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Analysis of a Single Server Queue with Working Vacation and Vacation Interruption

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Abstract. In this paper, an M/M/1 queue with working vacation and vacation interruption is investigated. The server is supposed to take a working vacation whenever the system becomes empty and if there are at least N customers waiting in the system at a service completion instant, vacation interruption happens and the server resumes a normal working period. A matrix geometric approach is employed to obtain the stationary distribution for mean queue length. Moreover, we have derived the distributions for the mean queue length and the mean waiting time and obtained their stochastic decomposition structures if N=2. Finally, numerical examples are presented.

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1. Introduction

During the last three decades, researchers have extensively analyzed the vacation queuing models and successfully utilized in numerous applied problems. In the classical vacation queuing models, the server completely ceases service during a vacation and such a policy may lead to the dissatisfaction of the customers and ultimately to the loss of costumer base. However, there are many situations where the server does not remain completely inactive during a vacation. But provides service to the queue at a lower rate. This idea was first utilized by Servi and Finn [9]. Servi and Finn [9] introduced a class of semi vacation polices, where the server does not completely stops working during a vacation, but it will render service at a lower rate to the queuing system. This type of vacation is called a working vacation (WV). Servi and Finn [9] analyzed an M/M/1 queue with multiple working vacation policy and derived the probability generating function for the number of customers in the system and LST for waiting time distributions and utilized the results to analyze the system performance of gateway router in fiber communication networks. Subsequently, working vacation queues have received considerable attention in literature, such as Baba [1], Wu and Takagi [10], Liu et al. [7].

Generally, in working vacation policy, the server starts again his work at regular service rate after the end of vacation, only if the customers are waiting at the system. Definitely such speculations appears much more limited in real world situations. To come out of this restriction, Li and Tain [5] introduced the vacation interruption schedule in an M/M/1 queue

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with working vacations.

In this vacation policy, a server at the completion of vacation instantly ends his vacation and comes back to its normal working level if customers are waiting in the queue, instead he will continue his vacation till the system is non empty after vacation ends. Due to its strong application in the stochastic service models, it gives productive theoretical results in this area. Li et al. [6], Baba [2], Zhang and Hou [11], Gao and Liu [3], and Lee and Kim [4] are those who gave eminent papers in this area.

Although, in Li and Tian [5] service discipline, only the first arrival during a vacation period gets lower service rate. When the server switches from the lower service to the normal service rate during his vacation, the switching cost is incurred. The system has to face more additional cost if the service is mostly interrupted. Therefore, in practical application, the vacation interruption policy introduced by Li and Tian [5] has some limitations. In this paper modified vacation interruption policy is presented to reduce the switching cost of the system. In the modified vacation interruption policy the server at the completion of the vacation ends his vacation and restarts his work at normal rate only when at least N customers are queued in the system, else the server continues his vacation until at least the N customers are at the system.

The rest of the paper is structured as follows. In section 2, we discuss the model as a quasi birth and death process and obtain the steady state distribution of the queue length. Section 3 describes the stochastic decomposition structures of the number of customers in the system and waiting time for N = 2. In section 4, numerical illustrations are presented.

2. Model description

We consider a multiple working vacations policy in an *M/M/*1 queuing model under vacation interruption, where the server provides service to the customers at a reduced rate rather than stopping the service completely during his vacation period. The customers arrive according to a Poisson process with parameter λ. The server serves the customers at an exponential rate μ during a regular busy period and service discipline is first come first served (FCFS). The server begins a working vacation as soon as the system becomes unoccupied. The arriving customers during working vacation period are served at a rate lower than the regular service rate. The service times during the working vacation and vacation times are also assumed to be exponentially distributed with rates η and θ, respectively. The server is supposed to interrupt the vacation and returns back to the normal busy period, if there are at least N customers waiting in the system at a service completion instant during a working vacation period, otherwise, the sever will carry on with the vacation. Furthermore, if the server does not find any customer in queue after completing a working vacation he will take another working vacation, else he will resume his regular busy period instantly. The inter-arrival times, the service times, and the working vacation times all are taken to be mutually independent.

