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# A mathematically simple method based on definition for computing eigenvalues, generalized eigenvalues and quadratic eigenvalues of matrices

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**Abstract.**In this paper, a fundamentally new method, based on the definition, is introduced for numerical computation of eigenvalues, generalized eigenvalues and quadratic eigenvalues of matrices. Some examples are provided to show the accuracy and reliability of the proposed method. It is shown that the proposed method gives other sequences than that of existing methods but they still are convergent to the desired eigenvalues, generalized eigenvalues and quadratic eigenvalues of matrices. These examples show an interesting phenomenon in the procedure: The diagonal matrix that converges to eigenvalues gives them in decreasing order in the sense of absolute value. Appendices A to C provide Matlab codes that implement the proposed algorithms. They show that the proposed algorithms are very easy to program.

**Keywords:** Eigenvalue; Generalized eigenvalue; Quadratic eigenvalue; Numerical computation; Iterative method.

## 1. Introduction

The eigenvalue problem for square matrices $A$, that is the determination of $\lambda$'s for which there are nontrivial solutions for $A\mathbf{x} = \lambda\mathbf{x}$, is a central topic in numerical linear algebra. This problem is inherently nonlinear and this leads to many computational problems. Computation of the eigenvalues $\lambda$ via the explicit construction of the characteristic equation

$$det(A - \lambda I) = 0 \tag{1}$$

is, except for very special cases, not an option since the coefficients of the characteristic equation cannot be computed from determinant evaluations in a numerically

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stable way [1] , and even if the characteristic equation could be determined accurately, then the computation of its roots, in finite precision, may be highly unstable since small perturbations in the coefficients may lead to large perturbations of the roots [2]. The numerical computation of the associated eigenvectors and generalized eigenvectors is even more delicate, in particular when eigenvectors of $A$ make small angles with each other. In the limiting case, when the matrix is defective, $A$ can be reduced to the Jordan canonical form, but arbitrary small perturbations in $A$ may yield a nondefective matrix [1]. This leads to many challenging numerical questions, which give rise to the central problem: how can we compute eigenvalues and eigenvectors in an efficient manner and how accurate are they? In fact, this was already recognized by Jacobi, who, in 1846, computed the eigenvalues of symmetric matrices by rotating the matrix to a strongly diagonally dominant one. Jacobi's techniques are still relevant and have led to popular and powerful algorithms [1]. Another longstanding method that is of great significance and serves as the basis for many algorithms is the Power iteration. The method is based on the idea that if a given matrix is repeatedly applied to a vector, and is properly normalized, then ultimately, it will lie in the direction of the eigenvector associated with the eigenvalues which are largest in absolute value [3]. The rate of convergence for the Power iteration depends on the ratio of the second largest eigenvalue (in absolute value) to the largest eigenvalue (in absolute value) and for many applications this leads to unacceptably slow convergence. The method can be problematic if one wants to compute a number of extremal eigenvalues. The Power iteration is still in use, but most frequently as (implicit) part of more efficient techniques, e.g., Krylov methods, inverse iteration, $QR$-method. We refer the reader to [1] for an excellent and complete overview on the research on computational aspects of the eigenvalue problem.

The topic of inclusion regions of matrix eigenvalues is worthy investigating in practice as well as in theory. For instance, in [7] and [8] chapters on this topic are particularly designed.
Let $A = (a_{ij}) \in C^{n\times n}$ and $r_i = \Sigma_{j\neq i} \mid a_{ij} \mid (i = 1, 2, ..., n)$. Brauer's theorem provides eigenvalue inclusion region composed of so-called Cassini's ovals in the complex plane

$$D_{ij} = \{z \in C :\mid z - a_{ii} \mid\mid z - a_{jj} \mid\leqslant r_i r_j\}, 1 \leqslant i \neq j \leqslant n. \tag{2}$$

Theorem 1. ([7], p. 380) All eigenvalues of A are contained in

$$\Omega = \bigcup_{i\neq j} D_{ij}. \tag{3}$$

It can be verified that $\Omega$ in (3) is contained in the union of the famous Gerschgorin's disks. As mentioned in literature (for example [9], p. 235), it is not substantially difficult to determine whether a point in complex plane locates in an oval or not. In [10] eigenvalues that locate on the boundary of $\Omega$ are discussed under the as-

sumption that $A$ is irreducible. It pointed out an error in an often quoted assertion of Brauer [11] and presented a simplified Brauer's theorem in terms of directed graph. Their results were clarified by Tam, Yang and Zhang in [12].

