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n-Jordan homomorphisms on C -algebras

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Abstract. Let $n \in \mathbb{N}$. An additive map $h : \mathcal{A} \longrightarrow \mathcal{B}$ between algebras $\mathcal{A}$ and $\mathcal{B}$ is called n-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in \mathcal{A}$. We show that every n-Jordan homomorphism between commutative Banach algebras is a n-ring homomorphism when $n < 8$. For these cases, every involutive n-Jordan homomorphism between commutative C^*-algebras is norm continuous.

Keywords: n-homomorphism; n-ring.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. An n-ring homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a map $h : \mathcal{A}$ $\mathcal{B}$ that is additive (i.e., $h(a + b) = h(a) + h(b)$ for all a b $\mathcal{A}$) and n-multiplicative (i.e., $h(a_1 a_2$ $a_n) = h(a_1)h(a_1)$ $h(a_n)$ for all a_1 a_2 a_n $\mathcal{A}$). The map $h : \mathcal{A}$ $\mathcal{B}$ is called n-Jordan homomorphism if it is additive and $h(a^n) = (h(a))^n$ for all a $\mathcal{A}$. It is clear that every n-ring homomorphism is n-Jordan homomorphism but the converse is not true. There are some examples of n-Jordan homomorphisms which are not n-ring homomorphisms (for example refer to [2]). It is shown in [2] that every n-Jordan homomorphism between commutative Banach algebras is also n-ring homomorphism when n 3 4 . For $n = 2$, the proof is simple and routine. For the non-commutative case, Zelazko in [9] showed that if $\mathcal{A}$ is a Banach algebra which need not be commutative, and $\mathcal{B}$ is a semisimple commutative Banach algebra, then each Jordan homomorphism $h : \mathcal{A}$ $\mathcal{B}$ is a ring homomorphism.

An n-ring homomorphism $h : \mathcal{A}$ $\mathcal{B}$ between C^*-algebras is said to be - n-ring homomorphism if $h(a^*) = h(a)^*$ for all a $\mathcal{A}$. Similarly one can de ne -n-Jordan homomorphism. If, in addition, h is linear, we say that h is *involutive n-ring (Jordan) homomorphism*.

One of the fundamental results in the study of C^*-algebras is that if $T : \mathcal{A}$ $\mathcal{B}$ is is a -homomorphism between C^*-algebras, then it is norm contractive [6, theorem 2.1.7]. In [4], authors ask: Is every involutive n-ring homomorphism between C^*-algebras continuous? Park and Trout in [7] answered this question and proved that every involutive n-ring homomorphism between C^*-algebras is in fact norm

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contractive. Some questions of automatic continuity for n-homomorphisms between Banach algebras were also investigated in [1, 5]. After that, Tomforde in [8, theorem 3.6] proved that if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-preserving ring homomorphism, then φ is contractive. Consequently, φ is also continuous.

In this paper, we prove that every n-Jordan homomorphism between commutative Banach algebras is n-ring homomorphism when $n \in \{5, 6, 7\}$ (for the case $n = 5$ this had been proved earlier by Eshaghi et al in [3] with a long proof). Finally, using these results, we show that every involutive n-Jordan homomorphism between commutative C^*-algebras is continuous.

2. Main Results

Theorem 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative algebras, and let $h : \mathcal{A} \to \mathcal{B}$ be an n-Jordan homomorphism. Then h is an n-ring homomorphism for $n \in \{3, 4, 5, 6, 7\}$.

Proof For the cases $n = 3, 4$, refer to [3]. As for $n = 5$, the map h is additive such that $h(x^5) = (h(x))^5$ for all $x \in \mathcal{A}$. Using this equality, we have

$$h\left(\sum_{k=1}^{4}\binom{5}{k}x^ky^{5-k}\right) = \sum_{k=1}^{4}\binom{5}{k}h(x)^kh(y)^{5-k} \tag{1}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x + z$ in (1), we obtain

