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Numerical Solution of Nonlinear System of Ordinary Differential Equations by the Newton-Taylor Polynomial and Extrapolation with Application from a Corona Virus Model

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Abstract. In this paper, we consider a nonlinear non autonomous system of differential equations. We linearize this system by the Newton's method and obtain a sequence of linear systems of ODE. We are going to solve this system on $[0, Nl] = \cup_{k=1}^{N}[(k-1)l, kl]$ for some positive integer N and a positive real $l > 0$. For this purpose, in the first step we solve the problem on $[0, l]$ .By knowing the solution on $[0, l]$, we solve the problem on $[l, 2l]$ and obtain the solution on $[0, 2l]$. We continue this procedure until $[0, Nl]$. In each partial interval $[(k-1)l, kl]$, first of all, we solve the problem by the extrapolation method and obtain an initial guess for the Newton-Taylor polynomial solutions. These procedures cause that the errors don't propagate. The sequence of linear systems in Newton's method are solved by a famous method called Taylor polynomial solutions, which have a good accuracy for linear systems of ODE. Finally, we give a mathematical model of the novel corona virus disease and illustrate accuracy and applicability of the method by some examples from this model and compare them by similar work, that simulate the numerical solutions.

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1. Introduction

In this paper we consider the following nonlinear non autonomous system of ordinary differential equations

$$\begin{cases} \dot{U}(\mathrm{t}) = f\left(t, U(\mathrm{t})\right), \quad t \in \left[0, N_I l\right], \\ U(0) = U_0, \end{cases} \tag{1}$$

where $U(\mathrm{t}) = \left(u_1(\mathrm{t}), ..., u_d(\mathrm{t})\right)^T$ is unknown vector of functions and

$$f\left(t, U(t)\right) = \left(f_1\left(t, U(t)\right), ..., f_d\left(t, U(t)\right)\right)^T, \tag{2}$$

is a known continuous vector valued function. We shall solve the problem on $[0, N_I l] = \cup_{k=1}^{N_I} I_k$,

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where $l > 0$ be the length of partial intervals, N_I is the number of intervals and $I_k = [(k-1)l, kl]$ is the k th partial interval of the partition.

The Newton-Taylor polynomial solutions technique for numerical solution of above system is described in [3] and we are going to modified the method by adding the extrapolation method for obtaining an initial guess for the starting of Newton’s method. This starting value is important, since the Newton's method converges much more rapidly than a simple iterative method, if the starting point is already close to a zero.

Organization of the paper are as follow: in Section 1, Newton's method is described. Section 2 explains using the Taylor polynomial solutions technique for solving a linear ODE system. By extrapolation method, we obtain an initial guess for Newton's method, hence we describe this technique in Section 3. In Section 4 flow-chart form of the total algorithm is given. In Section 5 we give a mathematical model of the novel corona virus. This model is in the form of nonlinear ordinary differential equations (1). Finally in Section 6 numerical results of the method are given by some examples originated on Section 5.

2. Newton’s method

In accordance with [3], one step of Newton's method is

$$\left(U^{(n+1)}(t)\right)' - f'\left(t,U^{(n)}(t)\right)U^{(n+1)}(t) = f\left(t,U^{(n)}(t)\right) - f'\left(t,U^{(n)}(t)\right)U^{(n)}(t) \tag{3}$$

where $f'\left(t,U^{(n)}(\mathrm{t})\right) = \left[\frac{\partial f_i}{\partial u_j}(t,U(\mathrm{t}))\Big|_{U(t)=U^{(n)}(t)}\right]_{d\times d}$ is the Frechet derivative or the Jacobian matrix. Newton's method for this model is applicable and passes all hypotheses of Kantorovich theorem [1,12], which guarantees the convergence of Newton's method, and described in [3]. For this purpose, as mentioned in [3], the operator

$$A\left(U(\mathrm{t})\right) = \left[\frac{\partial f_i}{\partial u_j}\left(t,U(\mathrm{t})\right)\right], \tag{4}$$

must be satisfies

$$\sup_{W\in l(V,V)} \left\|A'W\right\| < \infty, \quad \forall V,V \in D, \tag{5}$$

where $l(V,V)$ be the line segment between V and V , D is the biologically feasible domain [11]. For investigation of (5), we give some definitions and a lemma about linear and bilinear operators.

Definition 1.1 Let X and Y are normed linear spaces. We denote all continuous linear operators from X to Y by $\mathbb{L}(X,Y)$. The space $\mathbb{L}(X,Y)$ is itself a linear space.

Definition 1.2 Let $\mathbb{L}(X^2,Y)$ denotes all continuous linear operators $X \to \mathbb{L}(X,Y)$. The space $\mathbb{L}(X^2,Y)$ is again a linear space. An element of $\mathbb{L}(X^2,Y)$ is called a bilinear operator. If $X = Y = \mathbb{R}^d$, then $\mathbb{L}(X,Y)$ is the set of all $d\times d$ matrices. The bilinear operators are the operations which transform vectors in to matrices. These are denoted by $d\times d\times d$ arrays. Indeed, if the bilinear operator B has elements b_{ijk}, $i,j,k = 1,...,d$, then

$$\left(BU\right)_{ij} = \sum_{k=1}^{d} b_{ijk}u_k, \quad U = \left(u_1,...,u_d\right)^T, \tag{6}$$

defines a $d \times d$ matrix.

Lemma 1.3 Let $P:\mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$\begin{cases} P(U) = P(U(\mathrm{t})) = f\left(t, U(t)\right), \\ U(\mathrm{t}) = \left(u_1(\mathrm{t}),...,u_d(\mathrm{t})\right)^T. \end{cases}$$

Then $P'(U) = A(U)$, where A is defined by (4). Suppose $W \in \mathbb{R}^d$ is given, then the second Frechet derivative of P at W, is the bilinear operator $B = A'W = P''W$ and $(BV)_{ij} = (A'WV)_{ij} = (P''WV)_{ij} = \sum_{k=1}^{d} b_{ijk} v_k$, where $V = \left(v_1,...,v_d\right)^T$, $i, j = 1,...,d$, $b_{ijk} = \frac{\partial^2 f_i}{\partial u_j \partial u_k}(t, U(t))\Big|_{U(t)=W}$.

