

New characterizations of fusion bases and Riesz fusion bases in Hilbert spaces

F. Aboutorabi Goudarzi^{a*}, M. S. Asgari^a

^a*Department of Mathematics, Faculty of Science,
Central Tehran Branch, Islamic Azad University, Tehran, Iran.*

Received 20 June 2015; Revised 24 August 2015; Accepted 19 September 2015.

Abstract. In this paper we investigate a new notion of bases in Hilbert spaces and similar to fusion frame theory we introduce fusion bases theory in Hilbert spaces. We also introduce a new definition of fusion dual sequence associated with a fusion basis and show that the operators of a fusion dual sequence are continuous projections. Next we define the fusion biorthogonal sequence, Bessel fusion basis, Hilbert fusion basis and obtain some characterizations of them. we study orthonormal fusion systems and Riesz fusion bases for Hilbert spaces. we consider the stability of fusion bases under small perturbations. We also generalized a result of Paley-Wiener [16] to the situation of fusion basis.

© 2015 IAUCTB. All rights reserved.

Keywords: Fusion Frame; Riesz fusion basis; Exact fusion frame; Orthonormal fusion basis.

2010 AMS Subject Classification: Primary 42C15; Secondary 46C99.

1. Introduction

Frames for Hilbert spaces were first formally defined by Duffin and Schaeffer [8] in 1952 to study some deep problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [7] and popularized from then on. A frame is a redundant set of vectors in a Hilbert space with the property that provide usually non-unique representations of vectors in terms of the frame elements. Fusion frames which were considered recently as generalized frames in Hilbert spaces, were introduced by Casazza and Kutyniok in [5] and have quickly turned into an industry. Related approaches

*Corresponding author.

E-mail address: f.godarzi_1346@yahoo.com (F. Aboutorabi Goudarzi).

with a different focus were undertaken by M.Fornasier [9] and W.Sun [15], D. R. Larson et al. [12]. Bases, frames and fusion frames play important roles in many applications in mathematics, science, and engineering, including coding theory, filter bank theory, sigma-delta quantization, signal and image processing and wireless communications and many other areas. The main subject of this paper deals with fusion bases and resolution of the identity. The paper is organized as follows: Section 2, contains a new definition of fusion basis in a Hilbert space. In this section similar to basis theory we first establishes a simple criterion for determining when a complete set of closed subspaces is a fusion basis. Next we introduce the concepts of fusion biorthogonal sequence, Bessel fusion basis, Hilbert fusion basis and obtain some characterizations of them. In Section 3, we study orthonormal fusion bases and Riesz fusion bases for Hilbert spaces. We introduce a new definition of Riesz fusion basis and then give some characterizations of orthonormal fusion bases and Riesz fusion bases. In Section 4, we study the stability of fusion bases under small perturbations. we also generalized a result of Paley-Wiener [16] to the situation of fusion basis.

Throughout this paper, \mathcal{H}, \mathcal{K} are separable Hilbert spaces and I, I_j, J denote the countable (or finite) index sets and π_W denote the orthogonal projection of a closed subspace W of \mathcal{H} . We will always use \mathcal{R}_T and \mathcal{N}_T to denote range and the null spaces of an operator $T \in B(\mathcal{H}, \mathcal{K})$ respectively.

Let $\mathcal{W} = \{W_j\}_{j \in J}$ be a sequence of closed subspaces in \mathcal{H} , and let $\{\alpha_j\}_{j \in J}$ be a family of weights, i.e., $\alpha_j > 0$ for all $j \in J$. A sequence $\mathcal{W}_\alpha = \{(W_j, \alpha_j)\}_{j \in J}$ is a fusion frame, if there exist real numbers $0 < C \leq D < \infty$ such that,

$$C\|f\|^2 \leq \sum_{j \in J} \alpha_j^2 \|\pi_{W_j} f\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{H}. \quad (1)$$

The constant C, D are called the fusion frame bounds. If $C = D = \lambda$, the fusion frame is λ -tight and it is a Parseval fusion frame if $C = D = 1$, and it is α -uniform if $\alpha = \alpha_i = \alpha_j$ for all $i, j \in J$. If the right-hand inequality of (1) holds, then we say that \mathcal{W}_α is a Bessel fusion sequence with Bessel fusion bound D .

For each sequence $\mathcal{W} = \{W_j\}_{j \in J}$ of closed subspaces of \mathcal{H} , we define the Hilbert space associated with \mathcal{W} by

$$\left(\sum_{j \in J} \oplus W_j \right)_{\ell^2} = \left\{ \{g_j\}_{j \in J} : g_j \in W_j \text{ and } \sum_{j \in J} \|g_j\|^2 < \infty \right\}. \quad (2)$$

with inner product given by

$$\langle \{f_i\}_{i \in J}, \{g_i\}_{i \in J} \rangle = \sum_{i \in J} \langle f_i, g_i \rangle. \quad (3)$$

For more details about the theory and application of bases, frames and fusion frames we refer the reader to the books by Young [16], Christensen [6], the survey articles by Asgari et al. [1–4], Casazza [5], Karimizad [13], Gavruta [10], and Holub [11].

2. Fusion Schauder bases

The concept of Riesz decomposition that we call fusion Schauder basis, was first introduced by Casazza and Kutyniok in [5]. In this section, we develop the fusion basis theory

for Hilbert spaces. As a consequence we generalized some results of bases to fusion bases.

Definition 2.1 Let $\mathcal{W} = \{W_j\}_{j \in J}$ be a sequence of closed subspaces in \mathcal{H} , then $\{W_j\}_{j \in J}$ is called a fusion Schauder basis or simply a f-basis for \mathcal{H} if for any $f \in \mathcal{H}$ there exists an unique sequence $\{g_j : g_j \in W_j\}_{j \in J}$ such that $f = \sum_{j \in J} g_j$ with the convergence being in norm. If this series converges unconditionally for each $f \in \mathcal{H}$, we say that $\{W_j\}_{j \in J}$ is an unconditional f-basis.

