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Fixed Point Theorems for semi λ-subadmissible Contractions in b-Metric spaces

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Abstract. Here, a new certain class of contractive mappings in the $b-$metric spaces is introduced. Some fixed point theorems are proved which generalize and modify the recent results in the literature. As an application, some results in the $b-$metric spaces endowed with a partial ordered are proved.

Keywords: Fixed point, b-metric.

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1. Introduction

The existence of a fixed point is studied by many authors. The notion of b-metric space was first explained by Bakhtin in [2] and then widely utilized by Czerwik in [6] (this space is a metric type spaces defined by Khamsi and Hussain [18]). Since then, many researches deal with fixed point theory for single-valued and multi-valued mappings in b-metric spaces (see, [3, 6, 7] and references therein). Meanwhile, Samet *et al.* [30] presented the notions of α-ψ-contractive and α-admissible mappings and founded several fixed point theorems for such mappings outline under the complete metric spaces. Subsequently, Salimi *et al.* [28] and Hussain *et al.* [13] improved the concepts of α-ψ-contractive and α-admissible mappings and studied some fixed point theorems. In this paper, a new classes of contractive mappings is introduced in order to study some fixed point theorems in the $b-$metric spaces.

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Definition 1.1 [6] Let X be a nonempty set and $s \geqslant 1$. A function $d : X \times X \to \mathbb{R}^+$ is a $b-$metric if and only if for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ iff $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

Then the tripled (X, d, s) is called a $b-$metric space.

Definition 1.2 [5] Let (X, d) be a $b-$metric space. A sequence $\{x_n\}$ in X is called:
(a) b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to +\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(b) b-Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to +\infty$.

Proposition 1.3 [5, Remark 2.1] In a $b-$metric space (X, d) the following assertions hold:
p_1. A b-convergent sequence has a unique limit.
p_2. Each b-convergent sequence is b-Cauchy.
p_3. In general a $b-$metric is not continuous.

Lemma 1.4 [1] Let (X, d) be a $b-$metric space with $s \geqslant 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then

$$\frac{1}{s^2} d(x, y) \leqslant \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leqslant \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leqslant s^2 d(x, y).$$

In particular, if $x = y$ then $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$

$$\frac{1}{s} d(x, z) \leqslant \liminf_{n\longrightarrow\infty} d(x_n, z) \leqslant \limsup_{n\longrightarrow\infty} d(x_n, z) \leqslant s d(x, z).$$

For more details on b-metric spaces the reader can refer to [7]-[11].

Definition 1.5 [30] Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geqslant 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geqslant 1.$$

Definition 1.6 [16] Let T be an α-admissible mapping. We say that T is a triangular α-admissible mapping if $\alpha(x, y) \geqslant 1$ and $\alpha(y, z) \geqslant 1$ implies that $\alpha(x, z) \geqslant 1$.

Lemma 1.7 [16] Let T be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then

$$\alpha(x_m, x_n) \geqslant 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Definition 1.8 [12] Let $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. We say that T is an α-continuous mapping if for given $x \in X$ and sequence $\{x_n\}$ with $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ one has $Tx_n \to Tx$.

Definition 1.9 Let T be a self-mapping on X and let $\lambda : X \to [0, +\infty)$ be a function. We say that T is a semi λ-subadmissible mapping if

$$x \in X, \quad \lambda(x) \leqslant 1 \quad \Longrightarrow \quad \lambda(Tx) \leqslant 1.$$

***Example* 1.10** Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = x^3$. Suppose that $\lambda : \mathbb{R} \to \mathbb{R}^+$ is given by $\lambda(x) = e^x$ for all $x \in \mathbb{R}$. Then T is a semi λ-subadmissible mapping. Indeed, if $\lambda(x) = e^x \leqslant 1$ then $x \leqslant 0$ which implies that $Tx \leqslant 0$. Therefore $\lambda(Tx) = e^{Tx} \leqslant 1$.

Consistent with Khan *et al.* [17] we denote by Ψ the set of all function $\varphi : [0, +\infty) \to [0, +\infty)$ (which is called an altering distance function) if the following conditions hold:

- φ is continuous and non-decreasing.
- $\varphi(t) = 0$ if and only if $t = 0$.

Motivated by Kumam and Roldán [20] we introduce the following class of mappings which is suitable for our results.

Let Θ denote the set of all functions $\theta : R^{+^4} \to R^+$ satisfying:
(Θ_1) θ is continuous and increasing in all its variables;
(Θ_2) $\theta(t_1, t_2, t_3, t_4) = 0$ iff either $t_1 = 0$ or $t_4 = 0$.

2. Main Theorems

In this section we stat the Main results. The first theorem is based on [7, Theorem 4] and [27, Theorem 3].

Theorem 2.1 Let (X, d, s) be a complete b-metric space, T be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ and $\lambda : X \to [0, +\infty)$ be two functions. Suppose that the following assertions hold.