Proof See Example 5. 11 of [5]. ∎

Suppose $V, V \in D$, $W \in l(V,V)$ and are arbitrary, with the infinity norm of matrices we obtain

$$\begin{aligned} \left\|A'W\right\|_\infty &= \sup_{\|V\|_\infty=1} \left\|A'WV\right\| = \sup_{\|V\|_\infty=1} \max_{1\le i\le d} \sum_{j=1}^{d} \left|\left(A'WV\right)_{ij}\right| \\ &= \sup_{\|V\|_\infty=1} \max_{1\le i\le d} \sum_{j=1}^{d} \left|\sum_{k=1}^{d} b_{ijk} v_k\right| \qquad (7) \\ &\le \max_{1\le i\le d} \sum_{j=1}^{d} \sum_{k=1}^{d} \left|b_{ijk}\right| =: L_0, \end{aligned}$$

where b_{ijk} are defined by the Lemma 1.3, (7) implies that the linear operator A is Lipschitz continuous with the Lipschitz constant L_0. Whatever L_0 is small, the accuracy of the method is better. For more details about the k -linear operators for $k \in \mathbb{N}$, see [5].

3. Taylor polynomial solutions technique

The Taylor polynomial solutions technique for the following linear system of differential equations is described in [3].

$$P_0(t)U(\mathrm{t}) + P_1(t)U'(\mathrm{t}) = r(t) \qquad (8)$$

where P_0, P_1, r are known vector valued functions and $U(\mathrm{t}) = \left(u_1(t),...,u_d(t)\right)^T$ is unknown vector valued function. We briefly explain this technique and for more details the reader is referred to [3,8]. From (3) each step of Newton's method is in the form (8) with $P_1(t) = I_d$ (the $d \times d$ identity matrix), $P_0(t) = -f'(t, U^{(n)}(t))$, $U(t) = U^{(n+1)}(t)$ and $r(t) = f(t, U^{(n)}(t)) - f'(t, U^{(n)}(t))U^{(n)}(t)$. We are going to represent the solution by a truncated Taylor series

$$u_i(t) = \sum_{j=0}^{N_T} \frac{u_i^{(j)}(c)}{j!}(t-c)^j, \; i = 1,...,d, \; a \le c \le b. \qquad (9)$$

where $N_T \geq 1$ is the Taylor polynomial degree and $u_i^{(j)}(c)$ are the Taylor coefficients to be determined. The general solution of (8) obtains from the following linear system of algebraic equations

$$\mathbf{W}X = \mathbf{R}, \tag{10}$$

where

$$\mathbf{W} = \left[w_{ij}\right]_{(N_T+1)d\times(N_T+1)d} := \mathbf{P}_0\mathbf{T}M_0^* + \mathbf{P}_1\mathbf{T}M_1^*,$$

$$\mathbf{P}_i = \begin{bmatrix} P_i(t_0) & 0 & 0 & 0 \\ 0 & P_i(t_1) & 0 & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & P_i(t_{N_T}) \end{bmatrix}_{(N_T+1)\times(N_T+1)Blocks}, \quad i = 1,2,$$

$$\mathbf{T} = \left[T^*(t_0) \quad T^*(t_1) \quad \cdots \quad T^*(t_{N_T})\right]_{1\times(N_T+1)Blocks},$$

$$T^*(t) = \begin{bmatrix} T(t) & 0 & 0 & 0 \\ 0 & T(t) & 0 & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & T(t) \end{bmatrix}_{d\times d\ \ Blocks}, \quad T(t) = \left(1,(t-c),(t-c)^2,\dots,(t-c)^{N_T}\right),$$

$$M_i^* = \begin{bmatrix} M_i & 0 & 0 & 0 \\ 0 & M_i & 0 & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & M_i \end{bmatrix}_{d\times d\ \ Blocks}, \quad i = 1,2,$$

$$M_0 = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{N_T!} \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & 0 & 0 & \dots & \frac{1}{(N_T-1)!} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix}_{d\times 1Blocks}, \quad X_i = \left(u_i(c),u_i'(c),u_i''(c),\dots,u_i^{(N_T)}(c)\right)^T, \quad \mathbf{R} = \begin{bmatrix} r(t_0) \\ r(t_1) \\ \vdots \\ r(t_{N_T}) \end{bmatrix}_{(N_T+1)\times 1Blocks},$$

and

$$t_j = a + \frac{b-a}{N_T}j, \quad j = 0,1,\dots,N_T, \tag{11}$$

are the Taylor collocation points. For a particular solution of (8) which satisfies the following initial condition

$$U(a)=U_0=\left(u_1^{(0)},\ldots,u_d^{(0)}\right)^T, \tag{12}$$

we must replace the rows of the matrices $\mathbf{V}=T^*(a)M_0^*=[v_{ij}]_{d\times(N_T+1)d}$ and U_0, by the last rows of the matrices $\mathbf{W}$ and $\mathbf{R}$, respectively and obtain

$$\overline{W}X=\overline{R}, \tag{13}$$

where

$$\overline{W}=\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1,(N_T+1)\mathrm{d}} \\ w_{21} & w_{22} & \cdots & w_{2,(N_T+1)\mathrm{d}} \\ \vdots & \vdots & \cdots & \vdots \\ w_{N_Td,1} & w_{N_Td,1} & \cdots & w_{N_Td,(N_T+1)\mathrm{d}} \\ v_{11} & v_{12} & \cdots & v_{1,(N_T+1)\mathrm{d}} \\ v_{21} & v_{22} & \cdots & v_{2,(N_T+1)\mathrm{d}} \\ \vdots & \vdots & \cdots & \vdots \\ v_{d1} & v_{d2} & \cdots & v_{\mathrm{d},(N_T+1)\mathrm{d}} \end{bmatrix}, \qquad \overline{R}=\begin{bmatrix} r(t_0) \\ r(t_1) \\ \vdots \\ r(t_{N_T-1}) \\ u_1^{(0)} \\ u_2^{(0)} \\ \vdots \\ u_d^{(0)} \end{bmatrix}.$$

Convergence analyses of the method are given in [4] and its references.

