Journal of
Linear and Topological Algebra
Vol. 02, *No.* 04, 2013, 191- 200

Generalized notion of character amenability

A. Bodaghi[a*], F. Anousheh[b] and S. Etemad[c]

[a] *Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran.*
[b] *Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran.*
[c] *Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran.*

Received 21 September 2013; revised 7 December 2013; accepted 20 December 2013.

Abstract. This paper continues the investigation of the first author begun in part one. The hereditary properties of n-homomorphism amenability for Banach algebras are investigated and the relations between n-homomorphism amenability of a Banach algebra and its ideals are found. Analogous to the character amenability, it is shown that the tensor product of two unital Banach algebras is n-homomorphism amenable if and only if each one is n-homomorphism amenable.

Keywords: amenability, contractibility, n-homomorphism.

2010 AMS Subject Classification: 39B52, 47H10, 39B82, 46H25.

1. Introduction

In [6], Kaniuth, Lau and Pym introduced and investigated a large class of Banach algebras which they called ϕ-amenable Banach algebras (see also [7]) . Let $\Delta(\mathcal{A})$ be the set of all nonzero homomorphisms (characters) from a Banach algebra $\mathcal{A}$ onto $\mathbb{C}$. Given $\phi \in \Delta(\mathcal{A})$, a Banach algebra $\mathcal{A}$ is said to be ϕ-amenable if there exists $m \in \mathcal{A}^{**}$ such that $\langle m, \phi\rangle = 1$ and $\langle m, f\cdot a\rangle = \phi(a)\langle m, f\rangle$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Moreover, the notion of (right) character amenability was introduced and studied by Monfared [8]. Character amenability of $\mathcal{A}$ is equivalent to $\mathcal{A}$ being ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$ and $\mathcal{A}$ having

*Corresponding author.
E-mail address: abasalt.bodaghi@gmail.com (A. Bodaghi).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

a bounded right approximate identity. One of the fundamental results of Monfared was that the group algebra $L^1(G)$ is a (right) character amenable Banach algebra if and only if G is an amenable locally compact group. For $\phi \in \Delta(\mathcal{A})$, Hu et al. [5] introduced and studied the notion of ϕ-contractibility of $\mathcal{A}$. In fact, $\mathcal{A}$ is called ϕ-contractible if there exists a (right) ϕ-diagonal for $\mathcal{A}$; that is, an element $\mathbf{m}$ in the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\phi(\pi(\mathbf{m})) = \mathbf{1}$ and $a \cdot m = \phi(a)\mathbf{m}$ for all $a \in \mathcal{A}$, where π denotes the product morphism from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ into $\mathcal{A}$ given by $\pi(a \otimes b) = ab$ for all $a, b \in \mathcal{A}$.

Let $n \in \mathbb{N}$ and let $\mathcal{A}$ be a Banach algebra. A linear map $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is called n-homomorphism if $\varphi(a_1a_2\cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)$ for all $a_1, a_2, ..., a_n \in \mathcal{A}$. We denote the set of all nonzero n-homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ by $\Delta^{(n)}(\mathcal{A})$. It is clear that every character on $\mathcal{A}$ belongs to $\Delta^{(n)}(\mathcal{A})$, but the converse is not true, in general. However, each element of $\Delta^{(n)}(\mathcal{A})$ can induces a character on $\mathcal{A}$ [1, Theorem 2.3]. Let $m \in \mathcal{A}^{**}$. Consider $\phi \in \Delta^{(n)}(\mathcal{A})$ such that $\phi(u) = 1$ for some $u \in \mathcal{A}$. Then m is said to be *n-ϕ-mean* on $\mathcal{A}^*$ (at u) if $m(\phi) = 1$ and $m(f \cdot a) = \phi(u^n a)m(f)$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. $\mathcal{A}$ is called *n-ϕ-amenable* if there exists a n-ϕ-mean m on $\mathcal{A}^*$. We say $\mathcal{A}$ is *n-0-amenable* if $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$, for any Banach $\mathcal{A}$-bimodule X for which the left action of $\mathcal{A}$ on X is zero, where $\mathcal{H}^1(\mathcal{A}, X^*)$ is the first cohomology group of $\mathcal{A}$ with coefficients in X^*. Also $\mathcal{A}$ is called *n-homomorphism amenable* if $\mathcal{A}$ is called n-ϕ-amenable for all $\phi \in \Delta^{(n)}(\mathcal{A}) \cup \{0\}$. The first author [1] showed that $L^1(G)$ is a n-homomorphism amenable if and only if G is an amenable locally compact group.

In this paper we study the hereditary properties of n-homomorphism amenability for a Banach algebra. We also introduce the concept of *n-φ-contractibility* for Banach algebras and charactrize such Banach algebras.

2. Main Results

Let $\mathcal{A}$ be a Banach algebra, and X be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \longrightarrow X$ is called a *derivation* if $D(ab) = D(a) \cdot b + a \cdot D(b)$, for all $a, b \in \mathcal{A}$. For each $x \in X$, we define a map $D_x : \mathcal{A} \longrightarrow X$ by

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in \mathcal{A}).$$

It is easy to see that D_x is a derivation. Derivations of this form are called *inner derivations* [4].