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\lambda(x_0) \leqslant 1$.
(ii) T is α-continuous, triangular α-admissible and semi λ-subadmissible mapping.
(iii) For all $x, y \in X$ with $\alpha(x, y) \geqslant 1$

$$\psi(sd(Tx, Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x, y)) - \varphi(M(x, y))\Big] + \theta\Big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big) \tag{1}$$

where $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\lambda(x_0) \leqslant 1$. We define a sequence $\{x_n\}$ as follows

$$x_n = T^n x_0 = Tx_{n-1}$$

for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$ then $x_n = Tx_n$ and so x_n is a fixed point of f. Hence we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since T is a triangular α-admissible mapping then by Lemma 1.7

$$\alpha(x_m, x_n) \geqslant 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Also, since T is a semi λ-subadmissible mapping and $\lambda(x_0) \leqslant 1$ then $\lambda(x_1) = \lambda(Tx_0) \leqslant 1$. Again, since T is semi λ-subadmissible, then $\lambda(x_2) = \lambda(Tx_1) \leqslant 1$. Continuing this process

$\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then by (iii),

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leqslant \psi(sd(x_n, x_{n+1}))\\ &= \psi(sd(Tx_{n-1}, Tx_n))\\ &\leqslant \lambda(x_{n-1})\lambda(x_n)\Big[\psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n))\Big]\\ &+\theta\big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big)\\ &\leqslant \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n))\\ &+\theta\big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big)\end{aligned} \tag{2}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \tfrac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s}\right\}\\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1}, x_{n+1})}{2s}\right\}\\ &\leqslant \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})}{2s}\right\}\\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\}\\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\end{aligned} \tag{3}$$

and

$$\begin{aligned}&\theta\big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big)\\ &= \theta\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\big)\\ &= \theta\big(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\big) = 0.\end{aligned} \tag{4}$$

By (2)-(4) and the properties of ψ and φ we obtain

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leqslant \psi\Bigg(\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}\Bigg) - \varphi\Bigg(M(x_{n-1}, x_n)\Bigg)\\ &< \psi\Bigg(\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}\Bigg).\end{aligned} \tag{5}$$

Now if

$$\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} = d(x_n, x_{n+1}),$$

then by (5)

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leqslant \psi(d(x_n, x_{n+1})) - \varphi(M(x_{n-1}, x_n))\\ &< \psi(d(x_n, x_{n+1})),\end{aligned}$$

which is a contradiction. Hence

$$\max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} = d(x_{n-1}, x_n).$$

Therefore

$$\psi(d(x_n, x_{n+1})) \leqslant \psi(d(x_n, x_{n-1})) - \varphi(M(x_{n-1}, x_n)) < \psi(d(x_n, x_{n-1})). \tag{6}$$

Since ψ is a non-decreasing mapping, then $\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$ is a non-increasing sequence of positive numbers. Then there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Letting $n \to \infty$ in (6), we have

$$\psi(r) \leqslant \psi(r) - \varphi(\lim_{n\to\infty} M(x_{n-1}, x_n)) \leqslant \psi(r).$$

Therefore $\varphi(\lim_{n\to\infty} M(x_{n-1}, x_n)) = 0$ and hence $r = 0$, *i.e.*,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

Now, we show that $\{x_n\}$ is a $b-$Cauchy sequence in X. Assume the contrary, that $\{x_n\}$ is not a $b-$Cauchy sequence. Then there exists $\varepsilon > 0$ and two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ dna } d(x_{m_i}, x_{n_i}) \geqslant \varepsilon. \tag{8}$$

That is

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{9}$$

By using (8), (9) and the triangular inequality

$$\begin{aligned}\varepsilon &\leqslant d(x_{m_i}, x_{n_i}) \\ &\leqslant sd(x_{m_i}, x_{m_i-1}) + sd(x_{m_i-1}, x_{n_i}) \\ &\leqslant sd(x_{m_i}, x_{m_i-1}) + s^2 d(x_{m_i-1}, x_{n_i-1}) + s^2 d(x_{n_i-1}, x_{n_i}).\end{aligned}$$

Now, using (7) and taking the upper limit as $i \to \infty$

$$\frac{\varepsilon}{s^2} \leqslant \limsup_{i\longrightarrow\infty} d(x_{m_i-1}, x_{n_i-1}).$$

On the other hand

$$d(x_{m_i-1}, x_{n_i-1}) \leqslant sd(x_{m_i-1}, x_{m_i}) + sd(x_{m_i}, x_{n_i-1}).$$

Using (7), (9) and taking the upper limit as $i \to \infty$

$$\limsup_{i\longrightarrow\infty} d(x_{m_i-1}, x_{n_i-1}) \leqslant \varepsilon s.$$

Hence

$$\frac{\varepsilon}{s^2} \leqslant \limsup_{i\longrightarrow\infty} d(x_{m_i-1}, x_{n_i-1}) \leqslant \varepsilon s. \tag{10}$$

Again using the triangular inequality

$$d(x_{m_i-1}, x_{n_i}) \leqslant sd(x_{m_i-1}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}), \tag{11}$$

$$\varepsilon \leqslant d(x_{m_i}, x_{n_i}) \leqslant sd(x_{m_i}, x_{m_i-1}) + sd(x_{m_i-1}, x_{n_i}) \tag{12}$$

and

$$\varepsilon \leqslant d(x_{m_i}, x_{n_i}) \leqslant sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}). \tag{13}$$

Using (7) and (10) and taking the upper limit as $i \to \infty$ in (11) and (12) we get

$$\frac{\varepsilon}{s} \leqslant \limsup_{i\longrightarrow\infty} d(x_{m_i-1}, x_{n_i}) \leqslant \varepsilon s^2. \tag{14}$$

Again using (7) and (9) and taking the upper limit as $i \to \infty$ in (13)

$$\frac{\varepsilon}{s} \leqslant \limsup_{i\longrightarrow\infty} d(x_{m_i}, x_{n_i-1}) \leqslant \varepsilon. \tag{15}$$

