Journal of
Linear and Topological Algebra
Vol. 04*, No.* 02*,* 2015, 143- 152

On the boundedness of almost multipliers on certain Banach algebras

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Received 9 July 2015; Revised 2 October 2015; Accepted 5 November 2015.

Abstract. Almost multiplier is rather a new concept in the theory of almost functions. In this paper we discuss on the boundedness of almost multipliers on some special Banach algebras, namely stable algebras. We also define an adjoint and extension for almost multiplier.

Keywords: Almost multipliers, almost additive maps, dual map, stable normed algebras.

2010 AMS Subject Classification: 39B62.

1. Introduction

Let A be a commutative Banach algebra. A bounded linear operator $T : A \to A$ is said to be a left (right) multiplier of A if $T(ab) = T(a)b$ ($T(ab) = aT(b)$) holds for all $a, b \in A$ and T is multiplier if $T(a)b = aT(b)$. If A is semisimple and has bounded approximate identity, then the set $M(A)$ consists of all multipliers on A, is a commutative semisimple closed unital subalgebra of the algebra of all bounded linear operators on A, $\mathcal{B}(A)$ [6]. For each $a \in A$, the bounded operator $L_a : A \to A, \quad b \to ab$, is a multiplier of A. Somehow we can say these are the building blocks of all multipliers.

The theory of multipliers has a long history in functional analysis. It was initially introduced by Helganson in [5]. Successively the general theory of multipliers on faithful (defined bellow) Banach algebras has been developed by Wang in [10] and Birtal in [3]. This theory has a close relation to harmonic analysis. Let G be a locally compact abelian group and $L^1(G)$ be its associated group algebra. It is known that $M(G)$, the algebra of

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

all complex Borel regular measures on G, with the natural measure convolution as the multiplication, is the multiplier algebra of $L^1(G)$ [6]. Host and Parreau in [4] worked on special multipliers on $L^1(G)$, and studied about range closedness also Zaiem improved some results about the existence of bounded approximate identity in the range of such multipliers. Ulger in [9], studied extensively the theory of multipliers. He extend the results from group algebras to the general commutative semisimple Banach algebras.

A normed algebra A is left (resp. right) faithful if for all $x \in A$, the condition $xA = 0$ (resp. $Ax = 0$), implies that $x = 0$. The normed algebra A is faithful if it is left and right faithful. Obviously if A has a bounded approximate identity, then it is faithful.

In [8], Miura defined the concept of almost multipliers on the faithful normed algebras, something that he calls approximate multipliers, and study some of their properties. The first and the third author also in [1] and [2] developed the theory of almost multipliers on special type of normed algebras, namely stable normed algebras.

The behavior of an almost multiplier on a unital normed algebra is much different when we move on a non-unital one. The boundedness of an almost multiplier is the main problem, which is not known in non-unital case. We will study some conditions that an almost multiplier is bounded. In this paper we will study the notion of almost multipliers on the stable, not necessarily unital, (commutative) normed algebras. We also define an adjoint and extension for almost multiplier. The main difference between an almost multiplier and a multiplier is being linear. We will give a non linear but continuous almost multiplier at the end.

This paper is organized as follows. In the second section we will discuss on the algebra of almost multipliers and the adjoint of a not necessarily linear one. In the third section we will study the boundedness behavior of almost multipliers on the stable normed algebras, and we generalize some results.

2. almost multipliers and some new constructions

In this section we will study almost multipliers and some of their properties. All over this paper when we use the word "map" there is no assumption on linearity or something else.

Definition 2.1 Let A be a normed algebra. An almost left (resp. right) multiplier on A is a map $T : A \to A$ such that

$$\|T(xy) - T(x)y\| \leqslant \varepsilon\|x\|\|y\| (resp.\ \|T(xy) - xT(y)\| \leqslant \varepsilon\|x\|\|y\|)$$

for some $\varepsilon > 0$, and for all $x, y \in A$. We say that the map T is an almost multiplier on A if

$$\|T(x)y - xT(y)\| \leqslant \varepsilon\|x\|\|y\|$$

for all $x, y \in A$.