Example 2.2 For each $N \in \mathbb{N}$, let $\mathcal{H} = \mathbb{C}^N$ and let $\{e_i\}_{i=1}^N$ be the standard basis of \mathbb{C}^N . Define $W_j \subset \mathcal{H}$ by $W_j = \text{span}\{\sum_{i \neq j} e_i\}$, for all $1 \leq j \leq N$. Then $\{W_j\}_{j=1}^N$ is a f-basis for \mathcal{H} .

Proposition 2.3 Let $\mathcal{W} = \{W_j\}_{j \in J}$ be a f-basis for \mathcal{H} . Then $\dim \mathcal{H} = \sum_{j \in J} \dim W_j$.

Proof. Let $\{e_{ij}\}_{i \in J_j}$ be an orthonormal basis for W_j for all $j \in J$. We show that $\{e_{ij}\}_{j \in J, i \in J_j}$ is a basis for \mathcal{H} . Since $\{e_{ij}\}_{i \in J_j}$ is an orthonormal basis for W_j , hence every $g_j \in W_j$ has a unique expansion of the form $g_j = \sum_{i_j} \langle g_j, e_{ij} \rangle e_{ij}$. This implies that also every $f \in \mathcal{H}$ has a unique expansion of the form $f = \sum_{j \in J} \sum_{i \in J_j} \langle g_j, e_{ij} \rangle e_{ij}$. This shows that $\dim \mathcal{H} = \sum_{j \in J} \dim W_j$. ■

Corollary 2.4 Let $\{W_j\}_{j \in J}$ and $\{V_i\}_{i \in I}$ be f-bases for \mathcal{H} . Then $\sum_{j \in J} \dim W_j = \sum_{i \in I} \dim V_i$.

Proof. This follows immediately from the Proposition 2.3. ■

Let $\{W_j\}_{j \in J}$ be a f-basis for \mathcal{H} , then any $f \in \mathcal{H}$ has a unique expansion of the form $f = \sum_{j \in J} g_j$. Hence, every $g_j \in W_j$ is a linear operator of f . If we denote this linear operator by $P_{W_j} : \mathcal{H} \rightarrow W_j$, then $g_j = P_{W_j} f$ and we have $f = \sum_{j \in J} P_{W_j} f$. The sequence $\{P_{W_j}\}_{j \in J}$ is called the f-dual sequence of $\{W_j\}_{j \in J}$ and $\mathcal{W} = \{(W_j, P_{W_j})\}_{j \in J}$ is called f-basis system.

In the next theorem we show that the operators of a f-dual sequence are continuous projections.

Theorem 2.5 Let $\{W_j\}_{j \in J}$ be a f-basis for \mathcal{H} , with f-dual sequence $\{P_{W_j}\}_{j \in J}$. Then $P_{W_j} \in B(\mathcal{H}, W_j)$ and $P_{W_i} P_{W_j} = \delta_{ij} P_{W_j}$ for all $i, j \in J$, where δ_{ij} is the Kronecker delta.

Proof. Define the space

$$\mathcal{A} = \left\{ \{g_j\}_{j \in J} : g_j \in W_j \text{ and } \sum_{j \in J} g_j \text{ is convergent} \right\},$$

with the norm defined by:

$$\|\{g_j\}_{j \in J}\| = \sup_{\substack{0 < |F| < \infty \\ F \subset J}} \left\| \sum_{i \in F} g_i \right\| < \infty.$$

It is clear that \mathcal{A} endowed with this norm, is a normed space with respect to the pointwise operations. We show that the space \mathcal{A} is a complete. Let $\{u_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in \mathcal{A} . If $u_n = \{g_{nj}\}_{j \in J}$, then given any $\varepsilon > 0$, there exists a number N such that

$$\sup_{\substack{0 < |F| < \infty \\ F \subset J}} \left\| \sum_{i \in F} (g_{ni} - g_{mi}) \right\| < \varepsilon, \tag{4}$$

for all $j \in J$ and $m, n \geq N$. This yields

$$\|g_{nj} - g_{mj}\| \leq \sup_{\substack{0 < |F| < \infty \\ F \subseteq J}} \left\| \sum_{i \in F} (g_{ni} - g_{mi}) \right\| < \varepsilon$$

It follows that $\{g_{nj}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in W_j and thus convergent. Let $g_j \in W_j$ such that $g_j = \lim_{n \rightarrow \infty} g_{nj}$ and $u = \{g_j\}_{j \in J}$. From (4), by letting $m \rightarrow \infty$, we obtain

$$\sup_{\substack{0 < |F| < \infty \\ F \subseteq J}} \left\| \sum_{i \in F} (g_{ni} - g_i) \right\| < \varepsilon, \quad (5)$$

for all $n \geq N$. Moreover, for every finite subset $F \subset J$ we have

$$\begin{aligned} \left\| \sum_{i \in F} g_i \right\| &\leq \left\| \sum_{i \in F} (g_{Ni} - g_i) \right\| + \left\| \sum_{i \in F} g_{Ni} \right\| \\ &\leq \sup_{\substack{0 < |F| < \infty \\ F \subseteq J}} \left\| \sum_{i \in F} (g_{Ni} - g_i) \right\| + \sup_{\substack{0 < |F| < \infty \\ F \subseteq J}} \left\| \sum_{i \in F} g_{Ni} \right\|, \end{aligned}$$

which implies that $u \in \mathcal{H}$. Further (5) implies that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is convergent to u in \mathcal{A} . This proves that \mathcal{A} is a Banach space. Now define the mapping $T : \mathcal{A} \rightarrow \mathcal{H}$ by $T(\{g_j\}_{j \in J}) = \sum_{j \in J} g_j$. Since $\{W_j\}_{j \in J}$ is a f-basis for \mathcal{H} , T is linear, one-to-one and onto. On the other hand we have

$$\|T(\{g_j\}_{j \in J})\| = \left\| \sum_{j \in J} g_j \right\| \leq \sup_{\substack{0 < |F| < \infty \\ F \subseteq J}} \left\| \sum_{i \in F} g_i \right\| = \|\{g_j\}_{j \in J}\|.$$