Since $\alpha(x_{m_i-1}, x_{n_i-1}) \geqslant 1$, $\lambda(x_{m_i-1}) \leqslant 1$ and $\lambda(x_{n_i-1}) \leqslant 1$ then from (iii) we have

$$\begin{aligned}
&\psi(sd(x_{m_i}, x_{n_i})) = \psi(sd(Tx_{m_i-1}, Tx_{n_i-1})) \\
&\leqslant \lambda(x_{m_i-1})\lambda(x_{n_i-1})\Big[\psi(M(x_{m_i-1}, x_{n_i-1})) - \varphi(M(x_{m_i-1}, x_{n_i-1}))\Big] \\
&+\theta\Big(d(x_{m_i-1}, Tx_{m_i-1}), d(x_{n_i-1}, Tx_{n_i-1}), d(x_{m_i-1}, Tx_{n_i-1}), d(x_{n_i-1}, Tx_{m_i-1})\Big) \\
&\leqslant \psi(M(x_{m_i-1}, x_{n_i-1})) - \varphi(M(x_{m_i-1}, x_{n_i-1})) \\
&+\theta\Big(d(x_{m_i-1}, Tx_{m_i-1}), d(x_{n_i-1}, Tx_{n_i-1}), d(x_{m_i-1}, Tx_{n_i-1}), d(x_{n_i-1}, Tx_{m_i-1})\Big),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
M(x_{m_i-1}, x_{n_i-1}) &= \max\Big\{d(x_{m_i-1}, x_{n_i-1}), d(x_{m_i-1}, Tx_{m_i-1}), d(x_{n_i-1}, Tx_{n_i-1}), \\
&\qquad \frac{d(x_{m_i-1}, Tx_{n_i-1}) + d(Tx_{m_i-1}, x_{n_i-1})}{2s}\Big\} \\
&= \max\Big\{d(x_{m_i-1}, x_{n_i-1}), d(x_{m_i-1}, x_{m_i}), d(x_{n_i-1}, x_{n_i}), \\
&\qquad \frac{d(x_{m_i-1}, x_{n_i}) + d(x_{m_i}, x_{n_i-1})}{2s}\Big\},
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\theta\Big(d(x_{m_i-1}, Tx_{m_i-1}), d(x_{n_i-1}, Tx_{n_i-1}), d(x_{m_i-1}, Tx_{n_i-1}), d(x_{n_i-1}, Tx_{m_i-1})\Big) \\
&= \theta\Big(d(x_{m_i-1}, x_{m_i}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{n_i}), d(x_{n_i-1}, x_{m_i})\Big).
\end{aligned} \tag{18}$$

Taking the upper limit as $i \to \infty$ in (17) and (18) and using (7), (10), (14) and (15) we get

$$\begin{aligned}
\frac{\varepsilon}{s^2} &= \min\left\{\frac{\varepsilon}{s^2}, \frac{\frac{\varepsilon}{s}+\frac{\varepsilon}{s}}{2s}\right\} \leqslant \limsup_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}) \\
&= \max\{\limsup_{i\longrightarrow\infty} d(x_{m_i-1}, x_{n_i-1}), 0, 0, \\
&\quad \frac{\limsup_{i\longrightarrow\infty} d(x_{m_i-1}, x_{n_i}) + \limsup_{i\longrightarrow\infty} d(x_{m_i}, x_{n_i-1})}{2s}\} \\
&\leqslant \max\left\{\varepsilon s, \frac{\varepsilon s^2+\varepsilon}{2s}\right\} = \varepsilon s.
\end{aligned}$$

So

$$\frac{\varepsilon}{s^2} \leqslant \limsup_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}) \leqslant \varepsilon s, \tag{19}$$

and

$$\begin{aligned}
&\limsup_{i\to\infty} \theta\Big(d(x_{m_i-1}, Tx_{m_i-1}), d(x_{n_i-1}, Tx_{n_i-1}), d(x_{m_i-1}, Tx_{n_i-1}), d(x_{n_i-1}, Tx_{m_i-1})\Big) \\
&= \limsup_{i\to\infty} \theta\Big(d(x_{m_i-1}, x_{m_i}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{n_i}), d(x_{n_i-1}, x_{m_i})\Big) = 0.
\end{aligned} \tag{20}$$

Similarly

$$\frac{\varepsilon}{s^2} \leqslant \liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}) \leqslant \varepsilon s. \tag{21}$$

Now, taking the upper limit as $i \to \infty$ in (16) and using (8), (19) and (20) we have

$$\begin{aligned}
\psi(\varepsilon s) &\leqslant \psi(s\limsup_{i\longrightarrow\infty} d(x_{m_i}, x_{n_i})) \\
&\leqslant \psi(\limsup_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1})) - \liminf_{n\longrightarrow\infty} \varphi(M(x_{m_i-1}, x_{n_i-1})) \\
&\leqslant \psi(\varepsilon s) - \varphi(\liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1})),
\end{aligned}$$

which implies

$$\varphi(\liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1})) = 0,$$

so $\liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}) = 0$, which is a contradiction with (21). So $\{x_{n+1}\}$ is a $b-$Cauchy sequence in X. Since X is a complete $b-$metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Also, from (ii) we know T is an $\alpha-$continuous mapping. Hence $Tx_n \to Tx^*$ as $n \to \infty$. Then

$$d(x^*, Tx^*) \leqslant sd(x^*, Tx_n) + sd(Tx_n, Tx^*).$$

Letting $n \to \infty$ in the above inequality

$$d(x^*, Tx^*) \leqslant s \lim_{n\to\infty} d(x^*, Tx_n) + s \lim_{n\to\infty} d(Tx_n, Tx^*) = 0.$$

So $Tx^* = x^*$. ■

For self-mappings that are not continuous or $\alpha-$continuous we have the following result.

Theorem 2.2 Let (X, d, s) be a complete b-metric space, T be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ and $\lambda : X \to [0, +\infty)$ be two functions. Suppose that the following assertions hold.