Let Q(t) be the number of customers in the system at time t and J(t) be the status of the server, which is defined as follows:

$$J(t)=\begin{cases}0, & \text{When the system stays in a WV period at time t}\\ 1, & \text{When the system stays in non vacation period at time t}\end{cases}$$

then stochastic process $\{(Q(t),J(t)),t\geq 0\}$ is a quasi birth-and-death (QBD) process with the state space

$$S=\{(0,0)\}\bigcup\{(k,j),k\geq 1,j=0,1\}$$

where state $(k,0)$ represents that the system remains in a WV period and there are $k(k\geq 0)$ customers in the system; state $(k,1)$ represents that the system remains in a normal working level and there are $k(k \geq 1)$ customers in the system.

Using the lexicographical order for the states, the infinitesimal generator for the QBD process is

$$Q = \begin{pmatrix} A_{00} & A_{01} & & & & & & & \\ B_{10} & A & C & & & & & & \\ & B_2 & A & C & & & & & \\ & B & A & C & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & B_N & A & C & & & \\ & & & & B & A & C & & \\ & & & & B & A & C & & \\ & & & & & \ddots & \ddots & \ddots & \end{pmatrix} \tag{1}$$

where

$$A_{00} = -\lambda, \quad A_{01} = (\lambda, 0), \quad B_{10} = (\eta, \mu)^T,$$

$$A = \begin{pmatrix} -(\lambda+\eta+\theta) & \theta \\ 0 & -(\lambda+\mu) \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \eta \\ 0 & \mu \end{pmatrix}, \quad B_i = \begin{pmatrix} \eta & 0 \\ 0 & \mu \end{pmatrix}, \quad \mathrm{i} = 2, 3, \ldots$$

Lemma 2.1 If the system workload, the following matrix quadratic equation

$$R^2 B + RA + C = 0$$

has a minimal non-negative solution, which is denoted by R and is called rate matrix.

Obviously, we obtain

$$R = \begin{pmatrix} r & \dfrac{r(\lambda+\theta)}{\mu} \\ 0 & \rho \end{pmatrix} \tag{2}$$

where

$$r = \frac{\lambda}{\lambda+\theta+\eta}, \quad 0 < r < 1 \tag{3}$$

It is well know that the QBD process $(Q(t), J(t)),\ t \geq 0$ is ergodic iff the spectral radius SP(R) of the rate matrix R satisfies $\mathrm{SP}(\mathrm{R}) < 1$ and the linear system of equations $xB[R] = 0$ have positive solution, where x is $2N+1$-dimensional row vector, $B[R]$ is $2N+1$-order matrix and

$$B[R] = \begin{pmatrix} A_{00} & A_{01} & & & & & \\ B_{10} & A & C & & & & \\ & B_2 & A & C & & & \\ & & B & A & C & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & B_N & A & C \\ & & & & & B & RB + A \end{pmatrix}$$

According to the expression of R in Lemma 2.1 and Theorem 3.1.1 in Neuts [8], it is easy to verify that the QBD process is positive recurrent if and only if the system workload $\rho<1$.

2.1 *Stationary distribution of queue length*

If $\rho<1$, let (Q,J) be the stationary limit of the QBD process $\{(Q(t),J(t)),t\geq 0\}$ and define

$$\Pi_k=\left(\pi_{k0},\pi_{k1}\right),\quad k\geq 1$$
$$x=\left(\pi_{00},\Pi_1,\Pi_2,\ldots,\Pi_N\right),$$
$$\pi_{kj}=P\{Q=k,J=j\}=\lim_{t\to\infty}\{(Q(t)=k,J(t)=j\},\quad (k,j)\in S.$$

In order to drive the stationary distribution of (Q,J), define a series of numbers as

$$\begin{cases}\beta_0=1,\\ \beta_k=1-\frac{\eta r}{\lambda}+\frac{\eta}{\lambda}\beta_{k-1}+\frac{\theta}{\lambda}\sum_{v=0}^{k-1}\beta_N, & 1\leq k\leq N-1.\end{cases} \tag{4}$$