$$\begin{aligned} h\Bigg(&\sum_{k=0}^{4}\binom{4}{k}x^kz^{4-k}\,y + 2\sum_{k=0}^{3}\binom{3}{k}x^kz^{3-k}\,y^2 \\ &+2\sum_{k=0}^{2}\binom{2}{k}x^kz^{2-k}\,y^3 + xy^4 + zy^4\Bigg) \\ &= \sum_{k=0}^{4}\binom{4}{k}h(x)^kh(z)^{4-k}\,h(y) + 2\sum_{k=0}^{3}\binom{3}{k}h(x)^kh(z)^{3-k}\,h(y)^2 \\ &\quad +2\sum_{k=0}^{2}\binom{2}{k}h(x)^kh(z)^{2-k}\,h(y)^3 + h(x)h(y)^4 + h(z)h(y)^4 \end{aligned} \tag{2}$$

for all $x, y, z \in \mathcal{A}$. Combining (1) and (2) gives

$$\begin{aligned} &h(2x^3zy + 3x^2z^2y + 2xz^3y + 3x^2zy^2 + 3xz^2y^2 + 2xzy^3) \\ &\quad = 2h(x)^3h(z)h(y) + 3h(x)^2h(z)^2h(y) + 2h(x)h(z)^3h(y) \\ &\quad\quad +3h(x)^2h(z)h(y)^2 + 3h(x)h(z)^2h(y)^2 + 2h(x)h(z)h(y)^3 \end{aligned} \tag{3}$$

for all $x, y, z \in \mathcal{A}$. Substituting z by $-x$ in (3), we obtain

$$h(x^4y + 2x^2y^3) = h(x)^4h(y) + 2h(x)^2h(y)^3 \tag{4}$$

for all $x, y \in \mathcal{A}$. Now, if we replace y by $y+w$ in (4) and employ the same equality, we get

$$h(x^2y^2w + x^2yw^2) = h(x)^2h(y)^2h(w) + h(x)^2h(y)h(w)^2 \tag{5}$$

for all $x, y, w \in \mathcal{A}$. Replacing x by $x+u$ in (5), we have

$$h(xuy^2w + xuyw^2) = h(x)h(u)h(y)^2h(w) + h(x)h(u)h(y)h(w)^2 \tag{6}$$

for all $x, y, u, w \in \mathcal{A}$. Now, if we change y to $y+v$ in (6), we conclude

$$h(xuyvw) = h(x)h(u)h(y)h(v)h(w)$$

for all $x, y, u, v, w \in \mathcal{A}$. Therefore h is 5-ring homomorphism.

For the case $n = 6$, we assume that the map h is additive and $h(x^6) = (h(x))^6$ for all $x \in \mathcal{A}$. This fact implies the following equality if we replace x by $x+y$

$$h\left(\sum_{k=1}^{5}\binom{6}{k}x^ky^{6-k}\right) = \sum_{k=1}^{5}\binom{6}{k}h(x)^kh(y)^{6-k} \tag{7}$$

for all $x, y \in \mathcal{A}$. Commuting x by $x+z$ in (7), we obtain

$$\begin{aligned}
&h\Bigg(\sum_{k=0}^{5}\binom{5}{k}x^kz^{5-k}\Bigg)y + 15\Bigg(\sum_{k=0}^{4}\binom{4}{k}x^kz^{4-k}\Bigg)y^2\\
&\quad+20\Bigg(\sum_{k=0}^{3}\binom{3}{k}x^kz^{3-k}\Bigg)y^3 + 15\Bigg(\sum_{k=0}^{2}\binom{2}{k}x^kz^{2-k}\Bigg)y^4 + 6xy^5 + 6zy^5\\
&\quad= 6\Bigg(\sum_{k=0}^{5}\binom{5}{k}h(x)^kh(z)^{5-k}\Bigg)h(y) + 15\Bigg(\sum_{k=0}^{4}\binom{4}{k}h(x)^kh(z)^{4-k}\Bigg)h(y)^2\\
&\quad+20\Bigg(\sum_{k=0}^{3}\binom{3}{k}h(x)^kh(z)^{3-k}\Bigg)h(y)^3 + 15\Bigg(\sum_{k=0}^{2}\binom{2}{k}h(x)^kh(z)^{2-k}\Bigg)h(y)^4\\
&\quad+6h(x)h(y)^5 + 6h(z)h(y)^5
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Combining the above equality and (7), we get