4. Extrapolation method

Suppose $t\in[0,N_Il]$, by integration of $\dot{U}(\tau)=f(\tau,U(\tau)),\ \tau\in[0,t]$, the Eq. (1), reduces to

$$U(\mathrm{t})=U_0+\int_0^t f\left(\tau,U(\tau)\right)d\tau. \tag{14}$$

Eq. (14) is a special case of (2) from [2]. Suppose we have solved (14) on $[0,(k-1)l]$ for some $k\in\mathbb{N}$, and we are going to solve the problem on $[(k-1)l,kl]$. For this purpose, suppose $t_{k-1,i}=\left(k-1+\frac{i}{N_P}\right)l,\ i=0,1,\ldots,N_P$ are the mesh points of $[(k-1)l,kl]$ and $U((k-1)l)=U_{k-1}$ is known from the lag interval $[0,(k-1)l]$. Eq. (14) at $t=t_{k-1,i}$ yields that

$$U(t_{k-1,i})-\int_{(k-1)l}^{t_{k-1,i}} f\left(\tau,U(\tau)\right)d\tau=U_0+\int_0^{(k-1)l} f\left(\tau,U(\tau)\right)d\tau=U_{k-1}. \tag{15}$$

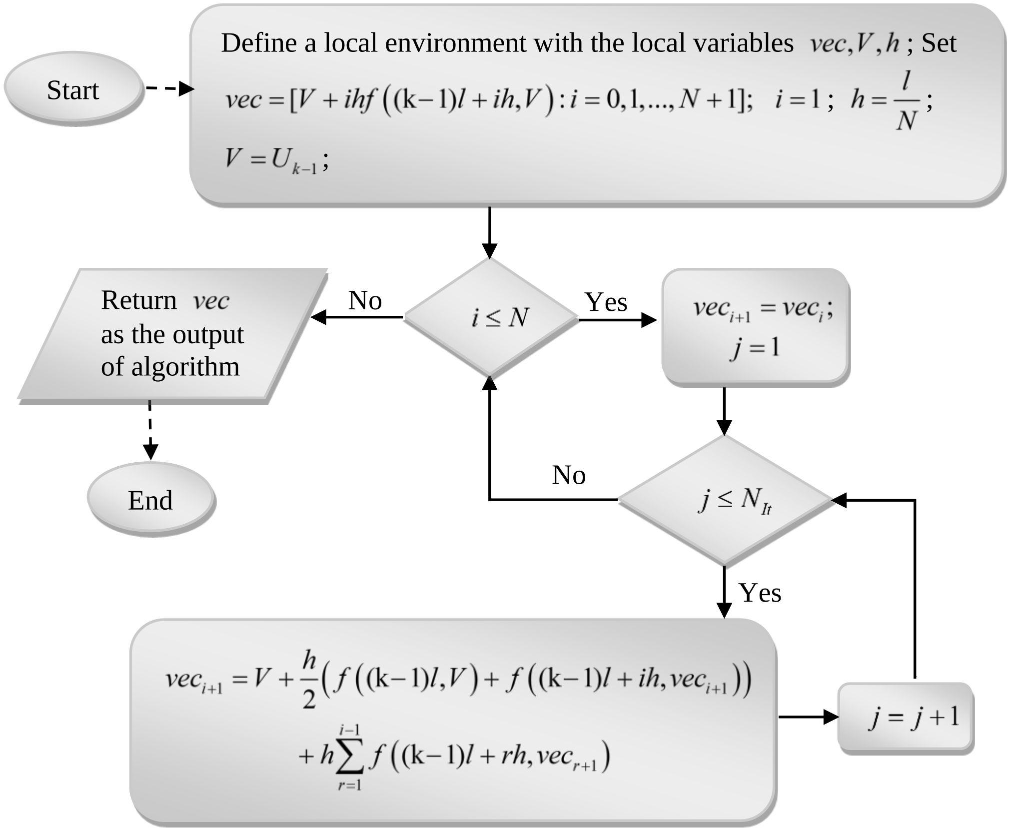
By using the composite trapezoidal rule we obtain

$$U_{k-1,i}=U_{k-1}+\frac{h}{2}\Big(f\left((k-1)l,U_{k-1}\right)+f\left(t_{k-1,i},U_{k-1,i}\right)\Big)+h\sum_{j=1}^{i-1} f\left(t_{k-1,j},U_{k-1,\mathrm{j}}\right), \quad i=1,\ldots,N_P, \tag{16}$$

where $U_{k-1,j}$ is the approximation of $U(t_{k-1,i})$ and $h=\dfrac{l}{N_P}$. For $i=1$, the $\sum$ term in the righthand side of (16) vanishes. The following algorithm approximate $U_{k-1,i}$, $i=1,...,N_P$, where $N_P \in \mathbb{N}$ denotes the number of partition and N_{It} is the number of iterations. This algorithm is similar with Algorithm 1 of [10] and a convergence analysis is given there.