Thus T is continuous and by open mapping theorem T^{-1} is also continuous. This shows that \mathcal{A} and \mathcal{H} are Banach spaces isomorphic. Now suppose that $f = \sum_{j \in J} g_j$ and $j \in J$ be arbitrary. Then we obtain

$$\|P_{W_j} f\| = \sup_{\substack{0 < |F| < \infty \\ F \subseteq J}} \left\| \sum_{i \in F} g_i \right\| = \|T^{-1} f\| \leq \|T^{-1}\| \|f\|.$$

This shows P_{W_j} is continuous and $\|P_{W_j}\| \leq \|T^{-1}\|$. Finally from $P_{W_i} g_j = \delta_{ij} g_j$ we have $P_{W_i} P_{W_j} = \delta_{ij} P_{W_j}$ for all $i, j \in J$. ■

Definition 2.6 Let $\{W_j\}_{j \in J}$ be a sequence of closed subspaces in \mathcal{H} . Then

- (i) $\{W_j\}_{j \in J}$ is called a complete set for \mathcal{H} , if $\mathcal{H} = \overline{\text{span}}\{W_j\}_{j \in J}$.
- (ii) A family of operators $\{Q_{W_j} \in B(\mathcal{H}, W_j) : j \in J\}$ is called a f-biorthogonal sequence of $\{W_j\}_{j \in J}$, if $Q_{W_i} g_j = \delta_{ij} g_j$ for all $i, j \in J$ and $g_j \in W_j$.

A direct calculation shows that $\{W_j\}_{j \in J}$ is a complete set for \mathcal{H} , if and only if

$$\{f : \pi_{W_j} f = 0, j \in J\} = \{0\}.$$

Moreover, if $\{Q_{W_j}\}_{j \in J}$ is a f-biorthogonal sequence of $\{W_j\}_{j \in J}$ then any Q_{W_j} is a projection from \mathcal{H} onto W_j and $Q_{W_i} \pi_{W_j} = \pi_{W_j} Q_{W_i}^*$.

Let $\{(W_j, P_{W_j})\}_{j \in J}$ be a f-basis system for \mathcal{H} . Then $P_{W_j}^*$ is a closed subspaces of \mathcal{H} ,

for all $j \in J$. The following theorem shows that the sequence $\{(P_{W_j}^*(W_j), P_{W_j}^*)\}_{j \in J}$ is also a f-basis system for \mathcal{H} .

Theorem 2.7 Let $\{(W_j, P_{W_j})\}_{j \in J}$ be a f-basis system for \mathcal{H} . Then $\{(P_{W_j}^*(W_j), P_{W_j}^*)\}_{j \in J}$ is also a f-basis system for \mathcal{H} .

Proof. We first prove that $\mathcal{H} = \overline{\text{span}}\{P_{W_j}^*(W_j)\}_{j \in J}$. To see this, let $f \perp \overline{\text{span}}\{P_{W_j}^*(W_j)\}_{j \in J}$, then we have $\|P_{W_j} f\|^2 = \langle f, P_{W_j}^* P_{W_j} f \rangle = 0$, which implies that $P_{W_j} f = 0$ for all $j \in J$. We also have $f = \sum_{j \in J} P_{W_j} f = 0$. Hence $\mathcal{H} = \overline{\text{span}}\{P_{W_j}^*(W_j)\}_{j \in J}$. It is easy to show that any $f \in \mathcal{H}$ has at least one representation of the form $f = \sum_{j \in J} P_{W_j}^* g_j$ for some sequence $\{g_j : g_j \in W_j\}_{j \in J}$. Now, we that this representation is unique. Assume that $\sum_{j \in J} P_{W_j}^* g_j = 0$, then we have $P_{W_i}^* g_i = P_{W_i}^* (\sum_{j \in J} P_{W_j}^* g_j) = 0$, which implies that $\{P_{W_j}^*(W_j)\}_{j \in J}$ is a f-basis for \mathcal{H} . Also, since $P_{W_i}^* P_{W_j}^* = \delta_{ij} P_{W_j}^*$ for all $i, j \in J$, $\{P_{W_j}^*\}_{j \in J}$ is the f-dual sequence of $\{P_{W_j}^*(W_j)\}_{j \in J}$. ■

Proposition 2.8 Every f-basis for a Hilbert space possesses a unique f-biorthogonal sequence.

Proof. Let $\{W_j\}_{j \in J}$ be a f-basis for \mathcal{H} with f-dual sequence $\{P_{W_j}\}_{j \in J}$. By definition this sequence is a f-biorthogonal sequence of $\{W_j\}_{j \in J}$. Moreover, if $\{Q_{W_j}\}_{j \in J}$ is another f-biorthogonal sequence of $\{W_j\}_{j \in J}$ then for any $f \in \mathcal{H}$ and $i \in J$ we have

$$Q_{W_i} f = \sum_{j \in J} Q_{W_j} P_{W_j} f = \sum_{j \in J} \delta_{ij} P_{W_j} f = P_{W_i} f.$$

Hence $Q_{W_i} = P_{W_i}$. ■

3. Orthonormal fusion bases and Riesz fusion bases

In this section, we develop a theory of orthonormal fusion bases and Riesz fusion bases for the Hilbert spaces.

Definition 3.1 Let $\{W_j\}_{j \in J}$ be a sequence of closed subspaces of \mathcal{H} . Then

- (i) $\{W_j\}_{j \in J}$ is called an orthonormal fusion system or simply a orthonormal f-system for \mathcal{H} , if $\{\pi_{W_j}\}_{j \in J}$ is a f-biorthogonal sequence of $\{W_j\}_{j \in J}$, that is

$$\pi_{W_i} g_j = \delta_{ij} g_j, \quad \forall i, j \in J, g_j \in W_j.$$

- (ii) $\{W_j\}_{j \in J}$ is called an orthonormal f-basis for \mathcal{H} , if it is a complete orthonormal f-system for \mathcal{H} .