Miura has given a trivial approximation property, that we mention in the following.

Theorem 2.2 [8, Theorem 1.1] Suppose A is a faithful complex Banach algebra. If the map $T : A \to A$ satisfies $T(0) = 0$ and

$$\|T(a)b - aT(b)\| \leqslant \varepsilon\|a\|^p\|b\|^p, \quad (a, b \in A)$$

for some $\varepsilon > 0$ and $p \neq 1$, then T is a multiplier on A.

Definition 2.3 We say that the complex normed algebra A is left (resp. right) stable if for $a \in A$, if there exists $M > 0$ such that $\|ab\| \leqslant M\|b\|$ $\;$ (*resp.* $\|ba\| \leqslant M\|b\|$) for all $b \in A$, then we have $\|a\| \leqslant M$.

The normed algebra A is stable if it is both left and right stable.

Stable Banach algebras is defined by the first and third authors in [2].

Theorem 2.4 [2]

(i) All unital Banach algebras are stable.
(ii) All Banach algebras with a bounded approximate identity with bound 1, are stable.
(iii) All stable Banach algebras are faithful.

Theorem 2.5 [1] Let A be a stable normed algebra. Then the set of all almost multipliers with the ordinary operations is an algebra.

2.1 *Dual Construction*

Unfortunately an almost multiplier need not to be linear, so working with its natural adjoint on the dual spaces is impossible. To remedy of this problem, we will work with a suitable substitutions. In the following we define almost additive maps.

Definition 2.6 Suppose A and B are normed algebras. Then the map $f : A \to B$, is almost additive if for some $\varepsilon > 0$,

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\| + \|y\|)$$

for all $x, y \in A$.

Theorem 2.7 [1] Let A be a stable normed algebra. Every almost (left or right) multiplier on A is almost additive.

Definition 2.8 For linear spaces A, B, with the same scalar field, a map $f : A \longrightarrow B$ is called homogeneous, if $f(\lambda x) = \lambda f(x)$, for all scaler λ and $x \in A$.

In this paper, by an almost linear map we mean a homogeneous almost additive map.

Let X and Y be normed spaces. We define $\mathcal{B}'(X,Y)$ the linear space of all homogeneous maps, $T : X \to Y$, such that $\sup_{\|x\|\leqslant 1} \|T(x)\|$ is finite. We denote $\mathcal{B}'(X,X)$ by $\mathcal{B}'(X)$ and $\mathcal{B}'(X,\mathbb{C})$ by X'. It is easy to see that for $T \in \mathcal{B}'(X,Y)$,

$$\|T\| := \sup_{\|x\|\leqslant 1} \|T(x)\|$$

defines a norm, also X' is a Banach space and X'' with the (first) Arens product is a Banach algebra.

According to the definition, the elements of $\mathcal{B}'(X,Y)$ are not necessary linear, but since for all $T \in \mathcal{B}'(X,Y)$ we have $\|T(x+y) - T(x) - T(y)\| \leqslant 2\|T\|(\|x\| + \|y\|)$, they are almost linear.

Definition 2.9 Let X be a normed space, and let T be an element of $\mathcal{B}'(X)$. We denote the adjoint of T by T' on X' and define it by $T'(x') = x' \circ T$. Similarly the second adjoint of T by T'' on X'' can be defined.

Theorem 2.10 Let A be a unital Banach algebra. If T is an almost multiplier on A then T'' is also an almost multiplier on A''.

Proof.
For x in the Banach algebra A, and $f \in A'$, by $\langle f, x\rangle$ and also $\langle x, f\rangle$ we denote the natural duality between A and A''.