Remark 4.1 In this paper a d-column vector $\mathbf{V}$ with i th component v_i, $i=1,...,d$ is denoted by $\mathbf{V}=[v_i : i=1,...,d]$. Each component v_i can be a vector or a matrix. In such cases we have a vector of vectors or a vector of matrices. For example, in the following algorithm the variable vec is a vector of vectors

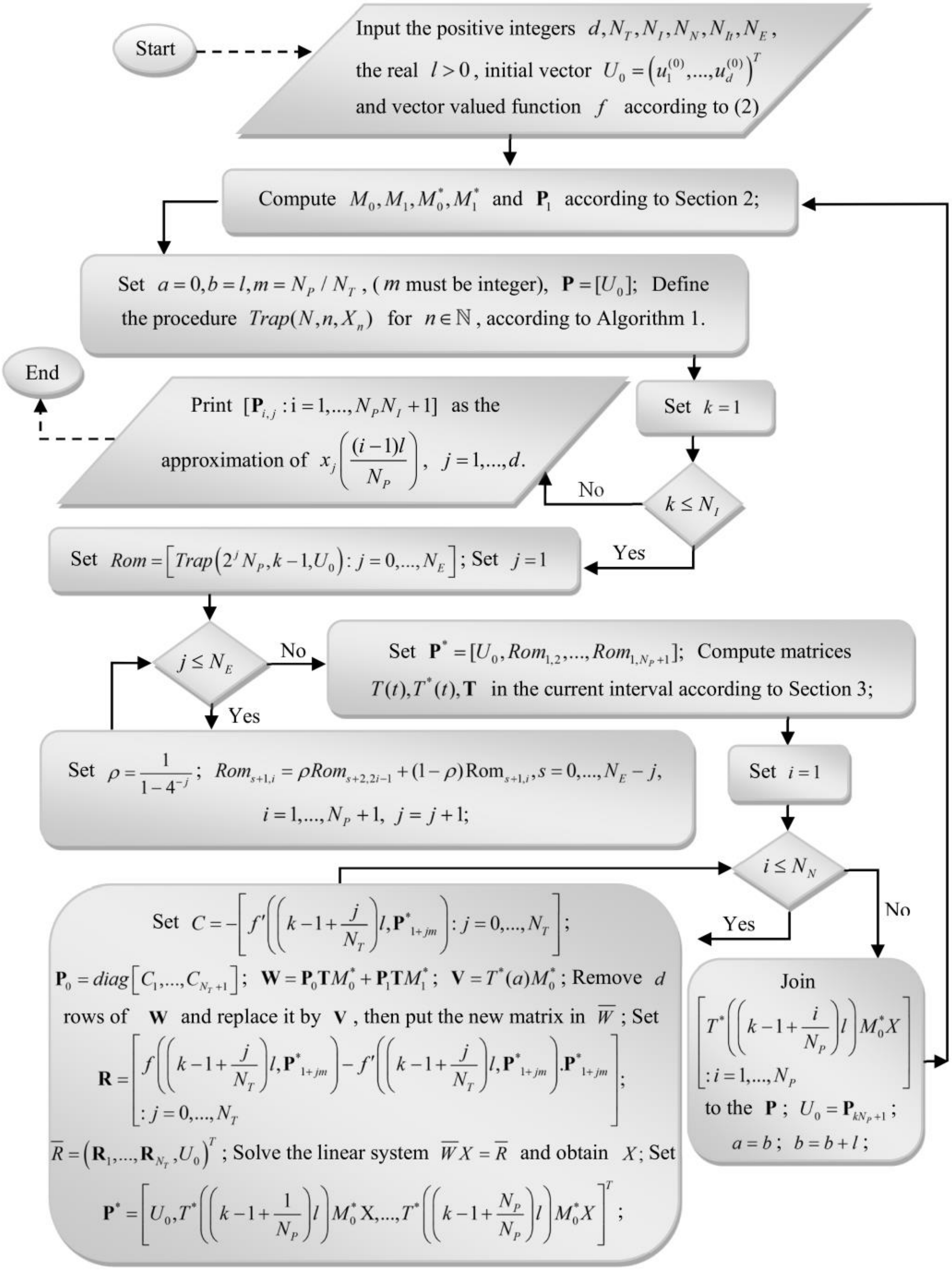
Algorithm 1. (Composite Trapezoidal iteration flow-chart)



We denote the output of Algorithm 1 by $Trap\left(N,k-1,U_{k-1}\right)$. The i th component of $Trap\left(N,k-1,U_{k-1}\right)$ is a crude approximation for $U(t_{k-1,i})$. After the above algorithm, we apply the Romberg extrapolation technique for numerical solution of (15). Indeed $Trap\left(N,k-1,U_{k-1}\right)$ is an initial guess for Romberg extrapolation and the solution obtained by Romberg extrapolation is an initial guess for Newton-Taylor polynomial solutions. These procedures cause that the initial guess of the Newton's method be near the exact solution and the Newton's iterations improve the final approximation. Romberg iterations in a simple case is described by algorithm 7.1 in [6], and we extend this method for vector case. For more details about extrapolation method and its convergence analysis see [7,9].

5. Total algorithm of the method

For illustration of superiority and applicability of the method we give the total algorithm for the numerical solution of the problem (1) by the proposed method. In this algorithm, N_N is the number of Newton's iterations, N_E is the number of extrapolations and other parameters are illustrated in the previous sections. In the last rectangle, each step of Newton's method is solved by the Taylor polynomial solutions, and we obtain the approximations on nodal points. This algorithm is denoted by the following flow-chart



6. The Corona virus model

Now we consider the following model of the novel corona virus epidemic in Wuhan China

$$\begin{cases} \dot{S}(t) = \Lambda - \beta_E(E)SE - \beta_I(I)SI - \beta_V(\mathrm{V})SV - \mu S, \\ \dot{E}(t) = \beta_E(E)SE + \beta_I(I)SI + \beta_V(\mathrm{V})SV - (\alpha+\mu)E, \\ \dot{I}(t) = \alpha E - (\omega+\gamma+\mu)I, \\ \dot{R}(t) = \gamma I - \mu R, \\ \dot{V}(t) = \xi_1 E + \xi_2 I - \sigma V, \end{cases} \tag{17}$$

where S,E,I,R,V are unknown functions of the model. Here S,E,I,R are susceptible, exposed, infected and recovered classes of human population respectively, and V is the concentration of the corona virus in the environmental reservoir. $\Lambda,\mu,\alpha,\gamma,\xi_1,\xi_2,\sigma$ are known positive constants, and $\beta_E(E),\beta_I(I),\beta_V(\mathrm{V})$ are the known functions of their arguments.