Numerous papers are devoted to the study of asymptotics of the distribution of eigenvalues and norms of random matrices. We shall mention papers [13] to [16] in which it was proved, apparently for the first time, that the spectral norm of a matrix with independent, identically distributed bounded random entries with zero mathematical expectations is equivalent in the probability sense to the square root of the order of the matrix as this order increases to infinity. In these papers probability inequalities for the norms of random matrices were obtained but the exact bounds were not proved. Along with these investigations, limit theorems for eigenvalues (in particular norms) of random matrices were proved. Although this is rather a new field of investigation, quite a few results have already been accumulated in papers [2] to [**?** ].

After reading [4], where the idea behind the $QR$ method and its relations to the power-family methods have been described elegantly, we were led to find the present method. First we thought it was somehow equivalent to the $QR$ algorithm but later, when we tested it independently on some problems, we found out that it gives a different sequence of numbers and, in general, it still gives all of the eigenvalues as the limit of the sequences

## 2. Basic Concept

This paper will be concerned with high-order collocation methods for the Fredholm-Volterra integral equations (FVIEs)

$$y(t) = g(t) + \int_a^b p(t,s)k(t,s,y(s))ds + \int_0^t p'(t,s)k'(t,s,y(s))ds, \qquad t \in [0,T] \quad (4)$$

where $y(t)$ is the unknow function whose value is to be determined in the interval $0 \leqslant t \leqslant T < \infty$, the kernels $k(t,s,y(s))$ and $k'(t,s,y(s))$ are lipschits continuous in their variable and $p(t,s)$ and $p'(t,s)$ are unbounded in the region of integration but integrable over $[0,T]$.

The following notation and methods were introduced in [4, 5] and will be used throughout this paper. The collocation methods generate , as approximation to the solution of (1) elements of the polynomial spline space

$$S_{m-1}^{(d)}(Z_N,T) := \{u \in C^{(d)}(I(T)) : u\mid_{\sigma_n} := u_n \in \pi_{m-1}, 0 \leqslant n \leqslant N-1\}, \qquad (5)$$

associated with a given partition

$$\Pi_N : 0 = t_0 < t_1 < \cdots < t_N = T, \quad N \geqslant 1 \tag{6}$$

of the interval $[0, T]$. Here, $\pi_{m-1}$ is the set of real polynomials of degree not exceeding $m-1$ and we have set $\sigma_0 := [t_0, t_1]$ and $\sigma_n := (t_n, t_{n+1}], n = 1, \ldots, N-1$, $Z_N := \{t_n : 1 \leqslant n \leqslant N-1\}$ (the set of interior grid points). The quantity $h$, $h := \max\{h_n := t_{n+1} - t_n : 0 \leqslant n \leqslant N-1\}$, is often called the diameter of the grid $\prod_N$. If $h_n \equiv h$ all $0 \leqslant n \leqslant N-1$, then the grid $\prod_N$ is called a uniform mesh.

The desired approximation to $y$ is the element $u \in S^{(d)}_{m-1}(Z_N, T)$ satisfying

$$u(t) = g(t) + \int_a^b p(t,s)k(t,s,u(s))ds + \int_0^t p'(t,s)k'(t,s,u(s))ds, \quad t \in X(N) \tag{7}$$

where $X(N) := \bigcup_{n=0}^{N-1} X_n$ with

$$X_N := \{t_{nj} := t_n + c_j h_n : 0 \leqslant c_1 < \cdots < c_m \leqslant 1\},$$

where $\{c_j\}_{j=1}^m$ are collocation parameters.

## 3. A generalized Grownwall-Type inequality

Throughout this paper, $c_i$ where $i$ is an integer, will denoted constants which are independent of $h$ .

**Definition 3.1.** Let $p_1(t,s) := p(t,s), p_1'(t,s) := p'(t,s)$ and set

$$\begin{array}{l} p_n(t,s) := \int_a^b p_1(t,\xi)p_{n-1}(\xi,s)d\xi \\ p_n'(t,s) := \int_0^t p_1'(t,\xi)p_{n-1}'(\xi,s)d\xi \ (t,s) \in S, n \geqslant 2 \end{array} \tag{8}$$

where $S := \{(t,s), 0 \leqslant s < t \leqslant T\}$. The functions $\{p_n, p_n', n = 1, 2, \ldots\}$ are called the iterated kernels associated with the given kernels $p$ and $p'$.

