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Generalization of Titchmarsh's Theorem for the Dunkl Transform in the Space $L^p(\mathbb{R}^d, w_l(x)dx)$

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Abstract. Using a generalized spherical mean operator, we obtain a generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the (φ, p)-Dunkl Lipschitz condition in the space $L^p(\mathbb{R}^d, w_l(x)dx)$, $1 < p \leqslant 2$, where w_l is a weight function invariant under the action of an associated reflection group.

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1. Introduction and Preliminaries

In [11], E.C.Titchmarsh's characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1 *[11] Let $f \in L^2(\mathbb{R})$. Then the following are equivalents:*
(i) $\|f(x+h) - f(x)\|_2 = O(h^\eta)$, *as* $h \to 0, 0 < \eta < 1$,
(ii) $\displaystyle\int_{|\lambda|\geqslant s} |\widehat{f}(\lambda)|^2 d\lambda = O\left(s^{-2\eta}\right)$, *as* $s \to \infty$,
where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we obtain a generalization of theorem 1.1 for the Dunkl transform on $\mathbb{R}^d$ in the space $L^p(\mathbb{R}^d, w_l(x)dx)$, $1 < p \leqslant 2$. For this purpose, we use a generalized spherical mean operator.

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We consider the Dunkl operators D_j, $1 \leqslant j \leqslant d$, on $\mathbb{R}^d$ which are the differential-difference operators introduced by Dunkl in [4]. These operators are very important in pure mathematics and in physics. The theory of Dunkl operators provides generalizations of various multivariable analytic structures, among others we cite the exponential function, the Fourier transform and the translation operator. For more details about these operators see ([3]-[5]). The Dunkl Kernel E_l has been introduced by Dunkl in [6]. This Kernel is used to define the Dunkl transform.
Let R be a root system in $\mathbb{R}^d$, W the corresponding reflection group, R_+ a positive subsystem of R (see [3],[5],[7]-[10]) and l a non-negative and W-invariant function defined on R. The Dunkl operator is defined for $f \in C^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} l(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{< \alpha, x >}, x \in \mathbb{R}^d (1 \leqslant j \leqslant d).$$

Here $<,>$ is the usual Euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and σ_α the reflection with respect to the hyperplane H_α orthogonal to α, and $\alpha_j =< \alpha, e_j >$, $(e_1, e_2, ..., e_d)$ being the canonical basis of $\mathbb{R}^d$.
We consider the weight function

$$w_l(x) = \prod_{\zeta \in R_+} | < \zeta, x > |^{2l(\alpha)}, x \in \mathbb{R}^d,$$

where w_l is W-invariant and homogeneous of degree 2γ where

$$\gamma = \gamma(R) = \sum_{\zeta \in R_+} l(\zeta) \geqslant 0.$$

The Dunkl kernel E_l on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C.F.Dunkl in [6]. For $y \in \mathbb{R}^d$, the function $x \mapsto E_l(x, y)$ is the unique solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{cases} D_j u(x, y) = y_j u(x, y), & \text{if} \quad 1 \leqslant j \leqslant d, \\ \qquad u(0, y) = 0, & \text{for all} \quad y \in \mathbb{R}^d, \end{cases}$$

E_l is called the Dunkl kernel.

Lemma 1.2 *[3] Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$*
1. $E_l(z, 0) = 1$, $\quad E_l(z, w) = E_l(w, z)$, $\quad E_l(\lambda z, w) = E_l(z, \lambda w)$.
2. For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}^d$,$x \in \mathbb{R}^d$,$z \in \mathbb{C}^d$, we have

$$|\partial_z^\nu E_l(x; z)| \leqslant |x|^{|\nu|} exp(|x||Rez|),$$

where

$$\partial_z^\nu = \frac{\partial^{|\nu|}}{\partial_{z_1}^{\nu_1} .... \partial_{z_d}^{\nu_2}}, |\nu| = \nu_1 + ... + \nu_d.$$

In particular $|\partial_z^\nu E_l(ix; z)| \leqslant |x|^{|\nu|}$ for all $x, z \in \mathbb{R}^d$.