Equilibrium analysis of the method with some numerical simulations are described in [11]. In this work we want to obtain a good accuracy of solutions and compare with [11], hence we rewrite Tables 1,2 of [11] for illustration of the parameters. This purpose is done in Table 1.

Table 1. Definitions and values of model parameter.

Parameter	Definition	Estimated mean value	Parameter	Definition	Estimated mean value
Λ	Influx rate	2.7123 per day	$1/\alpha$	Incubation period	7 days
β_{E0}	Transmission constant between S and E	3.11×10^{-8} person/day	ω	Disease-induced death rate	0.01 per day
β_{I0}	Transmission constant between S and I	3.11×10^{-8} person/day	γ	Recovery rate	1/15 per day
β_{V0}	Transmission constant between S and V	3.11×10^{-8} person/day	σ	Removal rate of virus	1 per day
c	Transmission adjustment coefficient	1.01×10^{-4}	ξ_1	Virus shedding rate by exposed people	2.30 per person per day per ml
μ	Natural death rate	3.01×10^{-5}	ξ_2	Virus shedding rate by exposed people	0 per person per day per ml

7. Numerical Results

Example 7.1 Associated with the data of Table 1, the initial condition is set as follows [10].

$$\left(S(0), E(0), I(0), R(0), V(0)\right)^T = \left(8998505, 1000, 475, 10, 10000\right)^T.$$

Figures 1,2,3 show the numbers of $S(\mathrm{t})$, $R(\mathrm{t})$, $V(\mathrm{t})$ during 150 days. In this example the transmission rates are $\beta_E(E) = \beta_{E0}$, $\beta_I(I) = \beta_{I0}$, $\beta_V(V) = \beta_{V0}$. Figure 4 shows the numbers of exposed and infected individuals during 150 days, which is similar to the simulation result in Figure 4 of [10]. But in our work the result is computed by a convergence method and has a better accuracy than a simulation. In this example value of the quantity L_0 in (7) is

$$L_0 = \max_{1 \le i \le 5} \sum_{j=1}^{5} \sum_{k=1}^{5} \left| b_{ijk} \right| = 2\left(\beta_{E0} + \beta_{I0} + \beta_{V0}\right) = 9.52 \times 10^{-8},$$

which is excellent for numerical computations. Here we put $l = 0.1$ and hence for obtaining a result on 150 days we set $N_I = 1500$. In this example we set $N_T = 4$, $N_N = 5, N_P = 8, N_E = 5, N_{It} = 5$.

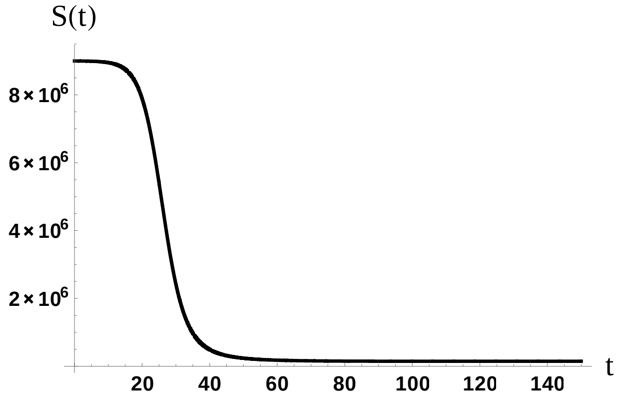


Figure 1. Variation of the $S(t)$ as a function of t for Example **7**.1.

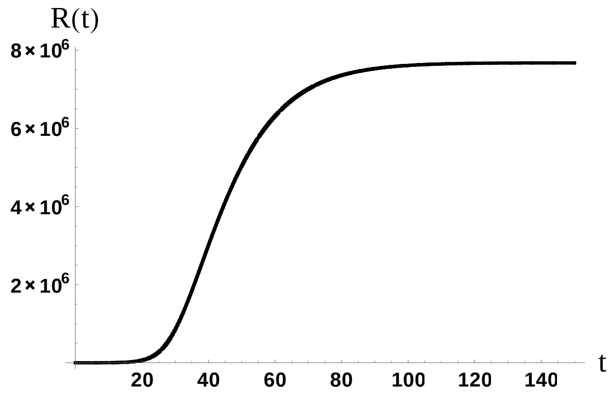


Figure 2. Variation of the $R(t)$ as a function of t for Example 7.1.

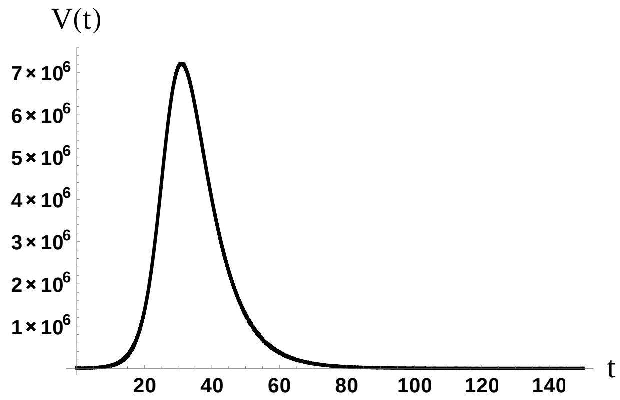


Figure 3. Variation of the $V(t)$ as a function of t for Example 7.1.

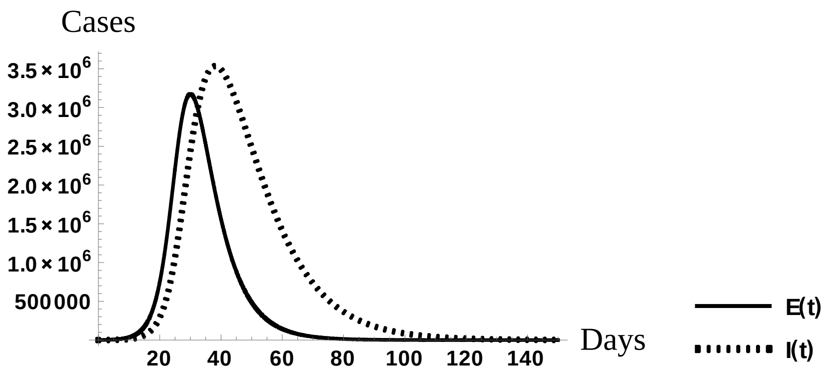


Figure 4. Variation of the $E(t), I(t)$ as functions of t for Example 7.1.