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\lambda(x_0) \leqslant 1$.
(ii) T is a triangular α-admissible and semi λ-subadmissible mapping.
(iii) For all $x, y \in X$ with $\alpha(x, y) \geqslant 1$

$$\psi(sd(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y))-\varphi(M(x,y))\Big]+\theta\Big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big),$$

where $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

(v) If $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$, $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\lambda(x_0) \leqslant 1$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Following the proof of the Theorem 2.1, we obtain that $\{x_n\}$ is a $b-$Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$ and $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since X is complete, there exists $x^* \in X$ such that the sequence $\{x_n\}$ b-converges to x^*. Using the assumption (v), we have $\alpha(x_n, x^*) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x^*) \leqslant 1$. By (iii)

$$\begin{aligned}
\psi(sd(x_{n+1}, Tx^*)) &= \psi(sd(Tx_n, Tx^*)) \\
&\leqslant \lambda(x_n)\lambda(x^*)\Big[\psi(M(x_n, x^*)) - \varphi(M(x_n, x^*))\Big] \\
&\quad +\theta\Big(d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n)\Big) \\
&\leqslant \psi(M(x_n, x^*)) - \varphi(M(x_n, x^*)) \\
&\quad +\theta\Big(d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n)\Big),
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
M(x_n, x^*) &= \max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \tfrac{d(x_n,Tx^*)+d(Tx_n,x^*)}{2s}\right\} \\
&= \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \tfrac{d(x_n,Tx^*)+d(x_{n+1},x^*)}{2s}\right\}
\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\theta\Big(d(x_n,Tx_n),d(x^*,Tx^*),d(x_n,Tx^*),d(x^*,Tx_n)\Big)\\&=\theta\Big(d(x_n,x_{n+1}),d(x^*,Tx^*),d(x_n,Tx^*),d(x^*,x_{n+1})\Big).\end{aligned} \tag{24}$$

Letting $n\to\infty$ in (23) and (24) and using lemma 1.4, we get

$$\begin{aligned}\tfrac{d(x^*,Tx^*)}{2s^2}&=\min\left\{d(x^*,Tx^*),\tfrac{d(x^*,Tx^*)}{2s^2}\right\}\leqslant\limsup_{n\longrightarrow\infty}M(x_n,x^*)\\&\leqslant\max\left\{d(x^*,Tx^*),\tfrac{sd(x^*,Tx^*)}{2s}\right\}=d(x^*,Tx^*),\end{aligned} \tag{25}$$

and

$$\theta\Big(d(x_n,Tx_n),d(x^*,Tx^*),d(x_n,Tx^*),d(x^*,Tx_n)\Big)\to 0 \text{ as } n\to\infty.$$

Similarly

$$\frac{d(x^*,Tx^*)}{2s^2}\leqslant\liminf_{n\longrightarrow\infty}M(x_n,x^*)\leqslant d(x^*,Tx^*). \tag{26}$$

Again, taking the upper limit as $i\to\infty$ in (22) and using lemma 1.4 and (25) we get

$$\begin{aligned}\psi(d(x^*,Tx^*)=\psi(s\frac{1}{s}d(x^*,Tx^*))&\leqslant\psi(s\limsup_{n\longrightarrow\infty}d(x_{n+1},Tx^*))\\&\leqslant\psi(\limsup_{n\longrightarrow\infty}M(x_n,x^*))-\liminf_{n\longrightarrow\infty}\varphi(M(x_n,x^*))\\&\leqslant\psi(d(x^*,Tx^*))-\varphi(\liminf_{n\longrightarrow\infty}M(x_n,x^*)).\end{aligned}$$

Hence, $\varphi(\liminf_{n\longrightarrow\infty}M(x_n,x^*))=0$. Then, $\liminf_{n\longrightarrow\infty}M(x_n,x^*)=0$ which is a contradiction. So, $x^*=Tx^*$. ■

Example 2.3 Let $X=\mathbb{R}$ be endowed with the $b-$metric

$$d(x,y)=\begin{cases}(|x|+|y|)^2, \text{ if } x\neq y\\ 0 \qquad\qquad\quad \text{if } x=y\end{cases}$$

for all $x,y\in X$. Define $T:X\to X$, $\alpha:X\times X\to[0,\infty)$ and $\lambda:X\to[0,\infty)$ by

$$Tx=\begin{cases}2x^3+\sin x, & \text{if } x\in(-\infty,0)\\ \frac{1}{8}x^2, & \text{if } x\in[0,1)\\ \frac{1}{8}x, & \text{if } x\in[1,2)\\ \frac{1}{4} & \text{if } x\in[2,+\infty)\end{cases} \qquad \alpha(x,y)=\begin{cases}2, \text{ if } x,y\in[0,+\infty)\\ 0, \text{ otherwise}\end{cases}$$

and $\lambda(x) = \begin{cases} 1, & \text{if } x \in [0,+\infty) \\ 2x^2+3, & \text{otherwise.} \end{cases}$