Let X be a Banach $\mathcal{A}$-bimodule. Then the dual space X^* of X is also a Banach $\mathcal{A}$-bimodule by the following module actions:

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \quad , \quad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \quad (a \in \mathcal{A}, x \in X, f \in X^*).$$

A Banach algebra $\mathcal{A}$ is said to be *amenable* if every continuous derivation from $\mathcal{A}$ into X^* is inner, for every Banach $\mathcal{A}$-bimodule X.

Let $\varphi \in \Delta^{(n)}(\mathcal{A})$ and take $u \in \mathcal{A}$ such that $\varphi(u) = 1$. If X is a Banach space, then X is a left Banach $\mathcal{A}$-module by the following action:

$$a \cdot x = \varphi(u^n a)x \qquad (a \in \mathcal{A}, x \in X). \tag{1}$$

Note that in this case, the right action of $\mathcal{A}$ on the dual $\mathcal{A}$-bimodule X^* is defined by $f \cdot a = \varphi(u^n a)f$ for all $a \in \mathcal{A}$ and $f \in X^*$.

From now on, we assume that $\mathcal{A}^{**}$, the second dual of the Banach algebra $\mathcal{A}$, is equipped with the first Arens product [2]. The canonical images of $a \in \mathcal{A}$ and $\mathcal{A}$ in $\mathcal{A}^{**}$ will be denoted by $\widehat{a}$ and $\widehat{\mathcal{A}}$, respectively. It is proved in [3, Theorem 4.1] that if

$\varphi \in \Delta^{(n)}(\mathcal{A})$, then $\varphi^{**} \in \Delta^{(n)}(\mathcal{A}^{**})$. Therefore each $\varphi \in \Delta^{(n)}(\mathcal{A})$ extends uniquely to element $\tilde{\varphi} = \varphi^{**} \in \Delta^{(n)}(\mathcal{A}^{**})$. It is well-known that $\ker\varphi$ is weak*-dense in $\ker\varphi^{**}$ and $\ker\varphi^{**} = (\ker\varphi)^{**}$ (for further details refer to [2]).

One should remember that if φ is a nonzero multiplicative linear functional on $\mathcal{A}$, then the left module structure (1) and the definition of n-φ-mean (n-homomorphism amenability) and φ-amenability (Character amenability) of $\mathcal{A}$ coincide (see [6] and [8]).

Theorem 2.1 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$ such that $\varphi(u) = 1$. Then the following are equivalent:

(i) $\mathcal{A}$ is n-φ-amenable (at u);
(ii) If X is a Banach $\mathcal{A}$-bimodule in which the left action is given by $a \cdot x = \varphi(u^n\, a)x$, for all $a \in \mathcal{A}$ and $x \in X$, then $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$;
(iii) If $\mathcal{A}$ acts on $(\ker\varphi)^*$ from left as $a \cdot f = \varphi(u^n a)f$ for all $f \in (\ker\varphi)^*$ and naturally from right, then every continuous derivation from $\mathcal{A}$ into $\ker\varphi^{**}$ is inner.

Proof. The equivalence of (i) and (ii) has been shown in [1, Theorem 2.1]. The implication (ii)$\Longrightarrow$(iii) is trivial. For (iii)$\Longrightarrow$(i), define an $\mathcal{A}$-bimodule structure on $X = \mathcal{A}$ by $a \cdot x = ax$ and $x \cdot a = \varphi(u^n a)x$ for every $a \in \mathcal{A}$ and $x \in X$. Therefore $\mathcal{A}^*$ is an $\mathcal{A}$-bimodule such that $a \cdot f = \varphi(u^n a)f$ and $f \cdot a$ is as usual for every $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Also the $\mathcal{A}$-bimodule structure on $\mathcal{A}^{**}$ is defined by $m \cdot a = \varphi(u^n a)m$ and $a \cdot m$ is as usual for every $m \in \mathcal{A}^{**}$ and $a \in \mathcal{A}$. It is easy to check that $\ker\varphi$ is an $\mathcal{A}$-submodule of X and $(\ker\varphi)^{\perp}$ is an $\mathcal{A}$-submodule of $\mathcal{A}^*$ and $(\ker\varphi)^* \cong \frac{\mathcal{A}^*}{(\ker\varphi)^{\perp}}$. So for any $g \in (\ker\varphi)^*$, $a \cdot g = \varphi(u^n a)g$ and the right action is the natural one. Define $D : \mathcal{A} \longrightarrow \mathcal{A}^{**}$ through

$$D(a) = a \cdot \widehat{u^n} - \widehat{u^n} \cdot a = \widehat{au^n} - \varphi(u^n a)\widehat{u^n} = (au^n - \varphi(u^n a)u^n)\widehat{}.$$

Therefore D is a derivation. We have

$$\varphi(au^n - \varphi(u^n a)u^n) = \varphi(au^n) - \varphi(u^n a)\varphi(u^n) = \varphi(au^n) - \varphi(u^n a) = 0.$$