Theorem 2.1 If $\rho<1$, the joint probability distribution of (Q,J) is

$$\pi_{k0}=\begin{cases}K\beta_{N-1-k}, & 0\leq k\leq N-1,\\ K, & k=N,\end{cases} \tag{5}$$

$$\pi_{k1}=\begin{cases}K\left(\frac{\lambda}{\mu}\beta_{N-1}-\frac{\eta}{\mu}\beta_{N-2}\right),k=1,\\ K\left\{\left(\frac{\lambda}{\mu}\right)^{k-1}\left(\frac{\lambda}{\mu}\beta_{N-1}-\frac{\eta}{\mu}\beta_{N-2}\right)+\frac{r(\lambda+\theta)}{\mu}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^j+\frac{\theta}{\lambda}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^j\sum_{v=k+1-j}^{N-1}\beta_{N-1-j}\right\},2\leq k\leq N,\end{cases}$$

$$\begin{cases}\pi_{k0}=\pi_{N0}r^{k-N}, & k>N,\\ \pi_{k1}=\pi_{N1}\rho^{k-N}+\frac{r(\lambda+\theta)}{\mu}+\sum_{v=0}^{k-N-1}r^v\rho^{k-N-1-v}, & k>N,\end{cases} \tag{6}$$

where the constant K can be determined by the normalization condition

$$\sum_{k=0}^{\infty}\pi_{k,0}+\sum_{k=1}^{\infty}\pi_{k,1}=1.$$

Proof. The linear system of equations $xB[R]=0$ can be rewritten as

$$-\lambda\pi_{00}+\eta\pi_{10}+\mu\pi_{11}=0. \tag{7}$$
$$\theta\pi_{10}-(\lambda+\mu)\pi_{11}+\mu\pi_{21}=0. \tag{8}$$
$$\lambda\pi_{k-1,0}-(\lambda+\theta+\eta)\pi_{k0}+\eta\pi_{k+1,0}=0,\quad 1\leq k\leq N-1. \tag{9}$$
$$\lambda\pi_{k-1,1}+\theta\pi_{k0}-(\lambda+\mu)\pi_{k1}+\mu\pi_{k+1,1}=0,\quad 2\leq k\leq N-1. \tag{10}$$
$$\lambda\pi_{N-1,0}-\frac{\lambda}{r}\pi_{N0}=0. \tag{11}$$
$$\lambda\pi_{N-1,1}+(\lambda+\theta)\pi_{N0}-\mu\pi_{N1}=0. \tag{12}$$

From (11), we have

$$\pi_{N,0}=r\pi_{N-1,0}. \tag{13}$$

Denoting $\pi_{N-1,0}=K$, then (5) is derived.

From (12), we have

$$\mu\pi_{N1}-\lambda\pi_{N-1,1}=(\lambda+\theta)\pi_{N0}. \tag{14}$$

Using (14) and (10), we recursively obtain

$$\mu\pi_{k1}=\lambda\pi_{k-1,1}+(\lambda+\theta)\pi_{N0}+\theta\sum\pi_{v0},\quad 2\leq k\leq N-1.$$

After manipulating, we recursively achieved

$$\pi_{k1}=\left(\frac{\lambda}{\mu}\right)^{k-1}\pi_{1,1}+\frac{(\lambda+\theta)}{\lambda}\pi_{N0}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j}+\frac{\theta}{\lambda}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j}\sum_{v=k+1-j}^{N-1}\pi_{v0},\quad 2\le k\le N. \tag{15}$$

Applying (5) in (7), we get

$$\pi_{11}=K\left[\frac{\lambda}{\mu}\beta_{N-1}-\frac{\eta}{\mu}\beta_{N-2}\right]. \tag{16}$$

Substituting (16) and (5) into (15), we have

$$\pi_{k1}=K\left\{\left(\frac{\lambda}{\mu}\right)^{k-1}\left(\frac{\lambda}{\mu}\beta_{N-1}-\frac{\eta}{\mu}\beta_{N-2}\right)+\frac{r(\lambda+\theta)}{\lambda}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j}\right.$$
$$\left.+\frac{\theta}{\lambda}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j}\sum_{v=k+1-j}^{N-1}\beta_{N-1-v}\right\},\ 2\le k\le N.$$

Furthermore, using the matrix geometric solution method [8], we obtain

$$\Pi_k=\Pi_N R^{k-N}=\left(\pi_{N,0},\pi_{N,1}\right)R^{k-N},\quad k>N.$$

From (2), we get

$$R^k=\begin{pmatrix} r^k & \dfrac{r(\lambda+\theta)}{\mu}\displaystyle\sum_{j=0}^{k-1}r^j\rho^{k-1-j} \\ 0 & \rho^k \end{pmatrix}.$$

Hence theorem is proved.