Example 3.2 Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for \mathcal{H} . Then

- (i) Define the subspace $W_j \subset \mathcal{H}$ ($j \in \mathbb{N}$) by

$$W_j = \text{span}\{e_{2j-1} + e_{2j}\} \quad \text{and} \quad \pi_{W_j} f = \frac{1}{2} \langle f, e_{2j-1} + e_{2j} \rangle (e_{2j-1} + e_{2j}).$$

Then it is easily checked that $\{W_j\}_{j \in \mathbb{N}}$ is an orthonormal f-system for \mathcal{H} . But it is not an orthonormal f-basis for \mathcal{H} .

(ii) Define the subspace $W_j \subset \mathcal{H}$ ($j \in \mathbb{N}$) by

$$W_j = \text{span}\{e_{2j-1}, e_{2j}\} \quad \text{and} \quad \pi_{W_j} f = \langle f, e_{2j-1} \rangle e_{2j-1} + \langle f, e_{2j} \rangle e_{2j}.$$

Then $\{W_j\}_{j \in \mathbb{N}}$ is an orthonormal f-basis for \mathcal{H} .

Theorem 3.3 Let $\{W_j\}_{j \in J}$ be an orthonormal f-system for \mathcal{H} , then the series $\sum_{j \in J} g_j$ converges if and only if $\{g_j\}_{j \in J} \in (\sum_{j \in J} \oplus W_j)_{\ell^2}$ and in this case $\|\sum_{j \in J} g_j\|^2 = \sum_{j \in J} \|g_j\|^2$.

Proof. For every finite subset $F \subset J$ we have

$$\left\| \sum_{j \in F} g_j \right\|^2 = \sum_{j \in F} \sum_{i \in F} \langle \pi_{W_i} g_j, g_i \rangle = \sum_{j \in F} \sum_{i \in F} \langle \delta_{ij} g_j, g_i \rangle = \sum_{j \in F} \|g_j\|^2.$$

From this the result follows. ■

Theorem 3.4 (Bessel's inequality) Let $\{W_j\}_{j \in J}$ be an orthonormal f-system for \mathcal{H} . Then

$$\sum_{j \in J} \|\pi_{W_j} f\|^2 \leq \|f\|^2 \quad \text{for all } f \in \mathcal{H}.$$

Proof. Let $f \in \mathcal{H}$. Fix $F \subset J$ with $|F| < \infty$. Then By Theorem 3.3 we have

$$\begin{aligned} \left\| f - \sum_{j \in F} g_j \right\|^2 &= \|f\|^2 - \sum_{j \in F} \langle \pi_{W_j} f, g_j \rangle - \sum_{j \in F} \langle g_j, \pi_{W_j} f \rangle + \sum_{j \in F} \|g_j\|^2 \\ &= \|f\|^2 - \sum_{j \in F} \|\pi_{W_j} f\|^2 + \sum_{j \in F} \|\pi_{W_j} f - g_j\|^2, \end{aligned}$$

for arbitrary vectors $g_j \in W_j$. In particular, if $g_j = \pi_{W_j} f$, then

$$\left\| f - \sum_{j \in F} \pi_{W_j} f \right\|^2 = \|f\|^2 - \sum_{j \in F} \|\pi_{W_j} f\|^2.$$

From this we have $\sum_{j \in F} \|\pi_{W_j} f\|^2 \leq \|f\|^2$, which implies that $\sum_{j \in J} \|\pi_{W_j} f\|^2 \leq \|f\|^2$. ■

Corollary 3.5 Let $\{W_j\}_{j \in J}$ be an orthonormal f-system for \mathcal{H} , then for all $f \in \mathcal{H}$ the series $\sum_{j \in J} \pi_{W_j} f$ converges and $\left\| f - \sum_{j \in J} \pi_{W_j} f \right\|^2 \leq \left\| f - \sum_{j \in J} g_j \right\|^2$ for all $\{g_j\}_{j \in J} \in (\sum_{j \in J} \oplus W_j)_{\ell^2}$.

Theorem 3.6 Let $\{W_j\}_{j \in J}$ be an orthonormal f-system for \mathcal{H} . Then the following conditions are equivalent:

- (i) $\{W_j\}_{j \in J}$ is an orthonormal f-basis for \mathcal{H} .
- (ii) $f = \sum_{j \in J} \pi_{W_j} f \quad \forall f \in \mathcal{H}$.
- (iii) $\|f\|^2 = \sum_{j \in J} \|\pi_{W_j} f\|^2 \quad \forall f \in \mathcal{H}$
- (iv) If $\pi_{W_j} f = 0$ for all $j \in J$, then $f = 0$.

Proof. The implication (i) \Rightarrow (ii) follows immediately from Corollary 3.5. The implications (ii) \Rightarrow (iii) \Rightarrow (iv) are obvious. To prove (iv) \Rightarrow (i) suppose that $f \perp \overline{\text{span}}\{W_j\}_{j \in J}$,

then for every $j \in J$ we have $\|\pi_{W_j} f\|^2 = \langle f, \pi_{W_j} f \rangle = 0$ which implies that $f = 0$. It follows that $\mathcal{H} = \overline{\text{span}\{W_j\}_{j \in J}}$. ■

Theorem 3.7 Let $\{(W_j, P_{W_j})\}_{j \in J}$ be a f-basis system for \mathcal{H} and let $T : \mathcal{H} \rightarrow \mathcal{K}$ be a bounded invertible operator such that $V_j = TW_j$ and $Q_{V_j} = TP_{W_j}T^{-1}$ for all $j \in J$. Then $\{(V_j, Q_{V_j})\}_{j \in J}$ is a f-basis system for \mathcal{K} .

