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Inverse eigenvalue problem of interval nonnegative matrices via lower triangular matrices

A. M. Nazari[a,*], M. Zeinali[b], H. Mesgarani[b], A. Nezami[a]

[a] *Department of Mathematics, Arak University, P.O. Box 38156-8-8349, Arak, Iran.*
[b] *Department of Mathematics, Shahid Rajaee University, Lavizan, Tehran, Iran.*

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Abstract. In this paper, for a given set of real interval numbers σ that satisfies in special conditions, we find an interval nonnegative matrix C^I such that for each point set δ of given interval spectrum σ, there exists a point matrix C of C^I such that δ is its spectrum. For this purpose, we use unit lower triangular matrices and especially try to use binary unit lower triangular matrices. We also study some conditions for existence solution to the problem.

Keywords: Interval matrix, interval arithmetics, inverse eigenvalue problem, nonnegative matrices.

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1. Introduction and preliminaries

A matrix L is unit lower triangular provided each entry on its main diagonal equals 1, and each entry above its main diagonal is zero. The inverse of a unit lower triangular matrix also is unit lower triangular and is easy to calculate. In Gaussian elimination method and LU factorization unit lower triangular matrices play a very important role. The binary unit lower triangular matrices is a unit lower triangular matrices that all entries below its main diagonal are 0 or 1.

Interval analysis is used to solve many robotic problems such as the clearance effect, robot reliability, motion planning, localization and navigation [6]. An interval matrix is a matrix whose entries are interval numbers. The use of interval numbers began in the

*Corresponding author.
E-mail address: a-nazari@araku.ac.ir (A. M. Nazari); m.zeinali64@yahoo.com (M. Zeinali); Hmesgarani@sru.ac.ir (H. Mesgarani); atiyeh.nezami@gmail.com (A. Nezami).

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first half of the twentieth century and is expanding every day. In 1965, logic was fuzzy by Zadeh and interval numbers were used [13]. In 1993, Rohn [10] found the inverse of interval matrices.

Now, we recall some definition of interval analysis and interval matrices. The summation, subtraction, multiplication and division of two interval numbers $\mathbf{b} = [\underline{b}, \overline{b}], \mathbf{a} = [\underline{a}, \overline{a}]$, respectively, are defined as follows:

- $\mathbf{a} + \mathbf{b} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$
- $\mathbf{a} - \mathbf{b} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$
- $\mathbf{a} \cdot \mathbf{b} = [\min\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}, \max\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}]$
- $\frac{\mathbf{a}}{\mathbf{b}} = \mathbf{a} \cdot \mathbf{b}^{'}$, $\mathbf{b}^{'} = [\frac{1}{\overline{b}}, \frac{1}{\underline{b}}]$ and $0 \notin \mathbf{b}$
  also the square of a interval number $\mathbf{a} = [\underline{a}, \overline{a}]$ is
- $\mathbf{a}^2 = \begin{cases} [\underline{a}^2, \overline{a}^2] & \text{if } 0 \leq \underline{a} \leq \overline{a}, \\ [\overline{a}^2, \underline{a}^2] & \text{if } \underline{a} \leq \overline{a} \leq 0, \\ [0, \max\{[\underline{a}^2, \overline{a}^2\}] & \text{if } \underline{a} \leq 0 \leq \overline{a}. \end{cases}$

The interval complex number z is similarly defined as follows:

$$z = [\underline{z}, \overline{z}] = \lambda + i\mu = [\underline{\lambda}, \overline{\lambda}] + i[\underline{\mu}, \overline{\mu}],$$

where λ is the real part and μ is the imaginary part of this interval complex number. The properties of interval complex numbers can be deduced based on the properties of interval real numbers defined above.

Definition 1.1 Let $\underline{A}$ and $\overline{A}$ be $n \times n$ real matrices, the following set

$$A^I = [\underline{A}, \overline{A}] = \{A : \underline{A} \leq A \leq \overline{A}\}$$

is called an $n \times n$ real interval matrix.The midpoint and the radius of A^I are denoted respectively by $A_c = \frac{\underline{A}+\overline{A}}{2}$ and $A_\Delta = \frac{\overline{A}-\underline{A}}{2}$.