Remark 2.1 The results are the same as those of [5] when $N=1$. Therefore, the model is the generalization of [5].

3. A special case of $\mathbf{N=2}$

It is difficult to obtain the stochastic decomposition structures of the mean queue length and mean waiting time at a steady state of this model as the distribution expressions of these indices are very complicated and hard to operate. Hence, we only analyze the stochastic decomposition structure of the steady state indices in a special case of $N=2$.

Suppose that $N=2$, then set of equations (7) to (12) takes the form

$$-\lambda\pi_{00}+\eta\pi_{10}+\mu\pi_{11}=0.$$
$$\theta\pi_{10}-(\lambda+\mu)\pi_{11}+\mu\pi_{21}=0.$$
$$\lambda\pi_{0,0}-(\lambda+\theta+\eta)\pi_{10}+\eta\pi_{2,0}=0.$$
$$\lambda\pi_{1,0}-\frac{\lambda}{r}\pi_{20}=0.$$
$$\lambda\pi_{1,1}+(\lambda+\theta)\pi_{20}-\mu\pi_{21}=0.$$

Assume that $\pi_{0,0}=K$, then we obtain

$$\Pi_1=K\left(\frac{\lambda}{\lambda+\theta+(1-r)\eta},\frac{\rho(\lambda+\theta+r\eta)}{\lambda+\theta+(1-r)\eta}\right).$$
$$\Pi_2=K\left(\frac{\rho\lambda}{\lambda+\theta+(1-r)\eta},\frac{\rho(\rho(\lambda+\theta+r\eta)+r(\lambda+\theta))}{\lambda+\theta+(1-r)\eta}\right).$$

Using the matrix geometric solution method [8], we obtain

$$\left(\pi_{k0},\pi_{k1}\right)=\left(\pi_{20},\pi_{21}\right)R^{k-2}$$

thus we have

$$\pi_{k0} = K\frac{\lambda}{\lambda+\theta+(1-r)\eta}r^{k-1}, \quad k \ge 2$$

$$\pi_{k1} = K\left[\frac{\rho(\lambda+\theta)}{\lambda+\theta+(1-r)\eta}\sum_{j=2}^{k-1} r^j \rho^{k-1-j} + \rho^{k-1}\frac{\rho(\lambda+\theta+r\eta)+r(\lambda+\theta)}{\lambda+\theta+(1-r)\eta}\right], \quad k \ge 2 \tag{17}$$

where K can be determined by the normalization condition

$$K = \left[1+\frac{\lambda}{(1-\tau)(\lambda+\theta+(1-r)\eta)}+\frac{\rho r(\lambda+\theta)}{(1-r)(1-\rho)(\lambda+\theta+(1-r)\eta)}\right.$$
$$\left.+\frac{\rho\left(\lambda+\theta-r\mu_v\right)}{(1-\rho)(\lambda+\theta+(1-r)\eta)}\right]^{-1}. \tag{18}$$

From (17), the probabilities that the system is in a working vacation period and in a regular busy period are as follows, respectively

$$P(J=0) = \sum_{k=0}^{\infty}\pi_{k0} = K\left[1+\frac{\lambda}{(1-r)(\lambda+\theta+(1-r)\eta)}\right].$$

$$P(J=1) = \sum_{k=1}^{\infty}\pi_{k1} = K\left[\frac{\rho r(\lambda+\theta)}{(1-r)(1-\rho)(\lambda+\theta+(1-r)\eta)}+\frac{\rho\left(\lambda+\theta-r\mu_v\right)}{(1-\rho)(\lambda+\theta+(1-r)\eta)}\right].$$

Theorem 3.1 If $\rho<1$ and $\eta>\mu$, the stationary queue length L in system can be decomposed into sum of two independent random variables: $Q=Q_0+Q_d$, where Q_0 is the stationary queue length of the classical M/M/1 queue without vacation and follows a geometric distribution with parameter $1-\rho$ and the additional queue length Q_d has a modified geometric distribution

$$P\{Q_d=\mathrm{k}\} = \begin{cases} K^*\phi_1, & K=0 \\ K^*\phi_2, & K=1 \\ K^*\phi_3(1-r)r^{k-1}, & K\ge 2 \end{cases} \tag{19}$$

where

$$\phi_1 = 1-r, \quad \phi_2 = \frac{(\mu-\eta)(1-r)\rho}{\lambda+\theta+(1-r)\eta}, \quad \phi_3 = \frac{(\mu-\eta)\rho}{\lambda+\theta+(1-r)\eta},$$

$$K^* = \left[(1-r)(1-\rho)+\frac{\lambda(1-\rho)}{(\lambda+\theta+(1-r)\eta)}+\frac{\rho r(\lambda+\theta)}{(\lambda+\theta+(1-r)\eta)}+\frac{\rho(1-r)(\lambda+\theta-r\eta)}{(\lambda+\theta+(1-r)\eta)}\right]^{-1}.$$