Let T be an almost multiplier on A, with corresponding $\varepsilon > 0$, and consider A'' with (the first) Arens product which is denoted by " $\cdot$ ". Fix $m, n \in A''$, $x, y \in A$ and $f \in A'$. We have $T''(m)\cdot n \in A''$, $n.f \in A'$ and $f.T(x) \in A'$ with the following natural definitions,

$$\langle T''(m) \cdot n, f\rangle = \langle T''(m), n.f\rangle = \langle m, T'(n.f)\rangle, \tag{1}$$

where,

$$\langle T'(n.f), x\rangle = \langle n.f, T(x)\rangle = \langle n, f.T(x)\rangle, \tag{2}$$

and,

$$\langle f.T(x), y\rangle = \langle f, T(x)y\rangle. \tag{3}$$

Similarly $m \circ T''(n) \in A''$, $f.x \in A'$ and $T''(n).f \in A'$, and

$$\langle m \cdot T''(n), f\rangle = \langle m, T''(n).f\rangle. \tag{4}$$

where

$$\langle T''(n).f, x\rangle = \langle T''(n), f.x\rangle = \langle n, T'(f.x)\rangle. \tag{5}$$

and

$$\langle T'(f.x), y\rangle = \langle f.x, T(y)\rangle = \langle f, xT(y)\rangle. \tag{6}$$

We will show that there exists $\varepsilon_{T''} > 0$ such that $\|T''(m)\cdot n - m\cdot T''(n)\| \leqslant \varepsilon_{T''}\|m\|\|n\|$. Since T is an almost multiplier, and each $f \in A'$, is almost linear so we have,

$$|\langle f, xT(y) - T(x)y\rangle - (\langle f, xT(y)\rangle - \langle f, T(x)y\rangle)| \leqslant 2\|f\|(\|xT(y)\| + \|T(x)y\|),$$

and then,

$$\begin{aligned} |\langle f, xT(y)\rangle - \langle f, T(x)y\rangle| &\leqslant \|f\|\|xT(y) - T(x)y\| + 4\|T\|\|f\|\|x\|\|y\| \\ &\leqslant \varepsilon\|f\|\|x\|\|y\| + 4\|T\|\|f\|\|x\|\|y\| \\ &= (\varepsilon + 4\|T\|)\|f\|\|x\|\|y\|. \end{aligned}$$

Now by 3 and 6 we have

$$|\langle T'(f.x), y\rangle - \langle f.T(x), y\rangle| \leqslant (\varepsilon + 4\|T\|)\|f\|\|x\|\|y\|,$$

and so

$$\|T'(f.x) - f.T(x)\| \leqslant (\varepsilon + 4\|T\|)\|f\|\|x\|. \tag{7}$$

Moreover

$$\begin{aligned} |\langle n, T'(f.x) - f.T(x)\rangle - (\langle n, T'(f.x)\rangle - \langle n, f.T(x)\rangle)| \\ \leqslant 2\|n\|(\|T'(f.x)\| + \|f.T(x)\|). \end{aligned} \tag{8}$$

Also we have

$$\begin{aligned} |\langle T'(f.x), y\rangle| &= |\langle f.x, T(y)\rangle| = |\langle f, xT(y)\rangle| \\ &\leqslant \|f\|\|xT(y)\| \leqslant \|f\|\|T\|\|x\|\|y\|, \end{aligned}$$

and $\|T'(f.x)\| \leqslant \|T\|\|f\|\|x\|$. Similarly, we can see $\|f.T(x)\| \leqslant \|T\|\|f\|\|x\|$. Now by 8,

$$\begin{aligned} |\langle n, T'(f.x) - f.T(x)\rangle - (\langle n, T'(f.x)\rangle - \langle n, f.T(x)\rangle)| \\ \leqslant 4\|T\|\|n\|\|f\|\|x\|. \end{aligned}$$

Also by 7

$$\begin{aligned} |\langle n, T'(f.x)\rangle - \langle n, f.T(x)\rangle| &\leqslant 4\|T\|\|n\|\|f\|\|x\| + \|n\|\|T'(f.x) - f.T(x)\| \\ &\leqslant 4\|T\|\|n\|\|f\|\|x\| + \|n\|(\varepsilon + 4\|T\|)\|f\|\|x\| \\ &\leqslant (\varepsilon + 8\|T\|)\|n\|\|f\|\|x\|. \end{aligned}$$