We denote by $L^p_l(\mathbb{R}^d) = L^p(\mathbb{R}^d, w_l(x)dx)$, $1 < p \leqslant 2$, the space of measurable functions on $\mathbb{R}^d$ with the norm

$$\|f\|_{p,l} = \left(\int_{\mathbb{R}^d} |f(x)|^p w_l(x)dx\right)^{\frac{1}{p}} < \infty,$$

and Δ_l the Dunkl Laplacian defined by

$$\Delta_l = \sum_{i=1}^{d} D_j^2.$$

The Dunkl transform is defined for $f \in L^1_l(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_l(x)dx)$ by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = c_l^{-1} \int_{\mathbb{R}^d} f(x)E_l(-i\xi, x)w_l(x)dx,$$

where the constant c_l is given by

$$c_l = \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}} w_l(z)dz.$$

The Dunkl transform shares several properties with its counterpart in the classical case, we mention here in particular that Plancherel's theorem holds in $L^2_l(\mathbb{R}^d)$, when both f and $\widehat{f}$ are in $L^1_l(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)E_l(ix, \xi)w_l(\xi)d\xi, x \in \mathbb{R}^d.$$

By Plancherel's theorem and the Marcinkiewicz interpolation theorem (see [12]), we get for $f \in L^p_l(\mathbb{R}^d)$ with $1 < p \leqslant 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}(f)\|_{q,l} \leqslant K\|f\|_{p,l}, \tag{1}$$

where K is a positive constant.
The generalized spherical mean value of $f \in L^p_l(\mathbb{R}^d)$ is defined by

$$M_h f(x) = \frac{1}{d_l}\int_{\mathbb{S}^{d-1}} \tau_x f(hy)d\eta_l(y), x \in \mathbb{R}^d, h > 0,$$

where τ_x Dunkl translation operator (see [10],[13]), η be the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set $d\eta_l(y) = w_l(y)d\eta(y)$ η_l is a W-invariant measure on $\mathbb{S}^{d-1}$ and $d_l = \eta_l(\mathbb{S}^{d-1})$.
We see that $M_h f \in L^p_l(\mathbb{R}^d)$ whenever $f \in L^p_l(\mathbb{R}^d)$ and

$$\|M_h f\|_{p,l} \leqslant \|f\|_{p,l},$$

for all $h > 0$.
For $\beta \geqslant \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_β defined

by

$$j_\beta(z) = \Gamma(\beta+1)\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n+\beta+1)}, \quad z \in \mathbb{C}.$$

LEMMA 1.3 *(Analog of lemma 2.9 in [2]) The following inequality is true*

$$|1 - j_\beta(x)| \geqslant c,$$

with $|x| \geqslant 1$, where $c > 0$ is a certain constant which depend only on β.

LEMMA 1.4 *[8] Let $f \in L_l^p(\mathbb{R}^d)$. Then*

$$\widehat{M_h f}(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi).$$

The first and higher order finite differences of $f(x)$ are defined as follows

$$Z_h f(x) = (M_h - I)f(x),$$

where I is the identity operator $L_l^p(\mathbb{R}^d)$.

$$Z_h^k f(x) = Z_h(Z_h^{k-1}f(x)) = (M_h - I)^k f(x) = \sum_{i=0}^{k}(-1)^{k-i}\tbinom{k}{i}M_h^i f(x),$$

where $M_h^0 f(x) = f(x)$, $M_h^i f(x) = M_h(M_h^{i-1}f(x))$, $i = 1, 2, ..$ and $k = 1, 2, ...$
Let $W_{p,l}^k$, $1 < p \leqslant 2$, be the Sobolev space constructed by the operator Δ_l, i.e.,

$$W_{p,l}^k = \{f \in L_l^p(\mathbb{R}^d) : \Delta_l^r f \in L_l^p(\mathbb{R}^d); r = 1, 2, ..., k\},$$

where $\Delta_l^0 f = f$, $\Delta_l^r f = \Delta_l(\Delta_l^{r-1}f)$.
In view ([3] or [5]) we can write

$$\widehat{D_j f}(y) = iy_j\widehat{f}(y), j = 1, ..., d; y \in \mathbb{R}^d. \tag{2}$$

From formula (2) and lemma 1.4, we obtain

$$\widehat{Z_h^k\Delta_l^r f}(\xi) = |\xi|^{2r}(j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1)^k\widehat{f}(\xi).$$

By (1) we get for $f \in W_{p,l}^k$,

$$\int_{\mathbb{R}^d}|\xi|^{2qr}|1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{qk}|\widehat{f}(\xi)|^q w_l(\xi)d\xi \leqslant K^q\|Z_h^k\Delta_l^r f(x)\|_{p,l}^q, \tag{3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2. Main Result

In this section we give the main results of this paper. We need first to define (φ, p)-Dunkl Lipschitz class.

Definition 2.1 *A function $f \in W^k_{p,l}$ is said to be in the (φ, p)-Dunkl Lipschitz class, denoted by $Lip(\varphi, p)$, if*

$$\|Z^k_h \Delta^r_l f(x)\|_{p,l} = O\left(\varphi(h)\right), \quad as \quad h \to 0, \gamma \geqslant 0,$$

where
i) $\varphi(t)$ a continuous increasing function on $[0, \infty)$,
ii) $\varphi(0) = 0$,
iii) $\varphi(ts) = \varphi(t)\varphi(s)$ for all $t, s \in [0, \infty)$.