Example 7.2 For obtaining a similar sample problem which has some exact components of solution vector, suppose $E(\mathrm{t}) = 3000000\exp\left[-\left(\ln[3000](t-5)^2\right)/25\right]$, then the following semi-linear non-autonomous system

$$\begin{cases} \dot{S}(t) = \Lambda - \left(\dfrac{3000000\beta_{E0}}{\exp\left[\dfrac{\ln[3000](t-5)^2}{25}\right]} + \beta_{I0} I + \beta_{V0} V + \mu \right) S, \\ \dot{I}(t) = \dfrac{3000000\alpha}{\exp\left[\dfrac{\ln[3000](t-5)^2}{25}\right]} - (\omega + \gamma + \mu) I, \\ \dot{R}(t) = \gamma I - \mu R, \\ \dot{V}(t) = \dfrac{3000000\xi_1}{\exp\left[\dfrac{\ln[3000](t-5)^2}{25}\right]} - \sigma V, \end{cases}$$

is similar with (17). The first equation of the above system is nonlinear and other ones are linear. The linear equations are analytically solvable and exact values of $I(\mathrm{t})$, $R(\mathrm{t})$, $V(\mathrm{t})$ are available. Associated with the data of Table 1, the initial condition is set as follows $\left(S(0), I(0), R(0), V(0)\right)^T = \left(8998505, 475, 10, 10000\right)^T$. We solve this problem by two methods on $[0,150]$: the proposed method with $N_N = 5$, $N_T = 4$, $N_E = 5$, $N_P = 8$, $N_{It} = 5$, $l = 0.1$, $N_I = 1500$ and the Runge-Kutta fourth-order method with step length $h = 0.1$. Suppose we denote approximation values of I, R, V by $\tilde{I}, R, V$ respectively, then Table 2 shows relative errors of $\tilde{I}, R, V$ at the points $t_i = 15i,\ i = 1,...,10$. As Figures 5-7 and Table 2 show, the accuracy of the proposed method is very good and there is not any error propagation in this long interval. Since $V(t)$ is near zero, relative error is not a good criterion in this case, hence we give absolute errors of V at the above points by the following vector

$$(8.72\times10^{-4}, 8.52\times10^{-10}, 5.79\times10^{-16}, 8.71\times10^{-23}, 4.84\times10^{-29}, 1.45\times10^{-35}, 3.99\times10^{-42}, 2.24\times10^{-48}, 4.48\times10^{-55}, 1.76\times10^{-61}).$$

Note that at the point $t = 15$, $V(t) = 2141.6808719768565 >> 0$, and at this point the relative error is a good criterion than the absolute error. Table 2 shows both methods have good accuracy, but the proposed method has symmetric values of relative errors for $\tilde{I}, R, V$ and these values is not symmetric in Runge-Kutta fourth-order method. Figure 8 shows that both methods give same shapes for the numbers of susceptible individuals during 150 days.

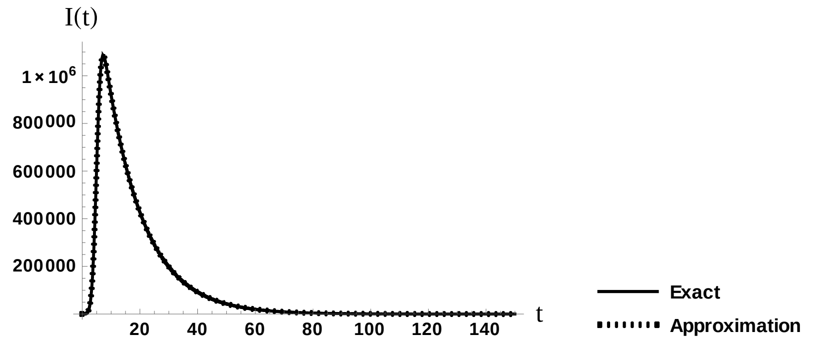


Figure 5. Variation of the $I(t)$ as a function of t for Example 7.2.

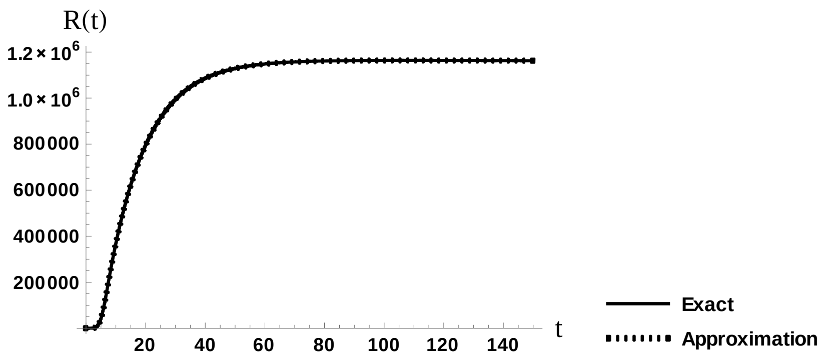


Figure 6. Variation of the $R(t)$ as a function of t for Example 7.2.