Now from 2, 5 and 7, we have

$$|\langle T''(n).f, x\rangle - \langle T'(n.f), x\rangle| \leqslant (\varepsilon + 8\|T\|)\|n\|\|f\|\|x\|,$$

and

$$\|T''(n).f - T'(n.f)\| \leqslant (\varepsilon + 8\|T\|)\|n\|\|f\|. \tag{9}$$

By a similar computation we have

$$\begin{aligned} |\langle m, T''(n).f - T'(n.f)\rangle - (\langle m, T''(n).f\rangle - \langle m, T'(n.f)\rangle)| \\ \leqslant 2\|m\|(\|T''(n).f\| + \|T'(n.f)\|) \end{aligned}$$

(10)

Since

$$|\langle T''(n).f, x\rangle| = |\langle n.f, T(x)\rangle| = |\langle n, f.T(x)\rangle| \\ \leqslant \|n\|\|f.T(x)\| \leqslant \|n\|\|f\|\|T\|\|x\|,$$

so $\|T''(n.f)\| \leqslant \|T\|\|n\|\|f\|$. Similarly we can see that $\|T'(n.f)\| \leqslant \|T\|\|n\|\|f\|$. Therefore by 10,

$$|\langle n, T'(f.x) - f.T(x)\rangle - (\langle n, T'(f.x)\rangle - \langle n, f.T(x)\rangle)| \\ \leqslant 4\|T\|\|m\|\|n\|\|f\|.$$

So by 9

$$|\langle m, T''(n).f\rangle - \langle m, T'(n.f)\rangle| \leqslant 4\|T\|\|m\|\|n\|\|f\| + \|m\|\|T''(n).f - T'(n.f)\| \\ \leqslant (\varepsilon + 12\|T\|)\|m\|\|n\|\|f\|,$$

and then by 1 and 4

$$|\langle T''(m) \cdot n, f\rangle - \langle m \cdot T''(n), f\rangle| \leqslant (\varepsilon + 12\|T\|)\|m\|\|n\|\|f\|,$$

and finally

$$\|T''(m) \cdot n - m \cdot T''(n)\| \leqslant (\varepsilon + 12\|T\|)\|m\|\|n\|.$$

This shows that T'' is an almost multiplier with corresponding $\varepsilon_{T''} = \varepsilon + 12\|T\|$ ■

Remark 1 *Since each arbitrary map T with finite $\|T\|$ is an almost multiplier, the above theorem can be proved much easier. Because in the middle of proof we can see that $\|T''\| \leqslant \|T'\| \leqslant \|T\|$. So*

$$\|T''(m) \cdot n - m \cdot T''(n)\| \leqslant 2\|T\|\|m\|\|n\|.$$

3. Comparing the algebra of homogeneous almost multipliers on a normed algebra A with $\mathcal{B}'(A)$

Suppose that A is a normed algebra. It is easy to see that every element of $\mathcal{B}'(A)$ is an almost multiplier. In this section we study conditions that make an almost multiplier to be an element of $\mathcal{B}'(A)$. Actually we will give some conditions that guarantees the boundedness of an almost multiplier.

Lemma 3.1 Let A be a unital normed algebra with unit e, and let T be a homogeneous almost multiplier (resp. almost left or right multiplier) with corresponding scaler $\varepsilon > 0$, then $T \in \mathcal{B}'(A)$.

Proof. The result comes from the trigonometric inequality,

$$\|T(a)\| - \|aT(e)\| \leqslant \|T(a)e - aT(e)\| \leqslant \varepsilon\|a\|.$$

■

Proposition 3.2 Let A be a normed algebra with a bounded approximate identity (e_n), bounded by M, and let T be a homogeneous almost multiplier, with corresponding scaler $\varepsilon > 0$, such that $(T(e_n))$ is a bounded sequence in A with bound M' then, $T \in \mathcal{B}'(A)$.