Proof. Using (17), the PGF of Q can be expressed as follows

$$Q(z) = \sum_{k=0}^{\infty}\pi_{k0}z^k + \sum_{k=1}^{\infty}\pi_{k1}z^k$$
$$= K\left\{1+\frac{\lambda}{\lambda+\theta+\eta(1-r)}\frac{z}{1-rz}+\frac{\rho(\lambda+\theta-\eta r)z}{\lambda+\theta+\eta(1-r)}+\frac{\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}\frac{r^2z^3}{(1-rz)(1-\rho z)}\right.$$

$$+\frac{\rho(\lambda+\theta-\eta r)+\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}\frac{z^2}{(1-\rho z)}\Bigg\}$$

$$=\frac{1-\rho}{1-\rho z}K^*\left\{(1-r)(1-\rho z)+\frac{\lambda}{\lambda+\theta+\eta(1-r)}\left((1-r)z+(r-\rho)\frac{(1-r)z^2}{1-rz}\right)\right.$$

$$\left.+\frac{\rho(\lambda+\theta-\eta r)z}{\lambda+\theta+\eta(1-r)}+\frac{\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}r(1-r)z^2+\frac{\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}\frac{r^2(1-r)z^3}{(1-rz)}\right\}$$

$$=\frac{1-\rho}{1-\rho z}K^*\left[\phi_1+\phi_2 z+\phi_3\frac{r(1-r)}{1-rz}z^2\right]$$

$$=\frac{1-\rho}{1-\rho z}Q_d(z)$$

It is easy to verify that $\phi_1+\phi_2+r\phi_3=\left(K^*\right)^{-1}$, therefore, $Q_d(z)$ is a PGF. Expanding $Q_d(z)$ in power series of z, we get the distribution of additional number of customers Q_d. With the stochastic decomposition structure in Theorem 2, we can easily get means

$$E\left(Q_d\right)=K^*\left[\phi_2+\frac{r(2-r)}{1-r}\phi_3\right],\quad E(Q)=\frac{\rho}{1-\rho}+E\left(Q_d\right).$$

Theorem 3.2 If $\rho<1$ and $\mu>\eta$, the waiting time W of an arrival can be decomposed into the sum of two independent variables: $W=W_0+W_d$, where W_0 is the waiting time of an arrival in a corresponding classical $M/M/1$ queue and is exponentially distributed with parameter $\mu(1-\rho)$ and W_d is the additional delay with the LST given by

$$W_d^*(s)=K^*\left\{1+1\frac{\alpha}{\alpha+s}\right\} \tag{20}$$

where

$$\alpha=\frac{\lambda(1-r)}{r},\quad 1=\phi_1+\phi_2-\phi_3\frac{1-r^2}{r},\quad 2=\frac{\phi_3}{r}.$$

Proof. The classical relationship between the PGF of Q and the LST of waiting time W is

$$Q(z)=W^*(\lambda(1-z)).$$

From Theorem 3.1, we get

$$Q(z)=\frac{1-\rho}{1-\rho z}K^*\left[\phi_1+\phi_2 z+\phi_3\frac{r(1-r)}{1-rz}z^2\right]. \tag{21}$$

Taking $z=1-\frac{s}{\lambda}$ in (20) and denoting $\frac{\lambda(1-r)}{r}=\alpha$, we get

$$W^*(s)=\frac{\mu(1-\rho)}{\mu(1-\rho)+s}K^*\left\{\phi_1+\phi_2\left(1-\frac{s}{\lambda}\right)+\phi_3\frac{(1-r)}{\lambda}\left[\frac{\frac{\lambda}{r}}{\alpha+s}-(2\lambda+\alpha)+s\right]\right\}$$

$$=\frac{\mu(1-\rho)}{\mu(1-\rho)+s}K^*\left[1+\quad 2\frac{\alpha}{\alpha+s}\right]$$

$$=\frac{\mu(1-\rho)}{\mu(1-\rho)+s}W_d^*(s)$$

It is easy to verify that $_1+_2=\phi_1+\phi_2+r\phi_3=(K^*)^{-1}$. Therefore, $W_d^*(s)$ is a LST.