If all interval entries of a real interval matrix ≥ 0, then A^I is called nonnegative interval matrix. The set of all real $n \times n$ interval matrices is denoted by $\mathbb{IR}^{n\times n}$ and the set of all $n \times n$ nonnegative interval matrices also is denoted by $\mathbb{NIR}^{n\times n}$.

Definition 1.2 Let A^I be an interval square matrix. Then $\Lambda(A^I) = \{\lambda \in \mathbb{C}; Ax = \lambda x, x \neq 0, A \in A^I\}$ is the set of eigenvalues of A^I.

When we say that the interval number of $\lambda_1^I = [\underline{\lambda_1}, \overline{\lambda_1}]$ is greater than or equal the interval number of $\lambda_2^I = [\underline{\lambda_2}, \overline{\lambda_2}]$, it means for all $a \in \lambda_1^I$ and $b \in \lambda_2^I$, $a \geqslant b$.

The eigenvalue of $n \times n$ nonnegative interval matrix A^I is called Perron interval eigenvalue of A^I if it is nonnegative and greater than or equal of all absolute value of eigenvalues of A^I and denoted by $\lambda_1 = [\underline{\lambda_1}, \overline{\lambda_1}]$. i.e. $[\underline{\lambda_1}, \overline{\lambda_1}] \geq \left|[\underline{\lambda_i}, \overline{\lambda_i}]\right|$ for $i = 2, 3, \cdots, n$, where $\left|[\underline{\lambda_i}, \overline{\lambda_i}]\right| = [\min\{\left|\underline{\lambda_i}\right|, \left|\overline{\lambda_i}\right|\}, \max\{\left|\underline{\lambda_i}\right|, \left|\overline{\lambda_i}\right|\}]$

The problem of finding the eigenvalue of interval matrices is one of the most pressing issues for mathematicians, and several papers have been written in recent years, for example [2–4, 11].

In 2018, Nazari et al. [9] started the inverse eigenvalue problem of nonnegative interval matrices, which is briefly denoted by NIIEP. They solved NIIEP for matrices of order at most 3. In this paper by helping unit lower triangular matrices and use similarity of matrices we try to solve the problem for order greater than 3. Nazari et al. [7] solved NIEP for any order of distance matrices via unit lower triangular matrices.

When we say that the interval spectrum σ^I is realizable by interval matrix C^I, it means, we can find an interval nonnegative matrix C^I such that for every point set δ of interval set of eigenvalues σ^I (for each interval element one point), there exist a point nonnegative matrix C of C^I such that δ is its spectrum.

Some necessary conditions for NIIEP on the list of complex interval number $\sigma = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \ldots, [\underline{\lambda_n}, \overline{\lambda_n}]\}$ to be the spectrum of a nonnegative interval matrix are listed below.
(1) The Perron eigenvalue $\max\{|[\underline{\lambda_i}, \overline{\lambda_i}]|; [\underline{\lambda_i}, \overline{\lambda_i}] \in \sigma\}$ belongs to σ (Perron-Frobenius theorem in interval case).
(2) The list σ is closed under complex conjugation.
(3) $s_k = \sum_{i=1}^n \left[\underline{\lambda_i}, \overline{\lambda_i}\right]^k \geq 0$.
(4) $)s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$ (JLL inequality in interval case) [1, 5].

In this paper, we only consider real interval spectrum and postpone the inverse eigenvalue problem for complex spectrum to another paper in the near future. The paper is organized as follows. First we solve the NIIEP in several cases where each element of σ is real, and σ has at least as many negative eigenvalues as positive eigenvalues. Then we solve the NIIEP in several cases where each element of σ is real and the number of negative elements of σ is less than the number of positive elements of σ.

2. Interval real spectrum

Let $k \leqslant 3$ and $\sigma^I = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \ldots, [\underline{\lambda_n}, \overline{\lambda_n}]\}$ be a given spectrum such that $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant \cdots \geqslant [\underline{\lambda_k}, \overline{\lambda_k}] \geqslant 0 > [\underline{\lambda_n}, \overline{\lambda_n}] \geqslant \cdots \geqslant [\underline{\lambda_{k+1}}, \overline{\lambda_{k+1}}]$. We try to construct a nonnegative interval matrix C^I such that it realizes spectrum σ. At first we solve the interval spectrum of Suleimanova. This spectrum has one positive eigenvalue and nonpositive another eigenvalues with nonnegative summation.