Proof. The statement comes from the following relations:

$$\begin{aligned}\|aT(e_n) - T(a)\| &= \|aT(e_n) - T(a)e_n + T(a)e_n - T(a)\| \\ &\leqslant \varepsilon\|a\|\|e_n\| + \|T(a)e_n - T(a)\|,\end{aligned}$$

and so

$$\begin{aligned}\|T(a)\| &\leqslant \|a\|\|T(e_n)\| + \varepsilon\|a\|\|e_n\| + \|T(a)e_n - T(a)\| \\ &\leqslant \|a\|(M' + \varepsilon M) + \|T(a)e_n - T(a)\|,\end{aligned}$$

and $\|T(a)e_n - T(a)\|$ tends to zero. ■

The above results show that each almost multiplier on a unital C^*-algebra is continues at zero, and somehow "bounded".

Let A be a unital C^*-algebra with unit e, and let T be an almost multiplier on A, with corresponding $\varepsilon > 0$. Then by definition $\|T(x) - T(e)x\| \leqslant \varepsilon\|x\|$. But by functional calculus on C^*-algebras we know that for any hermitian element $a \in A$,

$$\|a\| = \inf\{\lambda \geqslant 0, \ -\lambda e \leqslant a \leqslant \lambda e\}, \tag{11}$$

now let T be a positive map, i.e. maps positives into the the positive cone of A, by 11, there is a positive map $P : A \longrightarrow A$, such that

$$T(x) = T(e)x + \varepsilon\|x\|e + P(x), \quad x \in A_+$$

where A_+ is the positive cone of the C^*-algebra, A. This example gives us an imagination of the form of an almost multiplier, which is a linear map plus something else, on positives.

Theorem 3.3 Let A be a stable Banach algebra which has not necessarily unit or a bounded approximate identity, and let $T : A \to A$ be a homogeneous almost multiplier on A and there exists $\varepsilon > 0$ such that $\|T(x) - x\| \leqslant \varepsilon\|x\|$, then $T \in \mathcal{B}'(A)$ and can be extended to a homogeneous almost multiplier $\widetilde{T}$ on the $\mathcal{B}'(A^\sharp)$, where $A^\sharp$ is the unitization of A. Also $\|\widetilde{T}\| - 1 \leqslant \|T\| \leqslant \frac{1}{k}\|\widetilde{T}\|$, for some $k > 0$.

Proof. Let $T : A \longrightarrow A$ with corresponding ε_T be an almost multiplier. By the trigonometric inequality it is clear that T is a bounded map.

Now let $B(A)$ be the Banach algebra of all bounded linear maps on A. Since A is faithful the map $\phi : A \longrightarrow B(A), \quad a \to L_a$, is the natural embedding,.

In the following, we can see that $\phi(A)$ is a closed subalgebra of $\mathcal{B}(A)$, in norm topology. Suppose $L_{a_n} \to T$ in $\phi(A)$, therefore (L_{a_n}) is a Cauchy sequence. So for any arbitrary $\varepsilon > 0$ and $b \in A$ with norm less than 1, $\|a_n b - a_m b\| \leqslant \varepsilon$, for some $N > 0$ and every $m > n > N$. Since A is stable, we have $\|a_n - a_m\| \leqslant \varepsilon$. It means that there is an $a \in A$, such that $a_n \to a$. So $T = L_a$.

Now by open mapping theorem ϕ is an open and one-one map onto its range, therefore there exists $k > 0$, such that

$$k\|a\| \leqslant \|L_a\| \leqslant \|a\|.$$

Now we define $A^\sharp := \phi(A) \oplus^{\ell^1} \mathbb{C}I$, the unitization of A, which is a unital Banach algebra with the multiplication inherited from $B(A)$. Every elements of $A^\sharp$ is of the form $(L_a, \lambda I)$. Now for almost multiplier T on A, we define $\widetilde{T}$ on $A^\sharp$, by $\widetilde{T}(L_a + \lambda I) = L_{T(a)} + \lambda I$.