Also, define $\psi, \varphi : [0,\infty) \to [0,+\infty)$ and $\theta : [0,+\infty)^4 \to [0,+\infty)$ by $\psi(t) = t$, $\varphi(t) = \frac{3}{4}t$ and $\theta(t_1,t_2,t_3,t_4) = \min\{t_1,t_2,t_3,t_4\}$. Clearly (X,d,s) with $s=2$ is a complete $b-$metric space, $\psi,\varphi \in \Psi$ and $\theta \in \Theta$. Let $\alpha(x,y) \geqslant 1$, then $x,y \in [0,+\infty)$. On the other hand, $Tw \in [0,+\infty)$ for all $w \in [0,+\infty)$. Then $\alpha(Tx,Ty) \geqslant 1$. That is, T is an α-admissible mapping. Let $\alpha(x,y) \geqslant 1$ and $\alpha(y,z) \geqslant 1$. So $x,y,z \in [0,+\infty)$ *i.e.*, $\alpha(x,z) \geqslant 1$. Hence T is a triangular α-admissible mapping. Also, let $\lambda(x) \leqslant 1$. Thus $x \in [0,+\infty)$. That is, $\lambda(Tx) \leqslant 1$. Thus T is a semi λ-subadmissible mapping. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1}) \geqslant 1$ and $\lambda(x_n) \leqslant 1$ with $x_n \to x$ as $n \to \infty$. Then, $x_n \in [0,+\infty)$ for all $n \in \mathbb{N}$. Also $[0,+\infty)$ is a closed set. Then $x \in [0,+\infty)$. That is $\alpha(x_n,x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$. Clearly $\alpha(0,T0) \geqslant 1$ and $\lambda(0) \leqslant 1$.

Let $\alpha(x,y) \geqslant 1$. So $x,y \in [0,+\infty)$.

Now we consider the following cases:

- Let $x,y \in [0,1)$ then

$$\begin{aligned}
\psi(2d(Tx,Ty)) &= 2d(Tx,Ty) = 2(\tfrac{1}{8}x^2 + \tfrac{1}{8}y^2)^2 \\
&= \tfrac{1}{32}(x^2+y^2)^2 \\
&\leqslant \tfrac{1}{4}(x+y)^2 \\
&= \tfrac{1}{4}d(x,y) \\
&\leqslant \tfrac{1}{4}M(x,y) \\
&= \psi(M(x,y)) - \varphi(M(x,y)) \\
&= \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] + \theta(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)).
\end{aligned}$$

- Let $x,y \in [1,2)$ then

$$\begin{aligned}
\psi(2d(Tx,Ty)) = 2d(Tx,Ty) &= 2(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 \\
&= \tfrac{1}{32}(x+y)^2 \\
&\leqslant \tfrac{1}{4}(x+y)^2 \\
&= \tfrac{1}{4}d(x,y) \\
&\leqslant \tfrac{1}{4}M(x,y) \\
&= \psi(M(x,y)) - \varphi(M(x,y)) \\
&\leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] \\
&\quad + \theta(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)).
\end{aligned}$$

- Let $x,y \in [2,\infty)$ then

$$\begin{aligned}
\psi(2d(Tx,Ty)) &= 2d(Tx,Ty) = 2(\tfrac{1}{4} + \tfrac{1}{4})^2 \\
&= \tfrac{1}{2} \leqslant 1 \\
&= \tfrac{1}{4}(1+1)^2 \\
&\leqslant \tfrac{1}{4}(x+y)^2 \\
&= \tfrac{1}{4}d(x,y) \\
&\leqslant \tfrac{1}{4}M(x,y) \\
&= \psi(M(x,y)) - \varphi(M(x,y)) \\
&\leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] \\
&\quad + \theta(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)).
\end{aligned}$$

- Let $x \in [0,1)$ and $y \in [1,2)$ then

$$\begin{aligned}\psi(2d(Tx,Ty)) &= 2d(Tx,Ty) = 2(\tfrac{1}{8}x^2 + \tfrac{1}{8}y)^2 \\ &\leqslant 2(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 \\ &= \tfrac{1}{32}(x^2+y^2)^2 \\ &\leqslant \tfrac{1}{4}(x+y)^2 \\ &= \tfrac{1}{4}d(x,y) \leqslant \tfrac{1}{4}M(x,y) \\ &= \psi(M(x,y)) - \varphi(M(x,y)) \\ &= \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] \\ &\quad +\theta(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)).\end{aligned}$$

- Let $x \in [0,1)$ and $y \in [2,\infty)$ then

$$\begin{aligned}\psi(2d(Tx,Ty)) &= 2d(Tx,Ty) = 2t(\tfrac{1}{8}x^2 + \tfrac{1}{4})^2 \\ &\leqslant 2(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 \\ &= \tfrac{1}{32}(x+y)^2 \\ &\leqslant \tfrac{1}{4}(x+y)^2 \\ &= \tfrac{1}{4}d(x,y) \\ &\leqslant \tfrac{1}{4}M(x,y) \\ &= \psi(M(x,y)) - \varphi(M(x,y)) \\ &= \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] \\ &\quad +\theta(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)).\end{aligned}$$

- Let $x \in [1,2)$ and $y \in [2,\infty)$ then

$$\begin{aligned}\psi(2d(Tx,Ty)) &= 2d(Tx,Ty) = 2(\tfrac{1}{8}x + \tfrac{1}{4})^2 \\ &\leqslant 2(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 \\ &= \tfrac{1}{32}(x+y)^2 \\ &\leqslant \tfrac{1}{4}(x+y)^2 \\ &= \tfrac{1}{4}d(x,y) \\ &\leqslant \tfrac{1}{4}M(x,y) \\ &= \psi(M(x,y)) - \varphi(M(x,y)) \\ &\leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] \\ &\quad +\theta(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)).\end{aligned}$$

Therefore $\alpha(x,y) \geqslant 1$ implies

$$\psi(2d(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] + \theta(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx))$$

Hence, all conditions of Theorem 2.2 holds and T has a fixed point. Here, $x = 0$ is a fixed point of T.

Corollary 2.4 Let (X,d,s) be a complete b-metric space, T be a self-mapping on X and $\alpha : X \times X \to [0,\infty)$ and $\lambda : X \to [0,+\infty)$ be two functions. Suppose that the following assertions hold.