$$\begin{aligned}
&h\Bigg(6\Bigg(\sum_{k=1}^{4}\binom{5}{k}x^kz^{5-k}\Bigg)y + 15\Bigg(\sum_{k=1}^{3}\binom{4}{k}x^kz^{4-k}\Bigg)y^2\\
&\qquad+20\Bigg(\sum_{k=1}^{2}\binom{3}{k}x^kz^{3-k}\Bigg)y^3 + 30zy^4\Bigg)\\
&\qquad= 6\Bigg(\sum_{k=1}^{4}\binom{5}{k}h(x)^kh(z)^{5-k}\Bigg)h(y) + 15\Bigg(\sum_{k=1}^{3}\binom{4}{k}h(x)^kh(z)^{4-k}\Bigg)h(y)^2\\
&\qquad+20\Bigg(\sum_{k=1}^{2}\binom{3}{k}h(x)^kh(z)^{3-k}\Bigg)h(y)^3 + 30h(x)h(z)h(y)^4
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Changing z to $-x$ in the last equality, we obtain

$$h(x^4y^2 + x^2y^4) = h(x)^4h(y)^2 + h(x)^2h(y)^4 \tag{8}$$

for all $x, y \in \mathcal{A}$. Now, if we replace y by $y+t$ in (8), we conclude

$$\begin{aligned}
h(x^4yt + 2x^2yt^3 + 3x^2y^2t^2 + 2x^2y^3t) &= h(x)^4h(y)h(t) + 2h(x)^2h(y)h(t)^3\\
&\quad+3h(x)^2h(y)^2h(t)^2 + 2h(x)^2h(y)^3h(t)
\end{aligned} \tag{9}$$

for all $x, y, t \in \mathcal{A}$. Substituting t by $t+u$ in (9), we have

$$h(x^2yt^2u+x^2ytu^2+x^2y^2tu) = h(x)^2h(u)h(y)h(t)^2h(u)+h(x)^2h(u)h(y)h(t)h(u)^2 +h(x)^2h(y)^2h(t)h(u) \tag{10}$$

for all $x, y, t, u \in \mathcal{A}$. We replace u by $u+v$ in (10) to obtain

$$h(x^2ytuv) = h(x)^2h(u)h(y)h(t)h(v) \tag{11}$$

for all $x, y, t, u, v \in \mathcal{A}$. Finally if we change x to $x+w$ in (11), we get

$$h(xytuvw) = h(x)h(y)h(t)h(u)h(v)h(w)$$

The above equality shows that the map h is 6-ring homomorphism. Now, for $n=7$. Replacing x by $x+y$ in equality $h(x^7) = (h(x))^7$, we have

$$h\left(\sum_{k=1}^{6}\binom{7}{k}x^ky^{7-k}\right) = \sum_{k=1}^{6}\binom{7}{k}h(x)^kh(y)^{7-k} \tag{12}$$

for all $x, y \in \mathcal{A}$. Commuting x by $x+z$ in (12), we obtain

$$\begin{aligned}
&h\Bigg(7\left(\sum_{k=0}^{6}\binom{6}{k}x^kz^{6-k}\right)y+21\left(\sum_{k=0}^{5}\binom{5}{k}x^kz^{5-k}\right)y^2\\
&\quad+35\left(\sum_{k=0}^{4}\binom{4}{k}x^kz^{4-k}\right)y^3+35\left(\sum_{k=0}^{3}\binom{3}{k}x^kz^{3-k}\right)y^4\\
&\quad+21\left(\sum_{k=0}^{2}\binom{2}{k}x^kz^{2-k}\right)y^5+7xy^6+7zy^6\Bigg)\\
&=7\left(\sum_{k=0}^{6}\binom{6}{k}h(x)^kh(z)^{6-k}\right)h(y)+21\left(\sum_{k=0}^{5}\binom{5}{k}h(x)^kh(z)^{5-k}\right)h(y)^2\\
&+35\left(\sum_{k=0}^{4}\binom{4}{k}h(x)^kh(z)^{4-k}\right)h(y)^3+35\left(\sum_{k=0}^{3}\binom{3}{k}h(x)^kh(z)^{3-k}\right)h(y)^4\\
&+21\left(\sum_{k=0}^{2}\binom{2}{k}h(x)^kh(z)^{2-k}\right)h(y)^5+7h(x)h(y)^6+7h(z)h(y)^6
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Combining (12) and the above equality, we get