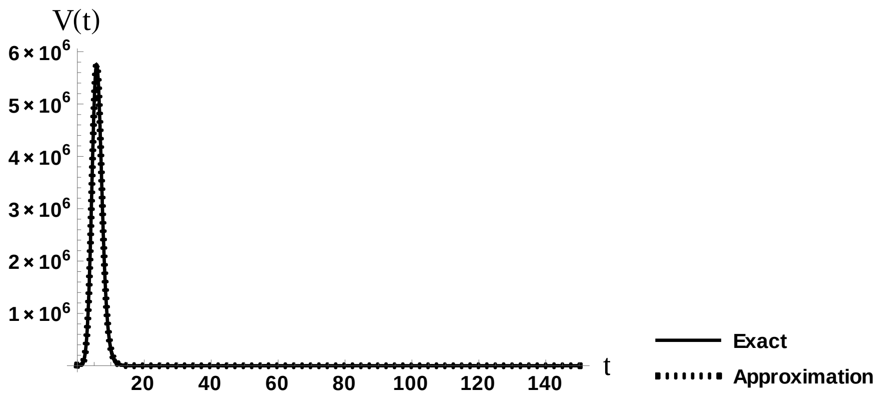


Figure 7. Variation of the $V(t)$ as a function of t for Example 7.2.

Table 2. Relative errors of $\tilde{I}, R, V$ at $t_i = 15i$ for Example 7.2.

i	Errors of Proposed method			Errors of Runge-Kutta fourth-order method		
	$\tilde{I}$	R	V	$\tilde{I}$	R	V
1	7.70×10^{-7}	6.93×10^{-8}	4.07×10^{-7}	1.05×10^{-10}	9.27×10^{-11}	5.92×10^{-6}
2	2.79×10^{-7}	4.00×10^{-7}	1.30×10^{-6}	7.16×10^{-11}	1.03×10^{-10}	1.95×10^{-5}
3	4.47×10^{-8}	1.63×10^{-7}	2.89×10^{-6}	3.82×10^{-11}	1.01×10^{-10}	3.31×10^{-5}
4	1.50×10^{-6}	3.30×10^{-6}	1.42×10^{-6}	4.79×10^{-12}	9.98×10^{-11}	4.67×10^{-5}
5	5.55×10^{-7}	2.56×10^{-6}	2.58×10^{-6}	2.86×10^{-11}	9.90×10^{-11}	6.03×10^{-5}
6	1.05×10^{-7}	2.22×10^{-6}	2.52×10^{-6}	6.20×10^{-11}	9.87×10^{-11}	7.39×10^{-5}
7	7.12×10^{-7}	2.24×10^{-7}	2.27×10^{-6}	9.54×10^{-11}	9.85×10^{-11}	8.74×10^{-5}
8	4.89×10^{-7}	1.83×10^{-6}	4.17×10^{-6}	1.29×10^{-10}	9.85×10^{-11}	1.01×10^{-4}
9	5.21×10^{-7}	9.16×10^{-7}	2.73×10^{-6}	1.62×10^{-10}	9.84×10^{-11}	1.15×10^{-4}
10	1.75×10^{-6}	3.05×10^{-6}	3.50×10^{-6}	1.96×10^{-10}	9.84×10^{-11}	1.28×10^{-4}

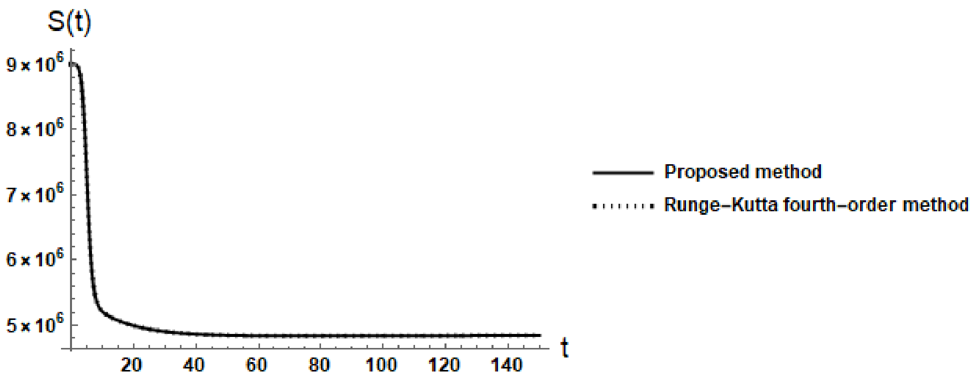


Figure 8. Variation of the $S(t)$ as a function of t for Example 7.2.

Example 7.3 In the Example 7.1, suppose the transmission rates be $\beta_E(E)=\frac{\beta_{E0}}{1+cE}$, $\beta_I(I)=\frac{\beta_{I0}}{1+cI}$, $\beta_V(\mathrm{V})=\frac{\beta_{V0}}{1+cV}$. Here we set $l=0.1$, and for obtaining a result during 300 days we set $N_I=3000$. Figures 9,10,11 show the numbers of $S(\mathrm{t})$, $R(\mathrm{t})$, $V(\mathrm{t})$ during 300 days. Figure 12 shows the numbers of exposed and infected individuals during 300 days, which is similar to the simulation result in Figure 2 of [10]. In this example, the quantity L_0 in (7) is as follows

$$
\begin{aligned}
L_0 &= \max_{1\le i\le d}\sum_{j=1}^{5}\sum_{k=1}^{5}\left|b_{ijk}\right| = 2\beta_{E0}c\left|x_1\right|\max_{cx_2\ge 0}\left|\frac{cx_2}{\left(1+cx_2\right)^3}-\frac{1}{\left(1+cx_2\right)^2}\right| \\
&+2\beta_{E0}\max_{cx_2\ge 0}\left|\frac{cx_2}{\left(1+cx_2\right)^2}-\frac{1}{1+cx_2}\right| + 2\beta_{I0}c\left|x_1\right|\max_{cx_3\ge 0}\left|\frac{cx_3}{\left(1+cx_3\right)^3}-\frac{1}{\left(1+cx_3\right)^2}\right| \\
&+2\beta_{I0}\max_{cx_3\ge 0}\left|\frac{cx_3}{\left(1+cx_3\right)^2}-\frac{1}{1+cx_3}\right| + 2\beta_{V0}c\left|x_1\right|\max_{cx_5\ge 0}\left|\frac{cx_5}{\left(1+cx_5\right)^3}-\frac{1}{\left(1+cx_5\right)^2}\right| \\
&+2\beta_{V0}\max_{cx_5\ge 0}\left|\frac{cx_5}{\left(1+cx_5\right)^2}-\frac{1}{1+cx_5}\right| \le 2\beta_0 cS_0(3m_1)+2\beta_0(3m_2)=6\beta_0(cS_0+1)=1.7\times 10^{-4}
\end{aligned}
$$