(i) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\lambda(x_0) \leqslant 1$.
(ii) T is a triangular α-admissible and semi λ-subadmissible mapping.
(iii) For all $x, y \in X$

$$\psi(s\alpha(x,y)d(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] + \theta\Big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\Big), \tag{27}$$

where $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\}.$$

(v) If $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$, $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$ then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

Proof. Let $\alpha(x,y) \geqslant 1$. Since ψ is increasing then from (iii)

$$\begin{aligned} \psi(sd(Tx,Ty)) &\leqslant \psi(s\alpha(x,y)d(Tx,Ty)) \\ &\leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] \\ &+\theta\Big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\Big). \end{aligned}$$

Therefore all conditions of Theorem 2.2 holds and T has a fixed point. ■

If in Corollary 2.4 we take $\alpha(x,y) = 1$ for all $x, y \in X$, then we have the following corollary.

Corollary 2.5 Let (X, d, s) be a complete b-metric space and T be a self-mapping on X and $\lambda : X \to [0, +\infty)$ be a function. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $\lambda(x_0) \leqslant 1$,
(ii) T is a semi λ-subadmissible mapping,
(iii) for all $x, y \in X$ we have

$$\psi(sd(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] + \theta\Big(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\Big) \tag{28}$$

where, $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\},$$

(v) if $\{x_n\}$ be a sequence such that $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$ then $\lambda(x) \leqslant 1$.

Then T has a fixed point.

3. Some results in $b-$metric spaces endowed with a graph

In this section, we show that many fixed point results in $b-$metric spaces endowed with a graph G (see [4]) can be deduced easily from our presented theorems.
As in [14], let (E, d, s) be a $b-$metric space and Δ denotes the diagonal of the Cartesian product of $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph, see [15, P.309], by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x,\ x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$.

Definition 3.1 [14] Let (X, d) be a metric space endowed with a graph G. We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G that is,

$$\text{for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G)$$

and T decreases the weights of the edges of G in the following way:

$$\exists\, \alpha \in (0,1) \text{ such that for all } x, y \in X, \quad (x, y) \in E(G) \Longrightarrow d(Tx, Ty) \leq \alpha d(x, y).$$

Definition 3.2 [14] A mapping $T : X \to X$ is called G-continuous if given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \,\text{as}\, n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \,\text{for all}\, n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$

Theorem 3.3 Let (X, d, s) be a complete b-metric space endowed with a graph G and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leqslant 1$,
(ii) T is G-continuous and semi λ-subadmissible mapping,
(iii) $\forall x, y \in X[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$
(iv) $\forall x, y, z \in X[(x, y) \in E(G) \,\text{and}\, (y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$
(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\psi(sd(Tx, Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x, y)) - \varphi(M(x, y))\Big] + \theta\Big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big)$$

where, $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

Proof. Define $\alpha : X^2 \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 2, \text{ if } (x, y) \in E(G) \\ \frac{1}{2}, \text{ otherwise.} \end{cases}$$

First we show that T is a triangular α-admissible mapping. Let $\alpha(x, y) \geqslant 1$ then $(x, y) \in E(G)$. From (iii) $(Tx, Ty) \in E(G)$. That is $\alpha(Tx, Ty) \geqslant 1$. Also let $\alpha(x, y) \geqslant 1$ and $\alpha(y, z) \geqslant 1$. So $(x, y) \in E(G)$ and $(y, z) \in E(G)$. From (iv) we get $(x, z) \in E(G)$, i.e. $\alpha(x, z) \geqslant 1$. Thus T is a triangular α-admissible mapping. Let T be G-continuous. So

$$x_n \to x \text{ as } n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$

That is,

$$x_n \to x \text{ as } n \to \infty \text{ and } \alpha(x_n, x_{n+1}) \geqslant 1 \text{ for all } n \in \mathbb{N} \text{ imply } Tx_n \to Tx$$

which implies that T is α-continuous. From (i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is $\alpha(x_0, Tx_0) \geqslant 1$. Let $\alpha(x, y) \geqslant 1$ then $(x, y) \in E(G)$. Now from (v) we have

$$\psi(sd(Tx, Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x, y)) - \varphi(M(x, y))\Big] + \theta\Big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big)$$

Hence all conditions of Theorem 2.1 are satisfied and T has a fixed point. ■

In Theorem 3.3 we take $\theta(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\}$.

Corollary 3.4 Let (X, d, s) be a complete b-metric space endowed with a graph G and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leqslant 1$,
(ii) T is G-continuous and semi λ-subadmissible mapping,
(iii) $\forall x, y \in X[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$
(iv) $\forall x, y, z \in X[(x, y) \in E(G) \text{ and } (y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$
(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\psi(sd(Tx, Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x, y)) - \varphi(M(x, y))\Big] + L\min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

where, $\psi, \varphi \in \Psi$, $L \geqslant 0$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

Theorem 3.5 Let (X, d, s) be a complete b-metric space endowed with a graph G and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leqslant 1$,
(ii) T is semi λ-subadmissible mapping,
(iii) $\forall x, y \in X[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$
(iv) $\forall x, y, z \in X[(x, y) \in E(G) \text{ and } (y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$
(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\psi(sd(Tx, Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x, y)) - \varphi(M(x, y))\Big] + \theta\Big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big) \tag{29}$$

where, $(\psi, \varphi \in \Psi)$, $\theta \in \Theta$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

(vi) if $\{x_n\}$ be a sequence in X such that $(x_n, x_{n+1}) \in E(G)$, $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$ then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

Proof. Define the mapping $\alpha : X^2 \to [0, +\infty)$ as in the proof of Theorem 3.3. Similar to the proof of Theorem 3.3 we can prove that the conditions (i)-(iii) of Theorem 2.2 are satisfied. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ and $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $(x_n, x_{n+1}) \in E(G)$ and $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. From (vi) we get $(x_n, x) \in E(G)$ and $\lambda(x) \leqslant 1$. That is $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$. Therefore all conditions of Theorem 2.2 holds and T has a fixed point. ■