$$\begin{aligned}
&h\Bigg(7\left(\sum_{k=1}^{5}\binom{6}{k}x^kz^{6-k}\right)y+21\left(\sum_{k=1}^{4}\binom{5}{k}x^kz^{5-k}\right)y^2\\
&\quad+35\left(\sum_{k=1}^{3}\binom{4}{k}x^kz^{4-k}\right)y^3+35\left(\sum_{k=1}^{2}\binom{3}{k}x^kz^{3-k}\right)y^4+42xzy^5\Bigg)\\
&\quad=7\left(\sum_{k=1}^{5}\binom{6}{k}h(x)^kh(z)^{6-k}\right)h(y)+21\left(\sum_{k=1}^{4}\binom{5}{k}h(x)^kh(z)^{5-k}\right)h(y)^2\\
&\quad+35\left(\sum_{k=1}^{3}\binom{4}{k}h(x)^kh(z)^{4-k}\right)h(y)^3+35\left(\sum_{k=1}^{2}\binom{3}{k}h(x)^kh(z)^{3-k}\right)h(y)^4\\
&\quad+42h(x)h(z)h(y)^5
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Letting z to be $-x$ in the above, we obtain

$$h(3x^2y^5+5x^4y^3+x^6y) = 3h(x)^2h(y)^5+5h(x)^4h(y)^3+h(x)^6h(y) \tag{13}$$

for all $x, y \in \mathcal{A}$. Now, if we replace y by $y+t$ in (13) and use the same equality, we conclude

$$h(x^2y^4t + 2x^2y^3t^2 + 2x^2y^2t^3 + x^2yt^4 + x^4y^2t + x^4yt^2)$$
$$= h(x)^2h(y)^4h(t) + 2h(x)^2h(y)^3h(t)^2 + 2h(x)^2h(y)^2h(t)^3$$
$$+h(x)^2h(y)h(t)^4 + h(x)^4h(y)^2h(t) + h(x)^4h(y)h(t)^2 \quad (14)$$

for all $x\ y\ t$ $\mathcal{A}$. Substituting t by $t + u$ in (14), we have

$$h(2x^2y^3tu + 3x^2y^2t^2u + 3x^2y^2tu^2 + 2x^2yt^3u + 3x^2yt^2u^2 + 2x^2ytu^3 + x^4ytu)$$
$$= 2h(x)^2h(y)^3h(t)h(u) + 3h(x)^2h(y)^2h(t)^2h(u)$$
$$+3h(x)^2h(y)^2h(t)h(u)^2 + 2h(x)^2h(y)h(t)^3h(u)$$
$$+3h(x)^2h(y)h(t)^2h(u)^2 + 2h(x)^2h(y)h(t)h(u)^3 + h(x)^4h(y)h(t)h(u)$$

for all $x\ y\ t\ u$ $\mathcal{A}$. We replace u by $u + v$ in the last equality to obtain

$$h(x^2y^2tuv + x^2yt^2uv + x^2ytu^2v + x^2ytuv^2)$$
$$= h(x)^2h(y)^2h(t)h(u)h(v) + h(x)^2h(u)h(y)h(t)^2h(u)h(v)$$
$$+h(x)^2h(y)h(t)h(u)^2h(v) + h(x)^2h(y)h(t)h(u)h(v)^2 \quad (15)$$

for all $x\ y\ t\ u\ v$ $\mathcal{A}$. Replacing v by $v + w$ in (15), we deduce

$$h(x^2ytuvw) = h(x)^2h(y)h(t)h(u)h(v)h(w) \quad (16)$$

for all $x\ y\ t\ u\ v\ w$ $\mathcal{A}$. Finally, if we change x to $x + z$ in (16), we get

$$h(xyztuvw) = h(x)h(y)h(z)h(t)h(u)h(v)h(w)$$

for all $x\ y\ z\ t\ u\ v\ w$ $\mathcal{A}$. Hence the map h is 7-ring homomorphism. ■