So $au^n - \varphi(u^n a)u^n \in ker\varphi$ and thus $D(a) \in \ker\varphi^{**}$. This shows that D is a derivation from $\mathcal{A}$ into $\ker\varphi^{**}$. By hypothesis there is $p \in \ker\varphi^{**}$ such that $D(a) = a \cdot p - p \cdot a = a \cdot p - \varphi(u^n a)p$ for all $a \in \mathcal{A}$. Let $m = \widehat{u^n} - p \in \mathcal{A}^{**}$. Then $a \cdot m = m \cdot a = \varphi(u^n a)m$ where $a \in \mathcal{A}$. So for each $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$, we have $m(f \cdot a) = (a \cdot m)(f) = \varphi(u^n a)m(f)$. Also,

$$m(\varphi) = \widehat{u^n}(\varphi) - p(\varphi) = \varphi(u^n) - p(\varphi) = \varphi(u)^n = 1.$$

Therefore m is a n-φ-mean on $\mathcal{A}$ (at u). This completes the proof. ■

It is shown in [1] that the definition of n-ϕ-amenability is independent from the choice of u and it is enough that the above Theorem holds for some $u \in \mathcal{A}$ with $\phi(u) = 1$.

Proposition 2.2 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$ such that $\varphi(u) = 1$. Then $\mathcal{A}$ is n-φ-amenable if and only if there exists a bounded net $(u_\alpha)_\alpha$ in $\mathcal{A}$ such that $\|au_\alpha - \varphi(u^n a)u_\alpha\| \to 0$ for all $a \in \mathcal{A}$ and $\varphi(u_\alpha) = 1$ for all α.

Proof. It is similar to the proof of [6, Theorem 1.4] with some modifications. ■

For any Banach algebra $\mathcal{A}$ and Banach $\mathcal{A}$-bimodule X, consider the following set

$$Z(\mathcal{A}, X^*) = \{f \in X^* : a \cdot f = f \cdot a \text{ for all } a \in \mathcal{A}\}.$$

The proof of the next result is similar to the proof of implication (i)⇒ (ii) from [6, Theorem 1.5], so it is omitted.

Proposition 2.3 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$ such that $\varphi(u) = 1$. If $\mathcal{A}$ is n-φ-amenable then for any Banach $\mathcal{A}$-bimodule X such that $a \cdot x = \varphi(u^n a)x$ for all $x \in X$ and $a \in \mathcal{A}$ and any Banach submodule Y of X, every element of $Z(\mathcal{A}, Y^*)$ extends to some element of $Z(\mathcal{A}, X^*)$.

Let $\mathcal{A}$ be a unital Banach algebra with identity e and $\varphi \in \Delta^{(n)}(\mathcal{A})$. Then $\varphi(e)^{n-1} = 1$, and thus $\varphi(e) = \omega_k$ for some k where $0 \leqslant k < n-1$ in which $\omega_k = e^{\frac{2k\pi}{n-1}i}$ is the $(n-1)$-th root of 1.

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ be the projective tensor product of $\mathcal{A}$ and $\mathcal{B}$. For $\varphi \in \Delta^{(n)}(\mathcal{A})$ and $\psi \in \Delta^{(n)}(\mathcal{B})$, consider $\varphi \otimes \psi$ by $(\varphi \otimes \psi)(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. It is clear that $\varphi \otimes \psi \in \Delta^{(n)}(\mathcal{A}\widehat{\otimes}\mathcal{B})$. Now, suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital with identities $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Let $f \in \Delta^{(n)}(\mathcal{A}\widehat{\otimes}\mathcal{B})$. Define $\varphi, \psi : \mathcal{A} \longrightarrow \mathbb{C}$ by $\varphi(a) := f(a \otimes e_{\mathcal{B}})$ and $\psi(b) := [f(e_{\mathcal{A}} \otimes e_{\mathcal{B}})]^{n-2} f(e_{\mathcal{A}} \otimes b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. One can easily check that $\varphi \in \Delta^{(n)}(\mathcal{A})$ and $\psi \in \Delta^{(n)}(\mathcal{B})$. Since f is a n-homomorphism, for each $a_1, \ldots, a_n \in \mathcal{A}$, we can write

$$\varphi(a_1 \cdots a_n) = f(a_1 \cdots a_n \otimes e_{\mathcal{B}}) = f(a_1 \otimes e_{\mathcal{B}}) \cdots f(a_n \otimes e_{\mathcal{B}}) = \varphi(a_1) \cdots \varphi(a_n)$$

and

$$\begin{aligned}
\psi(b_1 \cdots b_n) &= [f(e_{\mathcal{A}} \otimes e_{\mathcal{B}})]^{n-2} f(e_{\mathcal{A}} \otimes b_1 \cdots b_n) \\
&= [f(e_{\mathcal{A}} \otimes e_{\mathcal{B}})]^{(n-1)(n-2)} [f(e_{\mathcal{A}} \otimes e_{\mathcal{B}})]^{n-2} f(e_{\mathcal{A}} \otimes b_1 \cdots b_n) \\
&= [f(e_{\mathcal{A}} \otimes e_{\mathcal{B}})]^{(n)(n-2)} f(e_{\mathcal{A}} \otimes b_1) \cdots f(e_{\mathcal{A}} \otimes b_n) \\
&= \psi(b_1) \cdots \psi(b_n)
\end{aligned}$$

for all $b_1, \ldots, b_n \in \mathcal{B}$. On the other hand, we have

$$(\varphi \otimes \psi)(a \otimes b) = \varphi(a)\psi(b) = f(a \otimes e_{\mathcal{B}})[f(e_{\mathcal{A}} \otimes e_{\mathcal{B}})]^{n-2} f(e_{\mathcal{A}} \otimes b) = f(a \otimes b).$$