where $\left(x_1,x_2,x_3,x_4,x_5\right)^T=\left(S,E,I,R,V\right)^T$, $\left|x_1\right|=x_1=S\le S_0$, $\beta_0=\max\left\{\beta_{E0},\beta_{I0},\beta_{V0}\right\}$, $m_1=\max_{x\ge 0}\left|\frac{x}{(1+x)^3}-\frac{1}{(1+x)^2}\right|=1$, $m_2=\max_{x\ge 0}\left|\frac{x}{(1+x)^2}-\frac{1}{1+x}\right|=1$. This value of L_0 is an upper bound for Lipschitz constant. Even so this upper bound is excellent for numerical computations. Other parameters are similar to Example 7.1.

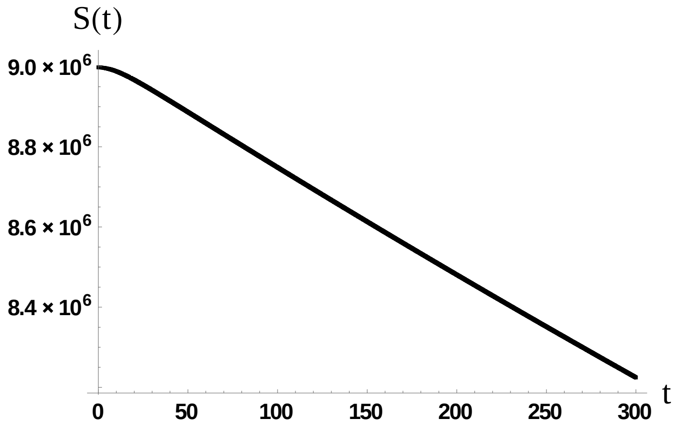


Figure 9. Variation of the $S(t)$ as a function of t for Example 7.3.

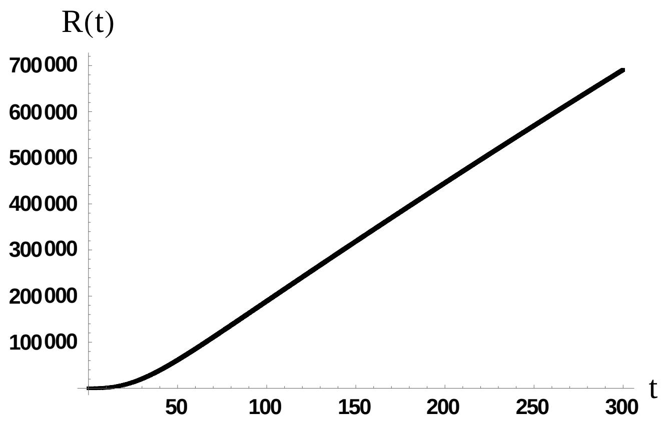


Figure 10. Variation of the $R(t)$ as a function of t for Example 7.3.

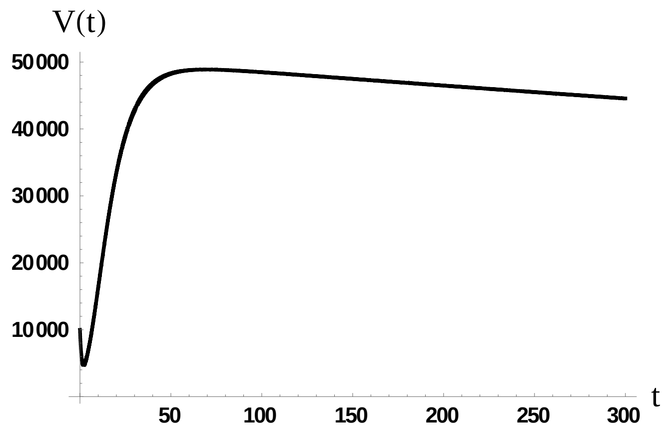


Figure 11. Variation of the $V(t)$ as a function of t for Example 7.3.

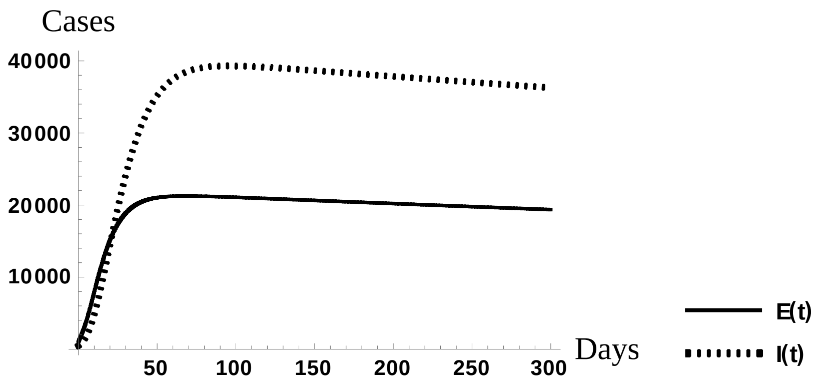


Figure 12. Variation of the $E(t), I(t)$ as functions of t for Example 7.3.

8. Conclusions

In this paper we consider a mathematical model for the novel corona virus, which is in the form (1). We consider the problem on union of many partial intervals. The proposed method solves the problem, interval by interval and the accuracy of the Newton's method caused that the propagation of the error doesn't appear. Although sufficient conditions for convergence of the Newton's method is referred to [3], but for applicability of the method it is necessary that the operator A in (4) is Lipschitz continuous and the initial guess $U^{(0)}$ is near the solution, which are true in our problem. As mentioned in the text and see in the Table 2, the proposed method has a good accuracy on long time interval which is required in medical phenomena. In future we are going to apply the proposed method for Influenza and some other infectious diseases models.

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