Now we show that $\widetilde{T}$ is an almost multiplier on $A^\sharp$. Let $a, a' \in A$ and $\lambda, \lambda' \in \mathbb{C}$, then

$$\begin{aligned}
&\|\widetilde{T}(L_a + \lambda I)(L_{a'} + \lambda' I) - (L_a + \lambda I)\widetilde{T}(L_{a'} + \lambda' I)\| \\
&\qquad = \|(L_{T(a)} + \lambda I)(L_{a'} + \lambda' I) - (L_a + \lambda I)(L_{T(a')} + \lambda' I)\| \\
&\qquad \leqslant \|L_{T(a)a'} + \lambda' L_{T(a)} + \lambda L_{a'} - L_{aT(a')} - \lambda' L_a - \lambda L_{T(a')}\| \\
&\qquad \leqslant \|T(a)a' - aT(a')\| + |\lambda'|\|T(a) - a\| + |\lambda|\|a' - T(a')\| \\
&\qquad \leqslant \varepsilon_T\|a\|\|a'\| + |\lambda'|\varepsilon_1\|a\| + |\lambda|\varepsilon_1\|a'\| \\
&\qquad \leqslant M(\|a\|\|a'\| + |\lambda'|\|a\| + |\lambda|\|a'\|) \\
&\qquad \leqslant M(\|a\|(\|a'\| + |\lambda'|) + |\lambda|\|a'\| + |\lambda||\lambda'|) \\
&\qquad = M(\|a\| + |\lambda|)(\|a'\| + |\lambda'|) \\
&\qquad \leqslant \frac{M}{k^2}(\|L_a\| + |\lambda|)(\|L_{a'}\| + |\lambda'|),
\end{aligned}$$

where M is a suitable positive real number. Now by the definition of the norm on the direct sum space we have

$$\|\widetilde{T}(L_a + \lambda I)(L_{a'} + \lambda' I) - (L_a + \lambda I)\widetilde{T}(L_{a'} + \lambda' I)\| \leqslant \frac{M}{k^2}\|L_a + \lambda I\|\|L_{a'} + \lambda' I\|.$$

It means that $\widetilde{T}$ is an almost multiplier and by Lemma 3.1 $\widetilde{T}$ is bounded on $A^\sharp$. But

$$\|T\| = \sup_{\|a\|\leqslant 1}\|T(a)\| \leqslant \frac{1}{k}\sup_{\|a\|\leqslant 1}\|\widetilde{T}(L_a)\| \leqslant \frac{1}{k}\sup_{\|L_a\|\leqslant 1}\|\widetilde{T}(L_a)\| = \frac{1}{k}\|\widetilde{T}\| < \infty,$$

also

$\|\widetilde{T}\| = \sup_{\|L_a + \lambda I\|\leqslant 1}\|\widetilde{T}(L_a + \lambda I)\| \leqslant \sup_{\|L_a\|\leqslant 1, |\lambda|\leqslant 1}(\|L_{T(a)}\| + |\lambda|) \leqslant \|T\| + 1$

■

Theorem 3.4 Let A be a commutative Banach algebra with a bounded approximate identity with bound 1, then each homogeneous uniformly continuous almost multiplier on A is an element of $\mathcal{B}'(A)$.

Proof. Let A be a commutative Banach algebra with a bounded approximate identity, namely (e_α) with bound 1, then for each multiplier φ the sequence $(L_{\varphi(e_\alpha)})$ converges to φ in strong operator topology. So the subalgebra $\{L_a | a \in A\}$ is dense in $M(A)$, the unital Banach algebra of multipliers, in the strong operator topology [6, Theorem 1.1.6].

For almost multiplier $T : A \longrightarrow A$ satisfying the assumption, we define $\widetilde{T} : M(A) \longrightarrow M(A)$, by $\widetilde{T}(\varphi) = \lim_{\alpha} L_{T(a_\alpha)}$, where $(a_\alpha) = (\varphi(e_\alpha))$. Also the above limit is taken in strong operator topology.