Summing up:

$$\Delta^{(n)}(\mathcal{A}\widehat{\otimes}\mathcal{B}) = \{\varphi \otimes \psi : \varphi \in \Delta^{(n)}(\mathcal{A}), \psi \in \Delta^{(n)}(\mathcal{B})\}.$$

The following theorem shows that n-$\varphi \otimes \psi$-amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ (at $u \otimes v$) is equivalent to n-φ-amenability of $\mathcal{A}$ (at u) and n-ψ-amenability of $\mathcal{B}$ (at v). The idea of the proof is taken from [6, Theorem 3.3].

Theorem 2.4 Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras, let $\varphi \in \Delta^{(n)}(\mathcal{A})$ and $\psi \in \Delta^{(n)}(\mathcal{B})$ such that $\varphi(u) = 1$ and $\psi(v) = 1$ for some $u \in \mathcal{A}$ and $v \in \mathcal{B}$. Then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is n-$\varphi \otimes \psi$-amenable *(*at $u \otimes v$*)* if and only if $\mathcal{A}$ is n-φ-amenable *(*at u*)* and $\mathcal{B}$ is n-ψ-amenable *(*at v*)*.

Proof. First, assume that m is an n-$\varphi \otimes \psi$-mean on $(\mathcal{A}\widehat{\otimes}\mathcal{B})^*$ (at $u \otimes v$). Define $\bar{m} \in \mathcal{A}^{**}$ by $\bar{m}(f) = m(f \otimes \psi)$, where $f \in \mathcal{A}^*$. Then $\bar{m}(\varphi) = m(\varphi \otimes \psi) = 1$. We know that

$\psi \cdot v^{n-1} = \psi$. Therefore

$$\begin{aligned}\bar{m}(f \cdot a) &= m(f \cdot a \otimes \psi) = m(f \cdot a \otimes \psi \cdot v^{n-1}) \\ &= m((f \otimes \psi) \cdot (a \otimes v^{n-1})) \\ &= \varphi(u^n a)\psi(v^n v^{n-1}) m(f \otimes \psi) \\ &= \varphi(u^n a)\bar{m}(f)\end{aligned}$$

for every $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Note that we have used the equality $\psi(v^n v^{n-1}) = \psi(v^n) = 1$ in the above statements. Hence $\mathcal{A}$ is n-φ-amenable. Similarly, one can show that $\mathcal{B}$ is n-ψ-amenable.

Conversely, let $\mathcal{A}$ be n-φ- amenable and let $\mathcal{B}$ be n-ψ-amenable. To prove the n-$\varphi \otimes \psi$-amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras with identities $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. By Theorem 2.1, (ii)$\Longrightarrow$(i) it suffices to show that if X is a Banach $\mathcal{A}\widehat{\otimes}\mathcal{B}$-bimodule such that $(a \otimes b) \cdot x = \varphi(u^n a)\psi(v^n b)x$ for all $x \in X$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then $H^1(\mathcal{A}\widehat{\otimes}\mathcal{B}, X^*) = \{0\}$. Let $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \longrightarrow X^*$ be a continuous derivation from $\mathcal{A}\widehat{\otimes}\mathcal{B}$ into X^*. Then it is obvious that the mapping $D_{\mathcal{A}} : \mathcal{A} \longrightarrow X^*$ defined by $D_{\mathcal{A}}(a) = D(a \otimes e_{\mathcal{B}})$ is a continuous derivation of $\mathcal{A}$ into X^*. Since $\mathcal{A}$ is n-φ-amenable, by Theorem 2.1, (i)$\Longrightarrow$(ii) there exists $f \in X^*$ such that

$$D_{\mathcal{A}}(a) = D_f(a \otimes e_{\mathcal{B}}) = (a \otimes e_{\mathcal{B}}) \cdot f - f \cdot (a \otimes e_{\mathcal{B}})$$

for every $a \in \mathcal{A}$. Then $\tilde{D} = D - D_f$ vanishes on $\mathcal{A} \otimes e_{\mathcal{B}}$. Since $\mathcal{A} \otimes e_{\mathcal{B}}$ and $e_{\mathcal{A}} \otimes \mathcal{B}$ commute,

$$(a \otimes e_{\mathcal{B}}) \cdot \tilde{D}(e_{\mathcal{A}} \otimes b) = \tilde{D}(a \otimes b) = \tilde{D}(e_{\mathcal{A}} \otimes b) \cdot (a \otimes e_{\mathcal{B}})$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Therefore $D_g(\mathcal{A} \otimes e_{\mathcal{B}}) = \{0\}$ for every $g \in \overline{\tilde{D}(e_{\mathcal{A}} \otimes \mathcal{B})}$, the w^*-clousre of $\tilde{D}(e_{\mathcal{A}} \otimes \mathcal{B})$ in X^*. Now, let Y be the annihilator of $\tilde{D}(e_{\mathcal{A}} \otimes \mathcal{B})$ in X. Considering X as a Banach $\mathcal{B}$-bimodule (as we did with $\mathcal{A}$ above), Y is a $\mathcal{B}$-submodule. Indeed, for $y \in Y$ and $b_1, b_2 \in \mathcal{B}$ we can write