Theorem 2.2 *Let $f \in W^k_{p,l}$. If $f(x)$ belong to $Lip(\varphi, p)$, then*

$$\int_{|\xi| \geqslant s} |\xi|^{2qr} |\widehat{f}(\xi)|^q w_l(\xi) d\xi = O\left(\varphi(s^{-q})\right), s \to \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Suppose that $f \in Lip(\varphi, p)$. Then

$$\|Z^k_h \Delta^r_l f(x)\|_{p,l} = O\left(\varphi(h)\right), \quad h \to 0.$$

From (3), we have

$$\|Z^k_h \Delta^r_l f(x)\|^q_{p,l} = \int_{\mathbb{R}^d} |\xi|^{2qr} |1 - j_{\gamma + \frac{d}{2} - 1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi.$$

If $|\xi| \in [\frac{1}{h}, \frac{2}{h}]$ then $h|\xi| \geqslant 1$ and lemma 1.3 implies that

$$1 \leqslant \frac{1}{c^{qk}} |1 - j_{\gamma + \frac{d}{2} - 1}(h|\xi|)|^{qk}.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h} \leqslant |\xi| \leqslant \frac{2}{h}} |\xi|^{2qr} |\widehat{f}(\xi)|^q w_l(\xi) d\xi &\leqslant \frac{1}{c^{2k}} \int_{\frac{1}{h} \leqslant |\xi| \leqslant \frac{2}{h}} |\xi|^{2qr} |1 - j_{\gamma + \frac{d}{2} - 1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \\
&\leqslant \frac{1}{c^{2k}} \int_{\mathbb{R}^d} |\xi|^{2qr} |1 - j_{\gamma + \frac{d}{2} - 1}(h|\xi|)|^{qk} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \\
&= O\left((\varphi(h))^q\right) \\
&= O\left(\varphi(h^q)\right).
\end{aligned}$$

We obtain

$$\int_{s \leqslant |\xi| \leqslant 2s} |\xi|^{2qr} |\widehat{f}(\xi)|^q w_l(\xi) d\xi \leqslant C' \varphi(s^{-q}),$$

where C' is a positive constant. Now,

$$\begin{aligned}\int_{|\xi|\geqslant s}|\xi|^{2qr}|\widehat{f}(\xi)|^q w_l(\xi)d\xi &= \sum_{i=0}^{\infty}\int_{2^i s}^{2^{i+1}s}|\xi|^{2qr}|\widehat{f}(\xi)|^q w_l(\xi)d\xi\\ &\leqslant C'\left(\varphi(s^{-q})+\varphi((2s)^{-q})+\varphi((4s)^{-q})+\cdots\right)\\ &\leqslant C'\varphi(s^{-q})\left(1+\varphi(2^{-q})+\varphi((2^{-q})^2)+\varphi((2^{-2})^3)+\cdots\right)\\ &\leqslant C'\varphi(s^{-q})\left(1+\varphi(2^{-q})+\varphi^2(2^{-q})+\varphi^3(2^{-q})+\cdots\right)\\ &\leqslant K_\varphi\varphi(s^{-q}),\end{aligned}$$

where $K_\varphi = C'(1-\varphi(2^{-q}))^{-1}$ since $\varphi(2^{-q}) < 1$.
Consequently

$$\int_{|\xi|\geqslant s}|\xi|^{2qr}|\widehat{f}(\xi)|^q w_l(\xi)d\xi = O\left(\varphi(s^{-q})\right), \quad as \quad s\to\infty.$$

■

Corollary 2.3 *Let $f\in W^k_{p,l}$ and let*

$$\|Z^k_h\Delta^r_l f(x)\|_{p,l} = O\left(\varphi(h)\right), \quad as \quad h\to 0.$$

Then

$$\int_{|\xi|\geqslant s}|\widehat{f}(\xi)|^q w_l(\xi)d\xi = O\left(s^{-2qr}\varphi(s^{-q})\right), \quad as \quad s\to\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

3. Conclusions

In this work we have succeeded to generalize the theorem 1.1 for the Dunkl transform in the space $L^p(\mathbb{R}^d, w_l(x)dx)$. We proved that $f(x)$ belong to $Lip(\varphi,p)$ Then

$$\int_{|\xi|\geqslant s}|\widehat{f}(\xi)|^q w_l(\xi)d\xi = O\left(s^{-2qr}\varphi(s^{-q})\right), \quad as \quad s\to\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

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