Since for all multiplier φ, $L_{\varphi(e_\alpha)} \to \varphi$ in strong operator topology, It means $L_{\varphi(e_\alpha)}$ is a cauchy net. By theorem 2.4, A is a stable Banach algebra so $\varphi(e_\alpha)$ is a couchy net in norm topology. Since T is uniformly continuous we can conclude $T(\varphi(e_\alpha))$ is a couchy net in norm topology. So $L_{T(\varphi(e_\alpha))}$ is a couchy net in strong operator topology.

In the following we show that $\widetilde{T}$ is an almost multiplier. Let φ and ψ be in $M(A)$ and $\{L_{a_\alpha}\}$ and $\{L_{b_\alpha}\}$ be sequences converging to φ and ψ in strong operator topology, respectively. It is easy to see that $\|a_\alpha\| \leqslant K\|\varphi\|$ and $\|b_\alpha\| \leqslant K\|\psi\|$. Now for $a \in A$,

$$\widetilde{T}(\varphi)(\psi(a)) = \lim_{\alpha} L_{T(a_\alpha)}\psi(a) = \lim_{\alpha} T(a_\alpha)b_\alpha a,$$

and similarly

$$\widetilde{T}(\psi)(\varphi(a)) = \lim_{\alpha} a_\alpha T(b_\alpha)a,$$

and so

$$\|\widetilde{T}(\varphi)(\psi(a)) - \widetilde{T}(\psi)(\varphi(a))\| \leqslant \varepsilon \lim_{\alpha} \|a_\alpha\|\|b_\alpha\|,$$

but by the boundedness of $(\|a_\alpha\|)$ and $(\|b_\alpha\|)$, it is easy to see that there is $M > 0$, such that

$$\|\widetilde{T}(\varphi)\psi - \widetilde{T}(\psi)\varphi\| \leqslant M\|\varphi\|\|\psi\|.$$

So $\widetilde{T}$ is an almost multiplier on the commutative unital Banach algebra $M(A)$ and so is bounded.

But $T = \widetilde{T}|_{\{L_a|a\in A\}}$, and so $\|T\| \leqslant \|\widetilde{T}\|$ and so is bounded. ■

By Theorem 3.4, we can see that under some assumption on the domain, every continuous almost multiplier which is not necessarily linear is bounded.

The linearity is the boundary between almost multipliers and bounded linear maps, as we can see in the following:

Theorem 3.5 Let A be a normed algebra with a bounded approximate identity, then each linear almost multiplier is an element of $\mathcal{B}'(A)$.

And so the set of all linear almost multipliers on A is equal to $\mathcal{B}(A)$.

Proof. Clearly each bounded linear map on A is an almost multiplier. Now we show that if T is a linear almost multiplier, with corresponding ε, then it is bounded. Let (a_n) be a null sequence in A. By Cohen factorization theorem, there exists $b \in A$ and a null sequence (c_n) in A, such that $a_n = bc_n$, $n \in \mathbb{N}$. Now we have

$$\|T(a_n) - T(b)c_n\| = \|T(bc_n) - T(b)c_n\| \leqslant \varepsilon\|b\|\|c_n\|.$$

It shows $T(a_n) \to 0$, and so T is continuous and therefore it is bounded, by linearity. ■

As we saw above, for an almost multiplier, linearity leads to continuity. Now we give an example of a continuous almost multiplier which is not linear.

***Example* 3.6** Let $\varepsilon > 0$ be a positive number. By the continuity of function $t \mapsto e^{it}$, there is a δ with $0 < \delta < 1$ corresponds to ε , such that $|t| < 2\pi(1-\delta)$ implies $|e^{it}-1| < \varepsilon$. With this δ, define the map $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} 0 & z = 0 \\ |z|e^{i\delta\theta} & z \in \mathbb{C} - \{0\}, \end{cases}$$

Where $\theta \in [0, 2\pi)$ denotes the argument of z. In [8], Miura and Takahasi showed that this map is an almost multiplier correspond to ε, which is not multiplier. It is easy to see that, this map is continuous on $\mathbb{C}$ but it is not linear.

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