The result of Theorem 3.2 indicates that additional delay W_d equals zero with probability $K^{*}1$ and follows an exponential distribution with parameter α with probability $K^{*}1$. It is easy to obtain

$$E(W_d) = K_2^{*}\frac{1}{\alpha} = \frac{1}{\lambda}E(Q_d), \quad E(W) = \frac{1}{\mu(1-\rho)} + E(W_d).$$

4. Numerical results

In this section, we illustrate the influence of the system parameters on the performance measures by presenting some numerical examples. Figures 1 and 2 depicts the expected queue length $E(Q)$ against the vacation service rate η for different values of θ and ρ respectively. In Figure 3, we present the state probability of the server for the change of η and different vacation rate θ. Figure 4 gives the comparison of our model (M/M/1/MWV+VI) with M/M/1/MWV (Liu et al. [7]) in terms of mean queue length. Figure 5 shows how the mean waiting time $E(W)$ changes with the mean vacation time and presents the comparison of the mean waiting time in our model with two different vacation policies i.e. the multiple vacation (MV) and the multiple working vacation (MWV). Finally, Figure 6 describes the impact of η and θ on the mean waiting time $E(W)$ of the customers. The main findings in this study are itemized as

- As explained in Figures 1 and 2, with the increase in vacation service rate η, the mean queue length $E(Q)$ apparently decreases. Meanwhile, when the vacation service rate η tends to $\mu = 2$, $E(Q)$ will approach to a constant value and the model reduces to the corresponding queue without vacation, regardless of how long the vacation times. Furthermore, when we increase the values of θ and ρ, the expected number of customers in the queue decreases and increases respectively.
- From Figure 3, the probability that the server stays in working vacation $P(J=0)$, evidently increases and the probability that the server remain in normal working level $P(J=1)$ decreases with an increase in vacation service rate η. Hence, the utilization level of the system idle time becomes larger. Moreover, the state probability of the server is also affected by the vacation rate θ. For example, when $\theta = 1.5$, $P(J=0)$ are evidently smaller than those when $\theta = 0.5$.
- From Figure 4, M/M/1/MWV (Liu et al. [7]) yields higher mean queue length $E(Q)$ when compared to our model M/M/1/MWV+VI for fixed η and hence the former resulted in more customers to wait. Therefore, the vacation interruption policy is appreciably more desirable in terms of $E(Q)$. Thus, we can attain a better service, if we consider vacation interruptions un- der working vacation policy so that we can make use of server productively and consequently decrease the waiting time of customers.
- As illustrated in Figure 5, increase in θ^{-1} leads to an increase in $E(W)$ and when θ^{-1} advances towards 0, $E(W)$ will arrive at a constant value i.e. our model becomes a classical *M/M/*1 queue. Moreover, MWV+VI policy performs better than the MV policy and the MWV policy, because the server will return back to a regular busy level more frequently if the mean vacation time is longer. Consequently, more customers are served at a higher rate.
- From Figure 6, $E(W)$ evidently decreases as θ and η increases. When η is

fixed, the mean waiting time $E(W)$ is bigger, if θ is smaller. Meanwhile, the vacation rate θ has small influence on the waiting time when vacation rate θ increases to the certain degree. When $\eta = \mu$, the system reduces to the model without vacations and $E(W)$ achieves a fixed value.

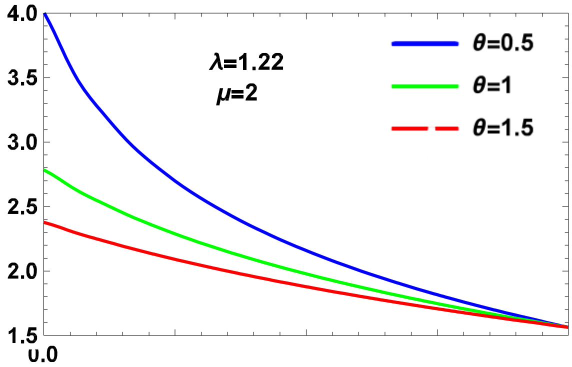

Figure 1. The effect of η on $E(Q)$ for different values of θ.