Proof. Suppose that $f \in \mathcal{K}$, then we can write $f = Tg$ for some $g \in \mathcal{H}$. By hypothesis g has an unique expansion to form $g = \sum_{j \in J} g_j$ for some sequence $\{g_j : g_j \in W_j\}_{j \in J}$ which implies that f has an unique expansion of the form $f = \sum_{j \in J} f_j$ with $f_j = Tg_j$ for all $j \in J$. We also have

$$Q_{V_j} f_j = TP_{W_j}T^{-1} f_j = T(\delta_{ij}T^{-1} f_j) = \delta_{ij} f_j$$

for arbitrary sequence $\{f_j : f_j \in V_j\}_{j \in J}$. From this the result follows. ■

Definition 3.8 Let $\{W_j\}_{j \in J}$ be a sequence of closed subspaces of \mathcal{H} . Then this sequence is called a Riesz fusion basis or simply Riesz f-basis for \mathcal{H} , if there is an orthonormal f-basis $\{V_j\}_{j \in J}$ for \mathcal{H} and a bounded invertible linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ such that $TV_j = W_j$ ($j \in J$). By Theorem 3.7 if $\{P_{W_j}\}_{j \in J}$ is f-dual sequence of $\{W_j\}_{j \in J}$. Then $P_{W_j} = T\pi_{V_j}T^{-1}$ for all $j \in J$.

Example 3.9 Let $\{f_j\}_{j \in J} = \{Te_j\}_{j \in J}$ be a Riesz basis for \mathcal{H} and let $W_j = \text{span}\{f_j\}$ ($j \in J$). Then $\{W_j\}_{j \in J}$ is a Riesz f-basis for \mathcal{H} . Since the sequence $\{V_j\}_{j \in J}$ defined by $V_j = \text{span}\{e_j\}$ ($j \in J$) is an orthonormal f-basis and $W_j = TV_j$ for all $j \in J$.

Corollary 3.10 If $\{(W_j, P_{W_j})\}_{j \in J}$ is a Riesz f-basis system for \mathcal{H} . Then

$$0 < \inf_{j \in J} \|P_{W_j}\| \leq \sup_{j \in J} \|P_{W_j}\| < \infty.$$

Proof. According to the definition we can write $\{W_j\}_{j \in J} = \{TV_j\}_{j \in J}$, where T is a bounded invertible operator on \mathcal{H} and $\{V_j\}_{j \in J}$ is an orthonormal f-basis for \mathcal{H} . For all $j \in J$ we have

$$\|T^{-1}\|^{-1} \|T\|^{-1} \leq \|P_{W_j}\| \leq \|T\| \|T^{-1}\|.$$

From this the result follows. ■

Proposition 3.11 Let $\{W_j\}_{j \in J} = \{TV_j\}_{j \in J}$ be a Riesz f-basis for \mathcal{H} and let $\{f_j : f_j \in V_j\}_{j \in J}$ and $\{g_j : g_j \in W_j\}_{j \in J}$ be two sequences such that $g_j = Tf_j$ ($j \in J$). Then the series $\sum_{j \in J} g_j$ converges if and only if $\sum_{j \in J} f_j$ is convergent.

Proof. This follows immediately from the fact that for each finite subset $F \subset J$ we have

$$\|T^{-1}\|^{-1} \left\| \sum_{j \in F} f_j \right\| \leq \left\| \sum_{j \in F} g_j \right\| \leq \|T\| \left\| \sum_{j \in F} f_j \right\|.$$

■

Definition 3.12 A family of bounded operators $\{T_j\}_{j \in J}$ on \mathcal{H} is a resolution of the identity on \mathcal{H} , if for any $f \in \mathcal{H}$ we have $f = \sum_{j \in J} T_j f$.

The following result shows another way to obtain a resolution of the identity from a Riesz f-basis.

Proposition 3.13 Let $\{W_j\}_{j \in J} = \{TV_j\}_{j \in J}$ be a Riesz f-basis for \mathcal{H} . Then

- (i) The sequence $\{S_j\}_{j \in J}$ defined by $S_j = T^{-1}\pi_{V_j}T$ ($j \in J$) is a resolution of the identity on \mathcal{H} .
- (ii) The sequence $\{U_j\}_{j \in J}$ defined by $U_j = T^*\pi_{V_j}(T^*)^{-1}$ ($j \in J$) is a resolution of the identity on \mathcal{H} .
- (iii) The sequence $\{R_j\}_{j \in J}$ defined by $R_j = (T^*)^{-1}\pi_{V_j}T^*$ ($j \in J$) is a resolution of the identity on \mathcal{H} .

Proof. This follows immediately from the definition. ■

To check Riesz f-baseness of a family of closed subspaces, we derive the following useful characterization.

Theorem 3.14 Let $\{W_j\}_{j \in J}$ be a sequence of closed subspaces of \mathcal{H} . Then the following statements are equivalent.

- (i) $\{W_j\}_{j \in J}$ is a Riesz f-basis for \mathcal{H} .
- (ii) There is an equivalent inner product on \mathcal{H} , with respect to which the sequence $\{W_j\}_{j \in J}$ becomes an orthonormal f-basis for \mathcal{H} .
- (iii) The sequence $\{W_j\}_{j \in J}$ is complete for \mathcal{H} and there exist positive constants A, B such that for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$ we have

$$A \sum_{j \in F} \|g_j\|^2 \leq \left\| \sum_{j \in F} g_j \right\|^2 \leq B \sum_{j \in F} \|g_j\|^2$$

Proof. (i) \Rightarrow (ii) Assume that $\{W_j\}_{j \in J}$ is a Riesz f-basis and write it in the form $W_j = TV_j$ ($j \in J$). Define a new inner product $\langle \cdot, \cdot \rangle_T$ on \mathcal{H} by $\langle f, g \rangle_T = \langle T^{-1}f, T^{-1}g \rangle$. If $\|\cdot\|_T$ is the norm generated by this inner product, then for all $f \in \mathcal{H}$ we have $\|T\|^{-1}\|f\| \leq \|f\|_T \leq \|T^{-1}\|\|f\|$, which implies that the new inner product is equivalent to the original one. Let $\{P_{W_j}\}_{j \in J}$ be f-dual sequence of $\{W_j\}_{j \in J}$, then we have

$$\begin{aligned} \langle P_{W_j}f, g \rangle_T &= \langle T\pi_{V_j}T^{-1}f, g \rangle_T = \langle \pi_{V_j}T^{-1}f, T^{-1}g \rangle \\ &= \langle T^{-1}f, \pi_{V_j}T^{-1}g \rangle = \langle f, P_{W_j}g \rangle_T. \end{aligned}$$

It follows that $P_{W_j} : \mathcal{H} \rightarrow W_j$ is an orthogonal projection with respect to $\langle \cdot, \cdot \rangle_T$. Hence $\{W_j\}_{j \in J}$ is an orthonormal f-basis with respect to inner product $\langle \cdot, \cdot \rangle_T$.