Corollary 3.6 Let (X, d, s) be a complete b-metric space endowed with a graph G and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leqslant 1$,
(ii) T is semi λ-subadmissible mapping,
(iii) $\forall x, y \in X[(x,y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$
(iv) $\forall x, y, z \in X[(x,y) \in E(G) \,\text{and}\, (y,z) \in E(G) \Rightarrow (x, z \in E(G)]$
(v) for all $x, y \in X$ with $(x,y) \in E(G)$ we have,

$$\psi(sd(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

where, $(\psi, \varphi \in \Psi)$, $L \geqslant 0$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

(vi) if $\{x_n\}$ be a sequence in X such that $(x_n, x_{n+1}) \in E(G)$, $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

4. Some results in $b-$metric spaces endowed with a partial ordered

The existence of fixed points in partially ordered sets has been considered by many authors (such as [19], [21–26] and [29] etc.). Later on, some generalizations of [26] are given in [27]. Several applications of these results to matrix equations are presented in [26].

Let X be a nonempty set. If (X, d, s) is a $b-$metric space and $(X, \preceq)$ be a partially ordered set, then $(X, d, s, \preceq)$ is called an ordered $b-$metric space. Two elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ hold. A mapping $T : X \to X$ is said to be non-decreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

In this section, we will show that many fixed point results in partially ordered $b-$metric spaces can be deduced easily from our obtained results.

Theorem 4.1 Let $(X, d, s, \preceq)$ be a complete ordered b-metric space and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $\lambda(x_0) \leqslant 1$,
(ii) T is continuous and semi λ-subadmissible mapping,
(iii) T is an increasing mapping,
(v) for all $x, y \in X$ with $x \preceq y$ we have,

$$\psi(sd(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y))-\varphi(M(x,y))\Big]+\theta\Big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big)$$

where, $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

Then T has a fixed point.

Proof. Define $\alpha : X^2 \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 2, \text{ if } x \preceq y \\ \frac{1}{2}, \text{ otherwise} \end{cases}$$

First, we prove that T is a triangular α-admissible mapping. Let $\alpha(x, y) \geqslant 1$, then $x \preceq y$. Since T is increasing, then we have $Tx \preceq Ty$. That is, $\alpha(Tx, Ty) \geqslant 1$. Suppose that $\alpha(x, y) \geqslant 1$ and $\alpha(y, z) \geqslant 1$. Then $x \preceq y$ and $y \preceq z$. Hence $x \preceq z$ i.e., $\alpha(x, z) \geqslant 1$. Therefore, T is a triangular α-admissible mapping. Since T is continuous then it is α-continuous too. From (i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. That is, $\alpha(x_0, Tx_0) \geqslant 1$. Let $\alpha(x, y) \geqslant 1$, then $x \preceq y$. Now, from (v) we have

$$\psi(sd(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y))-\varphi(M(x,y))\Big]+\theta\Big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big).$$

Hence, all conditions of Theorem 2.1 are satisfied and T has a fixed point. ■

If in Theorem 3.3 we take $\theta(t_1, t_2, t_3, t_4) = L\psi(\min\{t_1, t_4\})$ where $L \geqslant 0$, then we have the following Corollary.

Corollary 4.2 Let $(X, d, s, \preceq)$ be a complete ordered b-metric space and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $\lambda(x_0) \leqslant 1$,
(ii) T is continuous and semi λ-subadmissible mapping,
(iii) T is an increasing mapping,
(v) for all $x, y \in X$ with $x \preceq y$ we have,

$$\psi(sd(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] + L\psi(\min\{d(x,Tx), d(y,Tx)\})$$

where, $\psi, \varphi \in \Psi$, $L \geqslant 0$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

Then T has a fixed point.

If in Corollary 3.3 we take $\lambda(x) = 1$ for all $x \in X$, then we have the following Corollary.

Corollary 4.3 [27, Theorem 3] Let $(X, d, s, \preceq)$ be a complete ordered b-metric space and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,
(ii) T is continuous,
(iii) T is an increasing mapping,
(v) for all $x, y \in X$ with $x \preceq y$ we have,

$$\psi(sd(Tx, Ty)) \leqslant \psi(M(x, y)) - \varphi(M(x, y)) + +L\psi(\min\{d(x, Tx), d(y, Tx)\})$$

where, $\psi, \varphi \in \Psi$, $L \geqslant 0$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

Theorem 4.4 Let $(X, d, s, \preceq)$ be a complete partially ordered b-metric space and let T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $\lambda(x_0) \leqslant 1$,
(ii) T is a semi λ-subadmissible mapping,
(iii) T is an increasing mapping,
(iv) for all $x, y \in X$ with $x \preceq y$ we have,

$$\psi(sd(Tx, Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x, y)) - \varphi(M(x, y))\Big] + \theta\Big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big) \tag{30}$$

where, $(\psi, \varphi \in \Psi)$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

(v) if $\{x_n\}$ be an increasing sequence in X such that $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$ then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

Proof. Define the mapping $\alpha : X^2 \to [0, +\infty)$ as in the proof of Theorem 3.3. Analogous to the proof of Theorem 3.3 we can prove all the conditions (i)-(iii) of Theorem 2.2 are satisfied. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ and $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \preceq x_{n+1}$ and $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. From (v) we get, $x_n \preceq x$ and $\lambda(x) \leqslant 1$. That is, $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$. Therefore all conditions of Theorem 2.2 holds and T has a fixed point. ■