$$\langle \tilde{D}(e_{\mathcal{A}} \otimes b_1), (e_{\mathcal{A}} \otimes b_2) \cdot y \rangle = \langle \psi(v^n b_2)\tilde{D}(e_{\mathcal{A}} \otimes b_1), y \rangle = 0$$

and

$$\begin{aligned}\langle \tilde{D}(e_{\mathcal{A}} \otimes b_1), y \cdot (e_{\mathcal{A}} \otimes b_2) \rangle &= \langle (e_{\mathcal{A}} \otimes b_2) \cdot \tilde{D}(e_{\mathcal{A}} \otimes b_1), y \rangle \\ &= \langle \tilde{D}(e_{\mathcal{A}} \otimes b_2 b_1), y \rangle - \langle \tilde{D}(e_{\mathcal{A}} \otimes b_2) \cdot (e_{\mathcal{A}} \otimes b_1), y \rangle \\ &= -\langle \tilde{D}(e_{\mathcal{A}} \otimes b_2), (e_{\mathcal{A}} \otimes b_1) \cdot y \rangle = 0.\end{aligned}$$

Hence, X/Y is a Banach $\mathcal{B}$-bimodule satisfying $b \cdot (x + Y) = \psi(v^n b)(x + Y)$, $x \in X$, $b \in \mathcal{B}$ and $(X/Y)^* = \overline{\tilde{D}(e_{\mathcal{A}} \otimes \mathcal{B})}$. Since the restriction of $\tilde{D}$ to $e_{\mathcal{A}} \otimes \mathcal{B}$, that is $\tilde{D}|_{e_{\mathcal{A}} \otimes \mathcal{B}}$ defines a continuous derivation from $\mathcal{B}$ into $\overline{\tilde{D}(e_{\mathcal{A}} \otimes \mathcal{B})}$ and $\mathcal{B}$ is n-ψ-amenable, there is $g \in \overline{\tilde{D}(e_{\mathcal{A}} \otimes \mathcal{B})} \subseteq X^*$ such that

$$\tilde{D}(e_{\mathcal{A}} \otimes b) = D_g(e_{\mathcal{A}} \otimes b) = (e_{\mathcal{A}} \otimes b) \cdot g - g \cdot (e_{\mathcal{A}} \otimes b)$$

for all $b \in \mathcal{B}$. Also $g \in \overline{\tilde{D}(e_{\mathcal{A}} \otimes \mathcal{B})}$ implies that $D_g|_{\mathcal{A} \otimes e_{\mathcal{B}}} = 0$. Thus, $\tilde{D} - D_g$ is a continuous derivation of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ that vanishes on $\mathcal{A} \otimes e_{\mathcal{B}}$ and on $e_{\mathcal{A}} \otimes \mathcal{B}$. Since $(\mathcal{A} \otimes e_{\mathcal{B}}) \cup (e_{\mathcal{A}} \otimes \mathcal{B})$ generates $\mathcal{A}\widehat{\otimes}\mathcal{B}$, it follows that $\tilde{D} - D_g$ vanishes on all of $\mathcal{A}\widehat{\otimes}\mathcal{B}$. This shows that $D = D_f + D_g = D_{f+g}$, as required. Thus, $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is n-$\varphi \otimes \psi$-amenable (at $u \otimes v$). ■

Note that in the first part of the above proof, algebras being unital are not necessary. In analogy with φ-amenability, we have the following theorem for n-φ-amenability of the homomorphic image.

Theorem 2.5 Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $h : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous homomorphism with dense range. If $\varphi \in \Delta^{(n)}(\mathcal{B})$ and $\mathcal{A}$ is n-$\varphi \circ h$-amenable *(*at u*)*, then $\mathcal{B}$ is n-φ-amenable *(at $h(u)$)*.

Proof. Let $m \in \mathcal{A}^{**}$ satisfying $m(\varphi \circ h) = 1$ and $m(f \cdot a) = (\varphi \circ h)(u^n a)m(f)$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Define $m' \in \mathcal{B}^{**}$ by $m'(g) = m(g \circ h)$, $g \in B^*$. Then $m'(\varphi) = m(\varphi \circ h) = 1$. Since $h(\mathcal{A})$ is dense in $\mathcal{B}$, we have $m'(g \cdot b) = \varphi((h(u))^n b)m'(g)$ for all $b \in \mathcal{B}$ and $g \in \mathcal{B}^*$. It suffices to verify this equation for b of the form $b = h(a)$, $a \in \mathcal{A}$. Now we have

$$\langle (g \cdot h(a)) \circ h, a' \rangle = \langle g, h(a)h(a') \rangle = \langle g \circ h, aa' \rangle = \langle (g \circ h) \cdot a, a' \rangle$$

for all $a, a' \in \mathcal{A}$ and thus

$$\begin{aligned} m'(g \cdot b) &= m'(g \cdot h(a)) = m((g \cdot h(a)) \circ h) \\ &= m((g \circ h) \cdot a) = (\varphi \circ h)(u^n a)m(g \circ h) \\ &= \varphi(h(u^n a))m(g \circ h) = \varphi((h(u))^n h(a))m(g \circ h) \\ &= \varphi((h(u))^n b)m'(g) \end{aligned}$$

for all $g \in \mathcal{B}^*$ and $a \in \mathcal{A}$, as required. ∎

Corollary 2.6 Let $\mathcal{A}$ be a Banach algebra and $\mathcal{I}$ be an ideal in $\mathcal{A}$. If $\mathcal{A}$ is a n-homomorphism amenable, then so is $\mathcal{A}/\mathcal{I}$.