(ii) \Rightarrow (iii) Suppose that $\langle \cdot, \cdot \rangle_1$ is an equivalent inner product on \mathcal{H} and let $\{W_j\}_{j \in J}$ be an orthonormal f-basis with respect to $\langle \cdot, \cdot \rangle_1$. Then there exist positive constants m, M such that

$$m\|f\| \leq \|f\|_1 \leq M\|f\| \quad \forall f \in \mathcal{H}.$$

Now, for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$ we obtain

$$\begin{aligned} \frac{m^2}{M^2} \sum_{j \in F} \|g_j\|^2 &\leq \frac{1}{M^2} \sum_{j \in F} \|g_j\|_1^2 = \frac{1}{M^2} \left\| \sum_{j \in F} g_j \right\|_1^2 \\ &\leq \left\| \sum_{j \in F} g_j \right\|^2 \leq \frac{1}{m^2} \left\| \sum_{j \in F} g_j \right\|_1^2 \\ &= \frac{1}{m^2} \sum_{j \in F} \|g_j\|_1^2 \leq \frac{M^2}{m^2} \sum_{j \in F} \|g_j\|^2. \end{aligned}$$

(iii) \Rightarrow (i) Let $\{V_j\}_{j \in J}$ be an orthonormal f-basis for \mathcal{H} . Define the mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ by

$$T\pi_{V_j}f = \pi_{W_j}f, \quad \forall f \in \mathcal{H}, j \in J.$$

For any vectors $f_j \in V_j$ ($j \in J$) we have

$$\left\| T \left(\sum_{j \in J} f_j \right) \right\|^2 = \left\| \sum_{j \in J} \pi_{W_j} f_j \right\|^2 \leq B \sum_{j \in J} \|\pi_{W_j} f_j\|^2 \leq B \left\| \sum_{j \in J} f_j \right\|^2.$$

It follows that T is a bounded linear operator on \mathcal{H} . Similarly, define the mapping $S : \mathcal{H} \rightarrow \mathcal{H}$ by

$$S\pi_{W_j}f = \pi_{V_j}f, \quad \forall f \in \mathcal{H}, j \in J.$$

We also obtain

$$\left\| S \left(\sum_{j \in J} g_j \right) \right\|^2 = \left\| \sum_{j \in J} \pi_{V_j} g_j \right\|^2 \leq \sum_{j \in J} \|g_j\|^2 \leq \frac{1}{A} \left\| \sum_{j \in J} g_j \right\|^2,$$

for all vectors $g_j \in W_j$ ($j \in J$). Since $\{W_j\}_{j \in J}$ is complete, S is also a linear bounded operator on \mathcal{H} and $TS = ST = Id_{\mathcal{H}}$. Hence T is invertible and $TV_j = W_j$ for all $j \in J$. ■

Corollary 3.15 Let $\{W_j\}_{j \in J}$ be a sequence of closed subspaces of \mathcal{H} and let $\{e_{ij}\}_{i \in I_j}$ be an orthonormal basis for each subspace W_j for all $j \in J$. Then the following conditions are equivalent.

- (i) $\{W_j\}_{j \in J}$ is a Riesz f-basis for \mathcal{H} .
- (ii) $\{e_{ij}\}_{j \in J, i \in I_j}$ is a Riesz basis for \mathcal{H} .

Proof. (i) \Rightarrow (ii) Assume that $\{W_j\}_{j \in J}$ is a Riesz f-basis. By Theorem 3.14 there exist constants $A, B > 0$ such that for any finite subset $F \subset J$ and arbitrary vectors $g_j \in W_j$ we have

$$A \sum_{j \in F} \|g_j\|^2 \leq \left\| \sum_{j \in F} g_j \right\|^2 \leq B \sum_{j \in F} \|g_j\|^2.$$

Fix $G_j \subset I_j$ with $|G_j| < \infty$ and let $\{c_{ij}\}_{j \in F, i \in G_j}$ be arbitrary sequence. Then we compute

$$\begin{aligned} A \sum_{j \in F} \sum_{i \in G_j} |c_{ij}|^2 &= A \sum_{j \in F} \left\| \sum_{i \in G_j} c_{ij} e_{ij} \right\|^2 \leq \left\| \sum_{j \in F} \sum_{i \in G_j} c_{ij} e_{ij} \right\|^2 \\ &\leq B \sum_{j \in F} \left\| \sum_{i \in G_j} c_{ij} e_{ij} \right\|^2 = B \sum_{j \in F} \sum_{i \in G_j} |c_{ij}|^2. \end{aligned}$$

Now by Theorem 3.6.6 in [6], $\{e_{ij}\}_{j \in J, i \in I_j}$ is a Riesz basis for \mathcal{H} .

