergency of the proposed method numerically.

Example 4.1

Consider the following nonlinear age structured equation[10]:

$$\frac{\partial p(x,t)}{\partial t} + \frac{\partial p(x,t)}{\partial x} = -(1+P(t))p(x,t), \quad t \geqslant 0, \ 0 \leqslant x < A,$$

$$p(x,0) = \frac{e^{-x}}{2}, \qquad 0 \leqslant x < A, \tag{23}$$

$$p(0,t) = p(t), \qquad t \geqslant 0,$$

$$P(t) = \int_0^A p(x,t)dx, \qquad t \geqslant 0$$

where $A \to +\infty$. Also the exact solution of this equation is $\frac{e^{-x}}{2+t}$.

The HAM is applied, by using Eq.(16) then we get:

$$p_0(x,t) = \frac{e^{-x}}{2},$$

$$p_1(x,t) = \frac{hte^{-x}}{4},$$

$$p_2(x,t) = (t(t+2)h^2)e^{-x}/8 + hte^{-x}/4], \ldots$$

Table 1 shows the errors of the HAM, when h=-1, x=15 and t=0.1. Also error is calculated by $|\sum_{i=0}^{m} p_i - p|$. Figure. 1 shows the approximate and the exact solution of example 4.1 when m =4 ,h =-1, x ∈[0,20] and t ∈ [0,1].

Table 1. The errors of the HAM at x=15 and t=0.1.

n	Error
2	1.8208e-011
4	4.5521e-014
6	1.1380e-016
8	2.8442e-019
12	3.9705e-023

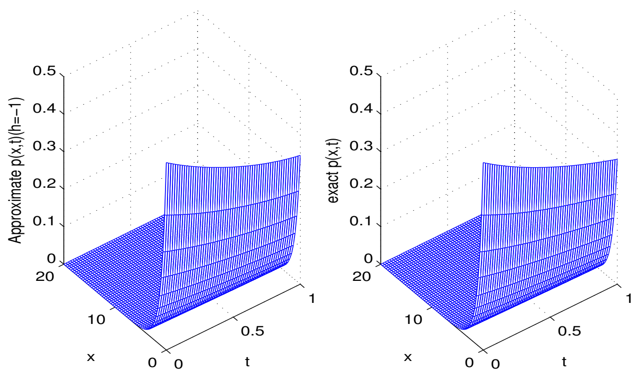

Figure 1. The approximate solution (m=4) and the exact solution of example 4.1 when $x \in$ [0 20] and $t \in$ [0,1].

5. Conclusion

This paper, regarding [10] has attempted to prove the convergency of the homotopy analysis method during a theorem for the mentioned equation. An example is given to verify the convergency of the HAM numerically. Consequently, the HAM can be applied to solve the nonlinear age-structured population models as an effective and valid scheme.

References

[1] S. Abbasbandy, The application of the homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A, **360 (1)** (2006) 109-13.

[2] S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation, Phys. Lett. A, **361 (6)** (2007) 478-83.

[3] S. Abbasbandy, Soliton solutions for the Fitzhugh–Nagumo equation with the homotopy analysis method, Applied Mathematical Modelling, **32** (2008) 2706-2714.

[4] S. Abbasbandy, E. Babolian and M. Ashtiani, Numerical solution of the generalized Zakharov equation by homotopy analysis method, Commun. Nonlinear Sci. Numer. Simulat, **14** (2009) 4114-4121.

[5] T. Allahviranloo and Sh. S. Behzadi, Application of Iterative Methods for Solving General Riccati Equation, Int. J. Industrial Mathematics, **4 (4)** (2012) 389-404.

[6] A. Alizadeh-Pahlavan, V. Aliakbar, F. Vakili-Farahani and K. Sadeghy, MHD flows of UCM fluids above porous stretching sheets using two-auxiliary-parameter homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul, **14 (2)** (2009) 473-88.

[7] A. K. Alomari, M. S. M. Noorani and R. Nazar, Explicit series solution of some linear and non linear Schridinger equations via the homotopy analysis method, Comunications in nonlinear science and Numerical Simulation, **14** (2009) 1196-1207.

[8] M. A. Fariborzi Araghi and A. Fallahzadeh, On the convergence of the homotopy analysis method for solving the Schrodinger equation J. Basic Appl. Sci. Res, **2 (6)** (2012) 6076-83.

[9] M. A. Fariborzi Araghi and S. Naghshband, On convergence of homotopy analysis method to solve the Schrodinger equation with a power law nonlinearity, Int. J. Industrial Mathematics, **5 (4)** (2013) 367-374.

[10] M. Ghoreishi, A. I. B. Md. Ismail, A. K. Alomari and A. Sami Bataineh, The comparison between Homotopy Analysis Method and Optimal Homotopy Asymptotic Method for nonlinear age-structured population models, Commun. Nonlinear Sci. Numer. Simulat, **17** (2012) 1163-1177.

[11] A. Jafarian, P. Ghaderi, AK. Golmankhaneh and D. Baleanu, Homotopy analysis method for solving coupled Ramani equations, Rom. J. Phys, **59 (1–2)** (2014) 26-35.

[12] A. Jafarian, P. Ghaderi, AK. Golmankhaneh and D. Baleanu, Analytical treatment of system of Abel integral equations by homotopy analysis method, Rom. Rep. Phys, **66 (3)** (2014) 603-11.

[13] S. J. Liao, Beyond perturbation:Introduction to the homotopy analysis method, Chapman and Hall/CRC Press, Boca Raton, (2003).

[14] S. J. Liao, Notes on the homotopy analysis method:some definitions and theorems, Comunications in nonlinear science and Numerical Simulation, **14** (2009) 983-997.

[15] S. Naghshband and M.A. Fariborzi Araghi, Solving generalized quintic complex Ginzburg–Landau equation by homotopy analysis method, Ain Shams Engineering Journal, doi:10.1016/j.asej.2016.01.015 (2016).

[16] S. Naghshband and M. A. Fariborzi Araghi, Approximation solution of the complex modified Korteweg-de Vries equation via the homotopy analysis method, Communications on Advanced Computational Science with Applications, **1** (2017) 12-23.