Corollary 4.5 Let $(X, d, s, \preceq)$ be a complete partially ordered b-metric space and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that, $x_0 \preceq Tx_0$ and $\lambda(x_0) \leqslant 1$,
(ii) T is a semi λ-subadmissible mapping,
(iii) T is an increasing mapping,

(iv) for all $x, y \in X$ with $x \preceq y$ we have,

$$\psi(sd(Tx,Ty)) \leqslant \lambda(x)\lambda(y)\Big[\psi(M(x,y)) - \varphi(M(x,y))\Big] + L\psi(\min\{d(x,Tx), d(y,Tx)\}) \tag{31}$$

where, $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

(v) if $\{x_n\}$ be an increasing sequence in X such that $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}\cup\{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

Corollary 4.6 [27, Theorem 4] Let $(X, d, s, \preceq)$ be a complete partially ordered b-metric space and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,
(iii) T is an increasing mapping,
(iv) for all $x, y \in X$ with $x \preceq y$ we have,

$$\psi(sd(Tx,Ty)) \leqslant \psi(M(x,y)) - \varphi(M(x,y)) + L\psi(\min\{d(x,Tx), d(y,Tx)\}) \tag{32}$$

where, $(\psi, \varphi \in \Psi)$, $L \geqslant 0$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

(v) if $\{x_n\}$ be an increasing sequence in X such that $x_n \to x$ as $n \to \infty$ then $x_n \preceq x$ for all $n \in \mathbb{N}\cup\{0\}$.

Then T has a fixed point.

5. Some integral type contractions

Let Φ denotes the set of all functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

- every $\phi \in \Phi$ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$,
- for any $\phi \in \Phi$ and any $\epsilon > 0$, $\int_0^\epsilon \phi(\tau)d\tau > 0$.

Note that if we take $\psi(t) = \int_0^t \phi(\tau)d\tau$ where $\phi \in \Phi$ then $\psi \in \Psi$.

Also note that if $\psi \in \Psi$ and $\theta \in \Theta$ then $\psi\theta \in \Theta$.

If in Theorem 2.1 we take $\psi(t) = \int_0^t \phi(\tau)d\tau$, $\varphi(t) = (1-r)\int_0^t \phi(\tau)d\tau$ for all $t \in [0, \infty)$ where $0 \leqslant r < 1$ and replace θ by $\psi\theta$ then we have the following theorem.

Theorem 5.1 Let (X, d, s) be a complete b-metric space, T be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ and $\lambda : X \to [0, +\infty)$ be two functions. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\lambda(x_0) \leqslant 1$,
(ii) T is α-continuous, triangular α-admissible and semi λ-subadmissible mapping,

(iii) for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$ we have

$$\int_0^{d(Tx,Ty)} \phi(\tau)d\tau \leqslant \frac{r\lambda(x)\lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)} \phi(\tau)d\tau \tag{33}$$

where, $0 \leqslant r < 1$, $\phi \in \Phi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

Theorem 5.2 Let (X, d, s) be a complete b-metric space, T be a self-mapping on X and $\alpha : X \times X \to [0, \infty)$ and $\lambda : X \to [0, +\infty)$ be two functions. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that, $\alpha(x_0, Tx_0) \geqslant 1$ and $\lambda(x_0) \leqslant 1$,
(ii) T is a triangular α-admissible and semi λ-subadmissible mapping,
(iii) for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$ we have

$$\int_0^{d(Tx,Ty)} \phi(\tau)d\tau \leqslant \frac{r\lambda(x)\lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)} \phi(\tau)d\tau \tag{34}$$

where, $0 \leqslant r < 1$, $\phi \in \Phi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\},$$

(v) if $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$, $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

Theorem 5.3 Let (X, d, s) be a complete b-metric space endowed with a graph G and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that, $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leqslant 1$,
(ii) T is G-continuous and semi λ-subadmissible mapping,
(iii) $\forall x, y \in X[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$
(iv) $\forall x, y, z \in X[(x, y) \in E(G)\,\text{and}\,(y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$
(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\int_0^{d(Tx,Ty)} \phi(\tau)d\tau \leqslant \frac{r\lambda(x)\lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)} \phi(\tau)d\tau \tag{35}$$

where, $0 \leqslant r < 1$, $\phi \in \Phi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

Theorem 5.4 Let (X, d, s) be a complete b-metric space endowed with a graph G and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leqslant 1$,
(ii) T is semi λ-subadmissible mapping,
(iii) $\forall x, y \in X[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$
(iv) $\forall x, y, z \in X[(x, y) \in E(G) \,\text{and}\, (y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$
(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\int_0^{d(Tx,Ty)} \phi(\tau)d\tau \leqslant \frac{r\lambda(x)\lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)} \phi(\tau)d\tau \tag{36}$$

where, $0 \leqslant r < 1$, $\phi \in \Phi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

(vi) if $\{x_n\}$ be a sequence in X such that $(x_n, x_{n+1}) \in E(G)$, $\lambda(x_n) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$ then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leqslant 1$.

Then T has a fixed point.

Theorem 5.5 Let $(X, d, s, \preceq)$ be a complete ordered b-metric space and T be a self-mapping on X. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $\lambda(x_0) \leqslant 1$,
(ii) T is continuous and semi λ-subadmissible mapping,
(iii) T is an increasing mapping,
(v) for all $x, y \in X$ with $x \preceq y$ we have

$$\int_0^{d(Tx,Ty)} \phi(\tau)d\tau \leqslant \frac{r\lambda(x)\lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau)d\tau + \int_0^{\theta\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)} \phi(\tau)d\tau \tag{37}$$

where, $0 \leqslant r < 1$, $\phi \in \Phi$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

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