(ii) \Rightarrow (i) Since $\{e_{ij}\}_{j \in J, i \in I_j}$ is a Riesz basis for \mathcal{H} , there exist constants $A, B > 0$ such that

$$A \sum_{j \in F} \sum_{i \in G_j} |c_{ij}|^2 \leq \left\| \sum_{j \in F} \sum_{i \in G_j} c_{ij} e_{ij} \right\|^2 \leq B \sum_{j \in F} \sum_{i \in G_j} |c_{ij}|^2,$$

where $G_j \subset I_j$ with $|G_j| < \infty$ and $\{c_{ij}\}_{j \in F, i \in G_j}$ is an arbitrary sequence. Now for every arbitrary vectors $g_j \in W_j$ we have

$$\begin{aligned} A \sum_{j \in F} \|g_j\|^2 &= A \sum_{j \in F} \sum_{i \in I_j} |\langle g_j, e_{ij} \rangle|^2 \leq \left\| \sum_{j \in F} \sum_{i \in I_j} \langle g_j, e_{ij} \rangle e_{ij} \right\|^2 \\ &\leq B \sum_{j \in F} \sum_{i \in I_j} |\langle g_j, e_{ij} \rangle|^2 = B \sum_{j \in F} \|g_j\|^2. \end{aligned}$$

Since $\left\| \sum_{j \in F} g_j \right\|^2 = \left\| \sum_{j \in F} \sum_{i \in I_j} \langle g_j, e_{ij} \rangle e_{ij} \right\|^2$, the result follows at once from Theorem 3.14. ■

The following result have proved by Gavruta in [10].

Theorem 3.16 Let $\{(V_j, \alpha_j)\}_{j \in J}$ be a fusion frame with fusion frame bounds C and D . Then $\{(TV_j, \alpha_j)\}_{j \in J}$ is a fusion frame with fusion frame bounds $C\|T\|^{-2}\|T^{-1}\|^{-2}$ and $D\|T\|^2\|T^{-1}\|^2$, where $T : \mathcal{H} \rightarrow \mathcal{H}$ is an invertible operator.

Corollary 3.17 If $\{W_j\}_{j \in J}$ is a Riesz f-basis for \mathcal{H} , then $\{(W_j, 1)\}_{j \in J}$ is a 1-uniform fusion frame with fusion frame bounds $\|T\|^{-2}\|T^{-1}\|^{-2}$ and $\|T\|^2\|T^{-1}\|^2$.

A fusion frame $\{(W_j, \alpha_j)\}_{j \in J}$ is called exact, if it ceases to be a fusion frame whenever anyone of its element is deleted.

Theorem 3.18 Let $\{W_j\}_{j \in J}$ be a Riesz f-basis for \mathcal{H} , then $\{(W_j, 1)\}_{j \in J}$ is a 1-uniform exact fusion frame for \mathcal{H} . But the opposite implication is not valid.

Proof. Let $\{e_{ij}\}_{i \in I_j}$ be an orthonormal basis for W_j ($j \in J$). By corollary 3.15 $\{e_{ij}\}_{j \in J, i \in I_j}$ is a Riesz basis for \mathcal{H} and hence it is an exact frame. Now by Lemma 4.5 in [5] $\{(W_j, 1)\}_{j \in J}$ is a 1-uniform exact fusion frame for \mathcal{H} . For the opposite implication is not valid suppose that $\{e_i\}_{i \in \mathbb{Z}}$ is an orthonormal basis for \mathcal{H} and define the subspaces W_1 and W_2 by

$$W_1 = \overline{\text{span}}\{e_i\}_{i \geq 0} \quad \text{and} \quad W_2 = \overline{\text{span}}\{e_i\}_{i < 0}.$$

Then $\{(W_1, 1), (W_2, 1)\}$ is a 1-uniform exact fusion frame but is not a Riesz f-basis for \mathcal{H} . ■

4. The stability of f-bases under perturbations

The stability of bases is important in practice and is therefore studied widely by many authors, e.g., see [16]. In this section we study the stability of f-bases for a Hilbert space \mathcal{H} . First we generalized a result of Paley-Wiener [16] to the situation of f-basis.

Theorem 4.1 Let $\{W_j\}_{j \in J}$ be a f-basis for \mathcal{H} and suppose that $\{V_j\}_{j \in J}$ is a family of closed subspaces of \mathcal{H} such that $\|\sum_{j \in F} (g_j - f_j)\| \leq \lambda \|\sum_{j \in F} g_j\|$, where $0 \leq \lambda < 1$ and $F \subset J$ is any finite subset and $g_j \in W_j, f_j \in V_j$. Then $\{V_j\}_{j \in J}$ is a f-basis for \mathcal{H} .

Proof. The assumption follows that the series $\sum_{j \in J} (g_j - f_j)$ converges if the series $\sum_{j \in J} g_j$ is convergent for all sequences $\{g_j : g_j \in W_j\}_{j \in J}$ and $\{f_j : f_j \in V_j\}_{j \in J}$. Define the mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ by $T\pi_{W_j}f = \pi_{W_j}f - \pi_{V_j}f$ for all $f \in \mathcal{H}$ and $j \in J$. Let $\{g_j : g_j \in W_j\}_{j \in J}$ be arbitrary sequence and $f = \sum_{j \in J} g_j$, then we have $\|Tf\| = \|\sum_{j \in J} (g_j - \pi_{V_j}g_j)\| \leq \lambda \|f\|$. It follows that T is a bounded linear operator and $\|T\| \leq \lambda < 1$. Thus the operator $Id_{\mathcal{H}} - T$ is invertible and $(Id_{\mathcal{H}} - T)W_j = V_j$ for all $j \in J$. Now the result follows from Theorem 3.7. ■

A family of subspaces $\{W_j\}_{j \in J}$ is called minimal, if $W_i \cap \overline{\text{span}}_{\substack{j \in J \\ j \neq i}} \{W_j\} = \{0\}$ for all $i \in J$.

Proposition 4.2 Let $\{W_j\}_{j \in J}$ be a sequence of closed subspaces of \mathcal{H} . Then

- (i) $\{W_j\}_{j \in J}$ has a f-biorthogonal sequence, if and only if it is minimal.
- (ii) The f-biorthogonal sequence of $\{W_j\}_{j \in J}$ is unique if and only if it is complete.