3. Applications

An element a of a C^*-algebra $\mathcal{A}$ is *positive* if a is hermitian, that is $a = a^*$, and (a) $\mathbb{R}^+$, where (a) is the spectrum of a. We write $a \geqslant 0$ to mean a is positive. Also a linear map $T : \mathcal{A}$ $\mathcal{B}$ between C^*-algebras is positive if $a \geqslant 0$ implies $T(a) \geqslant 0$ for all a $\mathcal{A}$. We say that the map T is *completely positive* if, for any natural number k, the induced map $T_k : M_k(\mathcal{A})$ $M_k(\mathcal{B}); T_k((a_{ij}))$ $(T(a_{ij}))$, on k k matrices is positive.

proposition 3.1. Let n $\mathbb{N}$ such that $2 \leqslant n \leqslant 7$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are commutative C^*-algebras. Let and be nonnegative real numbers and let $r\ s$ be real numbers, f be a map from $\mathcal{A}$ into $\mathcal{B}$, and let $r\ s$ be real numbers such that either $(r\ \ 1)(s\ \ 1) > 0$ and $s \geqslant 0$ or $(r\ \ 1)(s\ \ 1) > 0$, $s < 0$, and $f(0) = 0$. Assume that f satis es the system of functional inequalities

$$f(x + y + z^*) \quad f(x) \quad f(y) \quad f(z)^* \leqslant \ (\ x \ ^r + \ y \ ^r + \ z \ ^r)$$

$$f(x^n) \quad f(x)^n \leqslant \quad x \ ^{ns}$$

for all $x \ y$ $\mathcal{A}$. Then, there exists a unique -n-ring homomorphism $h : \mathcal{A}$ $\mathcal{B}$ such that

$$f(x) \quad h(x) \leqslant \frac{2}{2 \quad 2^r} \ x \ ^r$$

for all x $\mathcal{A}$.

Proof We can deduce the result from [3, theorm 2.1, theorem 2.2] and theorem 2. ■

The following theorem has been proved by Park and Trout in [7, theorm 3.2].

Theorem 3.1. Let $: \mathcal{A}$ $\mathcal{B}$ be an involutive n-homomorphism between C^*-algebras. If $n \geqslant 3$ is odd, then $\leqslant 1$, i.e., is norm-contractive.

corollary 3.1. Let n $3 \ 5 \ 7$, and let $\mathcal{A}$ and $\mathcal{B}$ be commutative C^*-algebras. If $h : \mathcal{A}$ $\mathcal{B}$ is an involutive n-Jordan homomorphism, then h $\leqslant 1$, i.e., h is norm contractive.

Proof For $n = 3$, The result follows from [2, theorm 2.1] and theorem 2 and for $n = 5 \ 7$, we can use theorem 2 and theorem 3. ■

For the even case, we need the following theorem which is proved in [7, theorm 2.3].

Theorem 3.2. Let $: \mathcal{A}$ $\mathcal{B}$ be an involutive n-homomorphism between C^*-algebras. If $n \geqslant 2$ is even, then is completely positive. Thus, is bounded.

corollary 3.2. Let n $4 \ 6$. If $h : \mathcal{A}$ $\mathcal{B}$ is an involutive n-Jordan homomorphism between commutative C^*-algebras, then h is completely positive. Thus, h is bounded.

Proof By using [2, theorm 2.1] and theorem 2 for $n = 4$ and theorems 2 and 3 for $n = 6$, we obtain the desired result. ■

Question. Let n be an arbitrary and xed natural number. Is every n-Jordan homomorphism between commutative algebras is also a n-ring homomorphism? If this is true, then every involutive n-Jordan homomorphism between commutative C^*-algebras is norm contractive. Is this true in the non-commutative case?

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