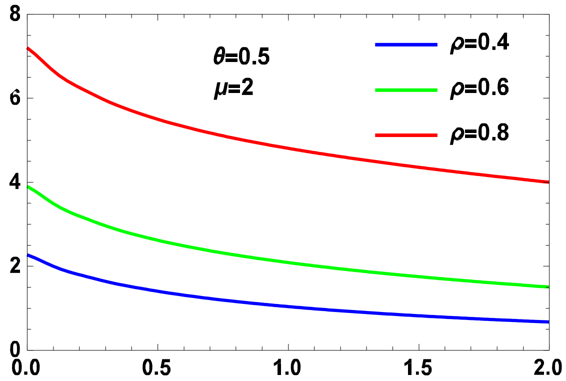

Figure 2. The effect of η on $E(Q)$ for different values of ρ.

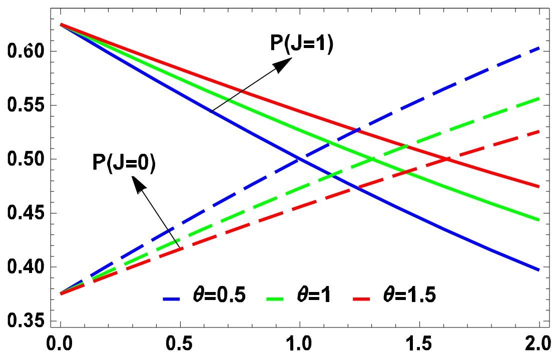

Figure 3. The state probability of the server with the change of η.

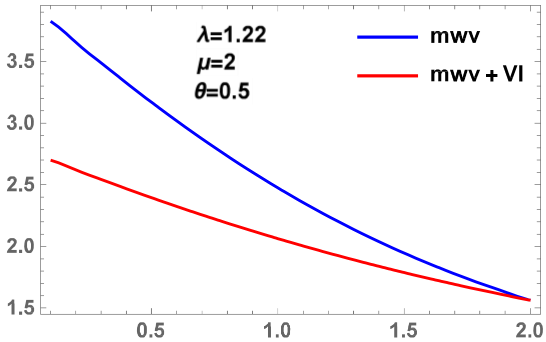

Figure 4. The comparison of models without and with vacation interruption.

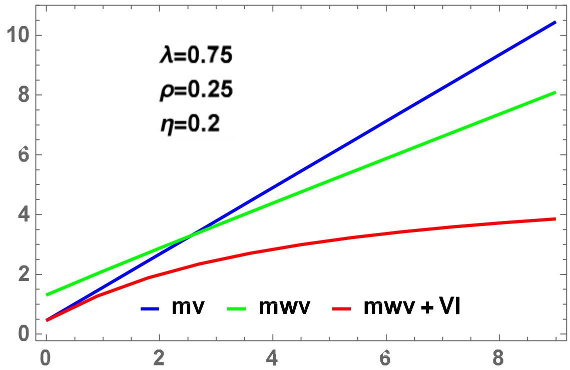

Figure 5. Comparisons among different models.

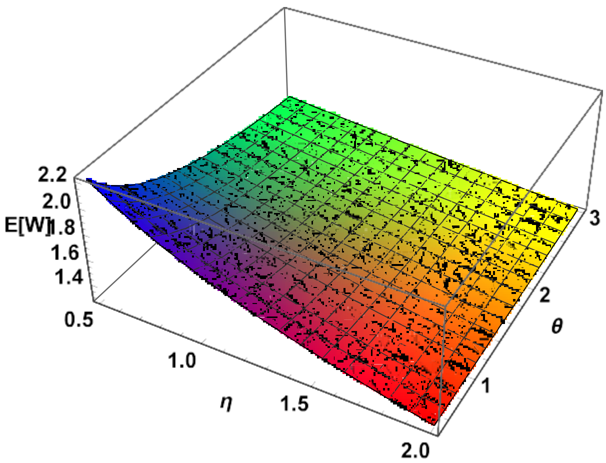

Figure 6. The relation of $E(W)$ with η and $\theta(\rho = 0.6)$.

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