Proof. For the proof of (i) suppose that $\{P_{W_j}\}_{j \in J}$ is a f-biorthogonal sequence of $\{W_j\}_{j \in J}$ and let $f \in W_i \cap \overline{\text{span}}_{\substack{j \in J \\ j \neq i}} \{W_j\}$ for any given $i \in J$. Then $f = g_i = \sum_{\substack{j \in J \\ j \neq i}} g_j$ for some sequence $\{g_j : g_j \in W_j\}_{j \in J}$. We also have

$$f = g_i = P_{W_i}g_i = \sum_{\substack{j \in J \\ j \neq i}} P_{W_i}g_j = \sum_{\substack{j \in J \\ j \neq i}} \delta_{ij}g_j = 0.$$

It follows that $f = 0$, that is $\{W_j\}_{j \in J}$ is minimal. For the opposite implication in (i), suppose that $\{W_j\}_{j \in J}$ is minimal, and let $\mathcal{H}_0 = \overline{\text{span}}\{W_j\}_{j \in J}$. It follows that $\{W_j\}_{j \in J}$ is a f-basis for \mathcal{H}_0 . Let $\{P'_{W_j}\}_{j \in J}$ is a f-dual sequence of $\{W_j\}_{j \in J}$ for \mathcal{H}_0 . If we define $P_{W_j} = P'_{W_j}\pi_{\mathcal{H}_0}$ ($j \in J$). Then $\{P_{W_j}\}_{j \in J}$ is a f-biorthogonal sequence for $\{W_j\}_{j \in J}$.

(ii) Let $\{P_{W_j}\}_{j \in J}$ be a f-biorthogonal sequence of $\{W_j\}_{j \in J}$. If $\{W_j\}_{j \in J}$ is not complete, then the sequence $\{Q_{W_j}\}_{j \in J}$ defined by $Q_{W_j} = P_{W_j} + P_{W_j}(Id_{\mathcal{H}} - \pi_{\mathcal{H}_0})$ ($j \in J$), is a f-biorthogonal sequence for $\{W_j\}_{j \in J}$. For the other implication in (ii), assume that $\{W_j\}_{j \in J}$ is complete. Let $\{g_j : g_j \in W_j\}_{j \in J}$ be arbitrary sequence and $\sum_{j \in J} g_j = 0$, then we have

$$g_i = \sum_{j \in J} \delta_{ij}g_j = \sum_{j \in J} P_{W_i}g_j = P_{W_i}(\sum_{j \in J} g_j) = 0.$$

This shows that $\{W_j\}_{j \in J}$ is a f-basis for \mathcal{H} . Now the conclusion follows from Proposition 2.8.

Theorem 4.3 Let $\{(W_j, P_{W_j})\}_{j \in J}$ be a f-basis system for \mathcal{H} and suppose that $\{V_j\}_{j \in J}$ is a family of closed subspaces of \mathcal{H} . If $\{Q_{V_j}\}_{j \in J}$ is a f-biorthogonal sequence of $\{V_j\}_{j \in J}$

such that

$$\left\| \sum_{j \in F} (P_{W_j} f - Q_{V_j} f) \right\| \leq \lambda \left\| \sum_{j \in F} P_{W_j} f \right\| \quad \forall f \in \mathcal{H},$$

for some constant $0 \leq \lambda < 1$ and any finite subset $F \subset J$. Then $\{V_j\}_{j \in J}$ is a f-basis for \mathcal{H} .

Proof. Define the mapping $T : \mathcal{H} \rightarrow \mathcal{H}$, by $TP_{W_j} f = P_{W_j} f - Q_{V_j} f$ for all $f \in \mathcal{H}$ and $j \in J$. Then as the proof of Theorem 4.1 the operator $Id_{\mathcal{H}} - T$ is invertible and $(Id_{\mathcal{H}} - T)W_j = V_j$ ($j \in J$). Now the claim follows from Theorem 3.7. ■

Acknowledgements

The authors would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help to improve the paper.

References

- [1] M. S. Asgari, G. Kavian, Expansion of Bessel and g-Bessel sequences to dual frames and dual g-frames, J. Linear and Topological Algebra, Vol. 02, No. 01, 2013, 51- 57
- [2] M. S. Asgari, New characterizations of fusion frames (frames of subspaces), Proc. Indian Acad. Sci. (Math. Sci.) 119 No. 3 (2009), 1-14.
- [3] M. S. Asgari, On the stability of fusion frames (frames of subspaces), Acta Math. Sci. Ser. B, 31(4), (2011), 1633-1642.
- [4] M. S. Asgari, Operator-valued bases on Hilbert spaces, J. Linear and Topological Algebra, Vol. 02, No. 04, (2013), 201-218.
- [5] P. G. Casazza and G. Kutyniok, Frames of subspaces, in Wavelets, Frames and Operator Theory (College Park, MD, 2003), Contemp. Math. 345, Amer. Math. Soc. Providence, RI, 2004, 87-113.
- [6] O. Christensen, An Introduction to frames and Riesz Bases, Birkhauser, Boston, 2003.
- [7] I. Daubechies, A. Grossmann and Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27(1986), 1271-1283.
- [8] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72,(2), (1952), 341-366.
- [9] M. Fornasier, Quasi-orthogonal decompositions of structured frames, J. Math. Anal. Appl. 289 (2004), 180-199.
- [10] P. Gavruta, On the duality of fusion frames, J. Math. Anal. Appl. 333 (2007), 871-879.
- [11] J.R. Holub, Pre-frame operators, Besselian frames and near-Riesz bases in Hilbert spaces, Proc. Amer. Math. Soc. 122 (1994) 779-785.
- [12] V. Kaftal, D. R. Larson and Sh. Zhang, Operator-valued frames, Trans. Amer. Math. Soc. 361 (2009), 6349-6385.
- [13] S. S. Karimizad, G-frames, g-orthonormal bases and g-Riesz bases, J. Linear and Topological Algebra, Vol. 02, No. 01, 2013, 25-33.
- [14] W. Rudin, Functional Analysis, McGrawHill. Inc, New York, (1991).
- [15] W. Sun, G-frames and G-Riesz bases, J. Math. Anal. Appl. (2006), 322, 437-452.
- [16] R. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 2001.