The following proposition shows the relationship between n-φ-amenability of a Banach algebra and its second dual. As we saw already, $\varphi \in \Delta^{(n)}(\mathcal{A})$ extends uniquely to $\tilde{\varphi} \in \Delta^{(n)}(\mathcal{A}^{**})$.

Proposition 2.7 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$. Then $\mathcal{A}$ is n-φ-amenable if and only if $\mathcal{A}^{**}$ is n-$\tilde{\varphi}$-amenable.

Proof. Let m be an n-φ-mean on $\mathcal{A}^*$ (at u). For each $p \in \mathcal{A}^{**}$ and $\lambda \in \mathcal{A}^{***}$, take bounded nets $(a_j) \in \mathcal{A}$ and $(f_k) \in \mathcal{A}^*$ with $\widehat{a}_j \xrightarrow{w^*} p$ and $\widehat{f}_k \xrightarrow{w^*} \lambda$. We identify m as an element $\widehat{m} \in \mathcal{A}^{****}$. Thus $\widehat{m}(\tilde{\varphi}) = 1$ and

$$\begin{aligned} \langle \widehat{m}, \lambda \cdot p \rangle &= \langle \lambda, p \cdot m \rangle = \lim_k \langle p \cdot m, f_k \rangle = \lim_k \langle p, m \cdot f_k \rangle \\ &= \lim_k \lim_j \langle m \cdot f_k, a_j \rangle = \lim_k \lim_j \langle m, f_k \cdot a_j \rangle \\ &= \lim_k \lim_j \varphi(u^n a_j) \langle m, f_k \rangle = \lim_j \varphi(u^n a_j) \lim_k \langle m, f_k \rangle \\ &= \tilde{\varphi}(u^n p) \langle \widehat{m}, \lambda \rangle. \end{aligned}$$

Consequently, $\mathcal{A}^{**}$ is n-$\tilde{\varphi}$-amenable.

Conversely, suppose that $\Phi \in \mathcal{A}^{****}$ satisfies $\Phi(\tilde{\varphi}) = 1$ and $\Phi(\lambda \cdot p) = \tilde{\varphi}(u^n p)\Phi(\lambda)$ for all $p \in \mathcal{A}^{**}$ and $\lambda \in \mathcal{A}^{***}$. Then, the restriction of Φ to $\mathcal{A}^*$ is an n-φ-mean on $\mathcal{A}^*$. ∎

To prove the next theorem, we need the following lemma.

Lemma 2.8 Let $\mathcal{A}$ be a Banach algebra and I be a closed ideal of $\mathcal{A}$. If $\varphi \in \Delta^{(n)}(\mathcal{A})$ such that $\bar{\varphi} = \varphi|_I \neq 0$, then n-φ-amenability of $\mathcal{A}$ implies n-$\bar{\varphi}$-amenability of I.

Proof. Since $\varphi|_I \neq 0$, there is $x \in I$ such that $\varphi(x) = 1$. By the paragraph after Theorem 2.1 in [1], we can suppose that $u \in I$. Assume that $m \in \mathcal{A}^{**}$ is n-φ-mean on $\mathcal{A}^*$ (at u). Then $m(f \cdot a) = 0$ for all $a \in I$ and $f \in I^\perp$ (since $f \cdot a = 0$), that is, $\varphi(u^n a)m(f) = 0$. It follows from $\varphi(u^2) \neq 0$ and $\varphi|_I \neq 0$ that $m(f) = 0$ for all $f \in I^\perp$. Thus m gives rise to a bounded linear functional $\tilde{m}$ on I^* defined by $\tilde{m}(g) = m(f)$, for $g \in I^*$, where f is an arbitrary element of $\mathcal{A}^*$ extending g. We have $\tilde{m}(\bar{\varphi}) = m(\varphi) = 1$. For any $g \in I^*$ and $a \in I$

$$\tilde{m}(g \cdot a) = m(f \cdot a) = \varphi(u^n a)m(f) = \varphi(u^n a)\tilde{m}(g).$$

Note that in the above equalities, $(f \cdot a)|_I = g \cdot a$. Therefore I is n-$\bar{\varphi}$-amenable. ■

Theorem 2.9 Suppose that $\mathcal{A}$ is a n-homomorphism amenable Banach algebra and I is a closed ideal of $\mathcal{A}$. Then, I is n-homomorphism amenable if and only if I has a bounded right approximate identity.