**Definition 3.2.** If the functions $p$ and $p'$ satisfies

$$(i)p(t,s) \geqslant 0, \qquad p'(t,s) \geqslant 0, (t,s) \in S \tag{9}$$

$$(ii) \int_a^b p(t,s)dt \leqslant c_1, \qquad \int_0^t p'(t,s)dt \leqslant c_1' \tag{10}$$

$$(iii)p_v(t,s) \leqslant c_2, \qquad p_v'(t,s) \leqslant c_2', (t,s) \in S \tag{11}$$

where $v$ is a certain integer, then $p$ and $p'$ are said to satisfy conditions $C$.

**Theorem 3.1.** Let $A \geqslant 0$ be a constant, and Let the function $x$ satisfy to condition $C$. The function $x(t)$ is defined as

$$x(t) = \kappa_n, \quad t \in [t_n, t_{n+1}], \quad 0 \leqslant n \leqslant N-1 \tag{12}$$

where the $t_n$ is given by (3) and $\kappa_n \geqslant 0$, if the function $x$ satisfies the integral inequality

$$x(t) \leqslant \int_a^b p(t,s)x(s)ds + \int_0^t p'(t,s)x(s)ds + A \quad t \in [0,T) \tag{13}$$

then it can be bounded by

$$x(t) \leqslant c_2 \int_a^b x(s)ds + c_2' \int_0^t x(s)ds + c_3 A \quad t \in [0,T) \tag{14}$$

furthermore, if $h := \max\{h_n := t_{n+1} - t_n, 0 \leqslant n \leqslant N-1\} \leqslant c_4/N$, then

$$\kappa := \max\{\kappa_n, 0 \leqslant n \leqslant N-1\} \leqslant c_5 A \tag{15}$$

*Proof.* Consider

$$x(s) \leqslant \int_a^b p(s,\lambda)x(\lambda)d\lambda + A_1 \tag{16}$$

$$x'(s) \leqslant \int_a^b p'(s,\lambda)x(\lambda)d\lambda + A_2 \tag{17}$$

Where $A_1, A_2 \geqslant 0$ and $A_1 + A_2 = A$.

Multiplying (13) by $p(t,s)$ and integrate from $a$ to $b$ and multiplying (14) by $p'(t,s)$ and integrate from 0 to $t$, so we have

$$\int_a^b p(t,s)x(s)ds \leqslant \int_a^b \int_a^b p(t,s)p(s,\lambda)x(\lambda)d\lambda ds + c_1 A_1$$

$$\int_0^t p'(t,s)x(s)ds \leqslant \int_0^t \int_0^s p'(t,s)p'(s,\lambda)x(\lambda)d\lambda ds + c_1' A_2$$

or

$$\int_a^b p(t,s)x(s)ds \leqslant \int_a^b p_2(t,s)x(s)ds + c_1A_1 \tag{18}$$

$$\int_0^t p'(t,s)x(s)ds \leqslant \int_0^t p_2'(t,s)x(s)ds + c_1'A_2 \tag{19}$$

By adding (15) and (16) we obtain

$$\int_a^b p(t,s)x(s)ds+\int_0^t p'(t,s)x(s)ds \leqslant \int_a^b p_2(t,s)x(s)ds+\int_0^t p_2'(t,s)x(s)ds+c_1A_1+c_1'A_2$$

From (10) we have

$$x(t) \leqslant \int_a^b p_2(t,s)x(s)ds + \int_0^t p_2'(t,s)x(s)ds + [(1+c_1)A_1 + (1+c_1')A_2]$$

Repeating the above procedure, we have

$$x(t) \leqslant \int_a^b p_\nu(t,s)x(s)ds + \int_0^t p_\nu'(t,s)x(s)ds + \sum_{j=0}^{\nu-1}[(1+c_1)A_1 + (1+c_1')A_2]^j$$

From (8)we have

$$x(t) \leqslant c_2\int_a^b x(s)ds + c_2'\int_0^t x(s)ds + c_3 \tag{20}$$

where $c_3 = \sum\limits_{j=0}^{\nu-1}[(1+c_1)A_1 + (1+c_1')A_2]^j$, nothing that $h \leqslant \frac{c_4}{N}$ from (9) and (17) we obtain

$$\kappa_n \leqslant c_2'c_4\sum_{i=0}^{n-1}\kappa_i'\frac{1}{N} + D$$

where $D = c_2c_4\sum_{i=a}^{b}\kappa_i\frac{1}{N} + c_3, 0 \leqslant n \leqslant N-1$. The above inequality is the standard discrete Gronwall inequality which yields (12).

## 4. Convergence of collection methods

The quadratic eigenproblem

$$\lambda^2 M\mathbf{x} + \lambda C\mathbf{x} + K\mathbf{x} = 0 \tag{21}$$

is equivalent to the generalized eigenprobem [5, 6]:

$$A\mathbf{z} = \lambda \mathbf{B}\mathbf{z}, \tag{22}$$

with

$$A = \begin{pmatrix} 0 & I \\ -K & -C \end{pmatrix}, \quad B = \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \lambda\mathbf{x} \end{pmatrix}, \tag{23}$$

which can be solved by the method described in previous section. Appendix C presents the Matlab program which carries out this task.
*Example. 4.1.* In this example, we take

$$M = \begin{pmatrix} 3.3685 & 5.5651 \\ 10.3814 & -2.1611 \end{pmatrix}$$

$$C = \begin{pmatrix} 14.7529 & 3.4679 \\ -6.3062 & 11.1767 \end{pmatrix}$$

and

$$K = \begin{pmatrix} -15.5876 & -7.7931 \\ 1.0561 & -1.8346 \end{pmatrix}$$

Performing 6 iterations of (**??**), with $A$ and $B$ given by (23), we obtain

$$D_6 = \begin{pmatrix} -2.4754 & 0 & 0 & 0 \\ 0 & 1.4514 & 0 & 0 \\ 0 & 0 & 0.8622 & 0 \\ 0 & 0 & 0 & 0.1828 \end{pmatrix}$$

The exact quadratic eigenvalues computed by Matlab are

$$Quad(M, C, K) = \begin{pmatrix} -2.4412 \\ 1.4706 \\ 0.8625 \\ 0.1828 \end{pmatrix}$$

*Example. 4.2.* In this example, we take

$$M = \begin{pmatrix} 1.6310 & 3.8064 & 4.8423 \\ -8.4379 & -3.4994 & -5.8690 \\ 4.7768 & 3.8880 & 6.3983 \end{pmatrix}$$

$$C = \begin{pmatrix} -8.8674 & -5.4116 & 4.1484 \\ -4.0984 & 5.1078 & -0.3077 \\ 0.6829 & -2.5514 & 1.2613 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 1.0896 & 2.3755 & 0.7897 \\ 2.4963 & -1.4248 & 3.5624 \\ -5.5997 & -3.3813 & -2.9438 \end{pmatrix}$$

Performing 7 iterations of (**??**), with $A$ and $B$ given by (23), we obtain

$$D_7 = \begin{pmatrix} 7.3333 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.9818 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.6554 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.7193 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4601 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1467 \end{pmatrix}$$

The exact quadratic eigenvalues computed by Matlab are

$$Quad(M, C, K) = \begin{pmatrix} 7.3302 \\ 1.6866 \\ -1.8004 \\ -0.7291 \\ -0.1468 \\ 0.4904 \end{pmatrix}$$

## Appendix A. Matlab program for computing eigenvalues

$A = input('A =');$
$it = input('number\ of\ iterations =');$
$[n, m] = size(A);$
$Q = eye(n);\ R = eye(n);$
$for\ i = 1 : it$

```
    [Q, Rp] = qr(A * Q * R);
    D = diag(diag(Rp));
    for j = 1 : n
        R(:, j) = Rp(:, j)/D(j, j);
    end
end
D
```

**Appendix B. Matlab program for computing generalized eigenvalues**

```
A = input('A =');
B = input('B =');
it = input('number of iterations =');
[n, m] = size(A);
Q = eye(n); R = eye(n);
for i = 1 : it
    V = inv(B) * Q * R;
    [Q, Rp] = qr(A * V);
    D = diag(diag(Rp));
    for j = 1 : n
        R(:, j) = Rp(:, j)/D(j, j);
    end
end
D
```

**Appendix C. Matlab program for computing quadratic eigenvalues**

```
M = input('M =');
C = input('C =');
K = input('K =');
[n, m] = size(M);
Temp1 = horzcat(zeros(n), eye(n));
Temp2 = horzcat(−K, −C);
A = vertcat(Temp1, Temp2);
Temp1 = horzcat(eye(n), zeros(n));
Temp2 = horzcat(zeros(n), M);
B = vertcat(Temp1, Temp2);
it = input('number of iterations =');
```