Proof. It is well-known that the existence of a bounded right approximate identity for $\mathcal{A}$ is equivalent to $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X for which the left module action of $\mathcal{A}$ is $a.x = 0$. For the converse, let I have a bounded right approximate identity (a_j). In view of Lemma 2.8 it is sufficient to show that every $\psi \in \Delta^{(n)}(I)$ extends to some $\tilde{\psi} \in \Delta^{(n)}(\mathcal{A})$. The kernel of ψ, say J, is a closed right ideal in $\mathcal{A}$. If $x \in J$ and $a \in \mathcal{A}$, then $xa = \lim xaa_j^{n-1}$. We also have $\psi(xaa_j^{n-1}) = \psi(x)\psi(aa_j)\psi(a_j)^{n-2} = 0$. Thus $xaa_j^{n-1} \in J$ and so $xa \in J$.

Let $u \in I$ such that $\psi(u) = 1$. Then u^{n-1} is an identity of I modulo J, that is, for any $x \in I$, $x - xu^{n-1} \in J$ and $x - u^{n-1}x \in J$. Define $\tilde{\psi} : \mathcal{A} \to \mathbb{C}$ by $\tilde{\psi}(x) = \psi(u^{n-1}x)$ for every $x \in \mathcal{A}$. We wish to show that $\tilde{\psi} \in \Delta^{(n)}(\mathcal{A})$ and $\tilde{\psi}|I = \psi$. First note that for each $x, y \in \mathcal{A}$ we have

$$u^{n-1}xu^{n-1}y - u^{n-1}xy = (u^{n-1}xu^{n-1} - u^{n-1}x)y \in J.$$

Now, for each $x_1, x_2, \cdots, x_n \in \mathcal{A}$ we get

$$\begin{aligned}
\tilde{\psi}(x_1)\tilde{\psi}(x_2)\cdots\tilde{\psi}(x_{n-1})\tilde{\psi}(x_n) &= \psi(u^{n-1}x_1)\psi(u^{n-1}x_2)\cdots\psi(u^{n-1}x_{n-1})\psi(u^{n-1}x_n)\\
&= \psi(u^{n-1}x_1u^{n-1}x_2\cdots u^{n-1}x_{n-1}u^{n-1}x_n)\\
&= \psi(u^{n-1}x_1u^{n-1}x_2\cdots u^{n-1}x_{n-1}x_n)\\
&= \cdots = \psi(u^{n-1}x_1x_2\cdots x_{n-1}x_n)\\
&= \tilde{\psi}(x_1x_2\cdots x_{n-1}x_n).
\end{aligned}$$

The above statements show that I is n-homomorphism amenable. ■

Definition 2.10 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$. $\mathcal{A}$ is said to be n-φ-contractible, if every continuous derivation $D : \mathcal{A} \longrightarrow X$ is inner, whenever X is a Banach $\mathcal{A}$-bimodule with left action of $\mathcal{A}$ over X is given by $a \cdot x = \varphi(u^n a)x$ with $\varphi(u) = 1$.

Theorem 2.11 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$ such that $\varphi(u) = 1$ for some $u \in \mathcal{A}$. Then the following are equivalent:

(i) $\mathcal{A}$ is n-φ-contractible (at u);
(ii) There exists $m \in \mathcal{A}$ such that $\varphi(m) = 1$ and $m \cdot a = \varphi(u^n a)m$ for all $a \in \mathcal{A}$.

Proof. (i)$\Longrightarrow$ (ii): Define an $\mathcal{A}$-bimodule structure on $X = \mathcal{A}$ by $a \cdot x = \varphi(u^n a)x, x \cdot a = xa$ for all $a \in \mathcal{A}$ and $x \in X$. Since $\varphi(D(a)) = \varphi(u^n a)\varphi(u^n) - \varphi(u^n a) = 0$ and $\ker \varphi$ is an $\mathcal{A}$-submodule of X,

$$D(a) = a \cdot u^n - u^n \cdot a = \varphi(u^n a)u^n - u^n \cdot a, (a \in \mathcal{A})$$

defines a continuous derivation from $\mathcal{A}$ into $\ker \varphi$. Due to n-φ-contractibility of $\mathcal{A}$, there exists $p \in \ker \varphi$ such that $D(a) = a \cdot p - p \cdot a$ for all $a \in \mathcal{A}$. Hence, the element $m = u^n - p \in \mathcal{A}$ has the required properties because $\varphi(m) = \varphi(u^n - p) = 1$ and

$$\varphi(u^n a)u^n - u^n \cdot a = D(a) = a \cdot p - p \cdot a = \varphi(u^n a)p - p \cdot a.$$

Therefore

$$\varphi(u^n a)m = \varphi(u^n a)(u^n - p) = (u^n - p) \cdot a = m \cdot a$$

for all $a \in \mathcal{A}$.