$[n, m] = size(A);$
$Q = eye(n);\ R = eye(n);$
$for\ i = 1 : it$
$\quad V = inv(B) * Q * R;$
$\quad [Q, Rp] = qr(A * V);$
$\quad D = diag(diag(Rp));$
$\quad for\ j = 1 : n$
$\quad\quad R(:, j) = Rp(:, j)/D(j, j);$
$\quad end$
$end$
$D$

## References


[1] G.H. Golub, H.A. van der Vorst, Eigenvalue computation in the 20th century, J. Comput. Appl. Math., 123 (2000), pp. 35-65.
[2] N. Papathanasiou, P. Psarrakos, On condition numbers of polynomial eigenvalue problems, Appl. Math. Comput., 4 (2010), pp. 1194-205.
[3] J.E. Roman, M. Kammerer, F. Merz and F. Jenko, Fast eigenvalue calculations in a massively parallel plasma turbulence code, Parallel Computing, 5-6 (2010), pp. 339-58.
[4] D.S. Watkins, Understanding the $QR$ Algorithm, SIAM Review, Vol. 24, No. 4. (Oct., 1982), pp. 427-440, Jstor.
[5] F. Gantmacher, The Theory of Matrices, Vols. I and II, Chelsea, New York, 1959.
[6] P. Lancaster, Lambda-Matrices and Vibrating Systems, Pergamon Press, Oxford, UK, 1969.
[7] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, London, 1985.
[8] A. S. Householder, The Theory of Matrices in Numerical Analysis, Blaisdell Publishing Company, New York, 1964.
[9] Chen Gongning, Matrix Theory with Applications ,Higher Education Publishing House ,Beijing, 1990. (in Chinese)
[10] Zhang Xian and Gu Dunhe, A note on A. Brauer's theorem, Linear Algebra Appl., 196 (1994) pp. 163-174.
[11] A. Brauer, Limits for the characteristic roots of a matrix IV, Duke Math. J., 19 (1952) pp. 75-91.
[12] Tam Bit-shun,Yang Shangjun and Zhang Xiaodong, *Invertibility of irreducible matrices*, Linear Algebra Appl., 259 (1996) pp. 39-70.
[13] G. Bennet, V. Goodman, and C. M. Newman, Norm of random matrices, Pac. J. Math., 59 (1975) pp. 359-365.
[14] B. S. Kashin, On the mean value of certain function connected with the convergence of orthogonal series, Anal. Math., 4 (1978) pp. 27-35.
[15] B. S. Kashin, On properties of random matrices associated with unconditional convergence almost everywhere, Dokl. Akad. Nauk SSSR, 254 (1980) pp. 1322-1325.
[16] R. M. Megrabian, On a characteristic of random matrices connected with unconditional convergence almost everywhere, Anal. Math. 14 (1988) pp. 37-47.
[17] Y. Q. Yin, Z. D. Bai and P. R. Krishnaiah, On limit of the largest eigenvalue of the large dimensional sample covariance matrix, Center for Multivariate Analysis, Teclm. Report No. 84-44, University of Pittsburgh, Pittsburgh, PA. (1984).
[18] Z. D. Bai and Y. Q. Yin, Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of Wigner matrix, Center for Multivariate Analysis, Techn. Report No. 87-05, University of Pittsburgh, Pittsburgh, PA (1987).
[19] S. Geman, A limit theorem for the norm of random matrices, Ann. Probab., 8, No. 2 (1980) pp. 252-261.
[20] K. W. Wachter, The strong limits of random matrix spectra for sample matrices of independent elements, Ann. Probab., 6, No. 1 (1978) pp. 1-18.
[21] V. L. Girko, Limit theorems for the sums of distribution functions of eigenvalues of random symmetric matrices, Ukr. Mat. Zh., 40, No. 1 (1989) pp. 23-29.

[22] V. L. Girko,Limit theorems for the distribution of the eigenvalues of random symmetric matrices, Teor. Veroyatn. Mat. Stat., 41 (1989) pp. 23-29.
[23] V. L. Girko, The Spectral Theory of Random Matrices [in Russian], Nauka, Moscow (1988).
[24] V. L. Girko, Limit theorems for the maximal and minimal eigenvalues of random symmetric matrices, Teor. Veroyatn. Primen., 35, No. 4 (1990) pp. 677-690.
[25] L. A. Pastur, Spectra of random self-adjoint operators, Usp. Mat. Nauk, 28, No. 1 (1973) pp. 3-63.