(ii)$\Longrightarrow$ (i): Suppose that $m \in \mathcal{A}$ with $\varphi(m) = 1$ and $m \cdot a = \varphi(u^n a)m$ for all $a \in \mathcal{A}$. Let X be an $\mathcal{A}$-bimodule with the left action $a \cdot x = \varphi(u^n a)x$ for all $a \in \mathcal{A}$ and $x \in X$. Put $x_0 = D(m)$. Then

$$\begin{aligned} x_0 \cdot a &= D(m) \cdot a = D(m \cdot a) - m \cdot D(a) \\ &= D(m \cdot a) - \varphi(u^n m)D(a) \\ &= \varphi(u^n a)D(m) - \varphi(u^2)\overbrace{\varphi(u)\ldots\varphi(u)}^{(n-2)-times}\varphi(m)D(a) \\ &= \varphi(u^n a)D(m) - \varphi(u^2)D(a). \end{aligned}$$

Thus

$$\varphi(u^2)D(a) = \varphi(u^n a)x_0 - x_0 \cdot a = a \cdot x_0 - x_0 \cdot a \quad (a \in \mathcal{A}).$$

Hence $D(a) = a \cdot x - x \cdot a$, where $x = \frac{1}{\varphi(u^2)}x_0$. Therefore, $\mathcal{A}$ is n-φ-contractible. ■

Proposition 2.12 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$ such that $\varphi(u) = 1$ for some $u \in \mathcal{A}$. If $\ker(\varphi)$ has a left identity, then $\mathcal{A}$ is n-φ-contractible (at u).

Proof. Choose $b \in \ker(\varphi)$ such that $ba = a$ for every $a \in \ker(\varphi)$. Let $m = u^n - bu^n$. We have

$$\varphi(m) = \varphi(u^n) - \varphi(bu^n) = \varphi(u^n) - \varphi(b)\varphi(u)^{n-2}\varphi(u^2) = 1.$$

If $a \in \ker(\varphi)$, then $ma = u^n a - bu^n a = u^n a - u^n a = 0$ (since $u^n a \in \ker(\varphi)$). But $\varphi(u^n a) = 0$ implies that $\varphi(u^n a)m = 0$ and so $ma = \varphi(u^n a)m$, for every $a \in \ker(\varphi)$. On the other hand

$$\begin{aligned} mu - \varphi(u^n u)m &= u^n u - bu^n u - \varphi(u^{n+1})(u^n - bu^n) \\ &= u^{n+1} - \varphi(u^{n+1})u^n - b(u^{n+1} - \varphi(u^{n+1})u^n) \\ &= t - bt = 0, \end{aligned}$$

where $t = u^{n+1} - \varphi(u^{n+1})u^n \in \ker(\varphi)$. Note that $\varphi(t) = \varphi(u^{n+1}) - \varphi(u^{n+1})\varphi(u^n) = \varphi(u^{n+1}) - \varphi(u^{n+1}) = 0$. As $\mathcal{A} = \mathbb{C}u \oplus \ker\varphi$, for every $a \in \mathcal{A}$, $ma = \varphi(u^n a)m$. Therefore by Theorem 2.11 that $\mathcal{A}$ is n-φ contractible. ■

Proposition 2.13 Let $\mathcal{A}$ be a Banach algebra and $\varphi \in \Delta^{(n)}(\mathcal{A})$ such that $\varphi(u) = 1$ for some $u \in \mathcal{A}$. If $\mathcal{A}$ is n-φ-contractible and has a left identity, then $\ker(\varphi)$ has a left identity.

Proof. It follows from Theorem 2.11 that there exist $m_1 \in \mathcal{A}$ such that $m_1 \cdot a = \varphi(u^n a)m_1$ for all $a \in \mathcal{A}$ and $\varphi(m_1) = 1$. Since $\mathcal{A} = \mathbb{C}u \oplus \ker\varphi$ and $\varphi(m_1) = 1$, we have $m_1 = u + a_1$ for some $a_1 \in \ker\varphi$. Suppose that $m_2 = \lambda u + a_2$ is a left identity for $\mathcal{A}$, where $a_2 \in \ker\varphi$ and $\lambda \in \mathbb{C}$. Put $e = a_2 - \lambda a_1$. We have $m_1 \cdot a = \varphi(u^2)\varphi(u)^{n-2}\varphi(a)m_1 = 0$ and $m_2 \cdot a = a$ for all $a \in \ker\varphi$. Therefore e is a left identity for $\ker\varphi$. ■

Theorem 2.14 Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $h : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous homomorphism with dense range. If $\varphi \in \Delta^{(n)}(\mathcal{B})$ and $\mathcal{A}$ is n-$\varphi \circ h$-contractible (at u), then $\mathcal{B}$ is n-φ-contractible (at $h(u)$).

Proof. Let $m \in A$ satisfying $(\varphi \circ h)(m) = 1$ and $m \cdot a = (\varphi \circ h)(u^n a)m$ for all $a \in \mathcal{A}$. Let $m' = h(m) \in \mathcal{B}$. So $\varphi(m') = \varphi(h(m)) = 1$. For each $b \in \mathcal{B}$, where $b = h(a)$, $a \in \mathcal{A}$ we have

$$\begin{aligned} m' \cdot b = m' \cdot h(a) &= h(m) \cdot h(a) \\ &= h(m \cdot a) = h((\varphi \circ h)(u^n a)m) \\ &= (\varphi \circ h)(u^n a)h(m) \\ &= \varphi((h(u))^n)h(a))m' \\ &= \varphi((h(u))^n b)m'. \end{aligned}$$

Now, density of the range and continuity of h implies that $\mathcal{B}$ is n-φ-contractible (at $h(u)$). ■

Acknowledgement

The authors sincerely thank the anonymous reviewer for his careful reading, constructive comments and fruitful suggestions to improve the quality of the manuscript.

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