Theorem 2.1 ([12] Suleimanova's Theorem in interval case). Assume that given set of real interval numbers $\sigma^I = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \ldots, [\underline{\lambda_n}, \overline{\lambda_n}]\}$ such that $[\underline{\lambda_1}, \overline{\lambda_1}] > 0 \geqslant [\underline{\lambda_n}, \overline{\lambda_n}] \geqslant [\underline{\lambda_{n-1}}, \overline{\lambda_{n-1}}] \geqslant \cdots \geqslant [\underline{\lambda_2}, \overline{\lambda_2}]$, and $s_1 = \sum_{i=1}^n [\underline{\lambda_i}, \overline{\lambda_i}] \geq 0$, then there exists a set of nonnegative interval matrices that realizes σ.

Proof. If characteristic polynomial of interval matrix is

$$P(\lambda) = \prod_{i=1}^n \left(\lambda - [\underline{\lambda_i}, \overline{\lambda_i}]\right) = \lambda^n - [\underline{a_{n-1}}, \overline{a_{n-1}}]\lambda^{n-1} - [\underline{a_{n-2}}, \overline{a_{n-2}}]\lambda^{n-2} - \cdots - [\underline{a_0}, \overline{a_0}]$$

and all $[\underline{a_i}, \overline{a_i}] \geqslant 0$ for $i = 0, 1, \cdots, n-1$, then it is easy to see that the following nonnegative interval companion matrix is solution of problem

$$C^I = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ & & \ddots & \\ 0 & 0 & 0 \cdots & 1 \\ [\underline{a_0}, \overline{a_0}] & [\underline{a_1}, \overline{a_1}] & \cdots & \left[\underline{a_{n-1}}, \overline{a_{n-1}}\right] \end{pmatrix}.$$

On the other hand, we construct the solution via unit lower triangular matrix. Let $n = 2$

and consider the upper interval triangular matrix

$$A^I = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] & \alpha_2 \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] \end{pmatrix},$$

where $\alpha_2 = [\underline{\alpha_2}, \overline{\alpha_2}]$ is interval number and 2×2 unite lower tiangular matrix $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Therefore, by similarity of matrices, the following matrix

$$C^I = LA^IL^{-1} = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] - \alpha_2 & \alpha_2 \\ [\underline{\lambda_1}, \overline{\lambda_1}] - \alpha_2 - [\underline{\lambda_2}, \overline{\lambda_2}] & \alpha_2 + [\underline{\lambda_2}, \overline{\lambda_2}] \end{pmatrix}, \tag{1}$$

has eigenvalues $[\underline{\lambda_1}, \overline{\lambda_1}]$ and $[\underline{\lambda_2}, \overline{\lambda_2}]$ and if $-\overline{\lambda_2} \le \underline{\alpha_2} \le \overline{\alpha_2} \le \underline{\lambda_1}$, then the matrix C^I is nonnegative. For $n = 3$, we consider

$$A^I = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] & \alpha_2 & \alpha_3 \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] & 0 \\ 0 & 0 & [\underline{\lambda_3}, \overline{\lambda_3}] \end{pmatrix},$$

where $\alpha_2 = [\underline{\alpha_2}, \overline{\alpha_2}]$, $\alpha_3 = [\underline{\alpha_3}, \overline{\alpha_3}]$ are interval numbers and assume that

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then the matrix

$$C^I = LA^IL^{-1} = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] - \alpha_2 - \alpha_3 & \alpha_2 & \alpha_3 \\ [\underline{\lambda_1}, \overline{\lambda_1}] - \alpha_2 - [\underline{\lambda_2}, \overline{\lambda_2}] - \alpha_3 & \alpha_2 + [\underline{\lambda_2}, \overline{\lambda_2}] & \alpha_3 \\ [\underline{\lambda_1}, \overline{\lambda_1}] - \alpha_2 - \alpha_3 - [\underline{\lambda_3}, \overline{\lambda_3}] & \alpha_2 & \alpha_3 + [\underline{\lambda_3}, \overline{\lambda_3}] \end{pmatrix}, \tag{2}$$

is similar to the matrix A^I and if $-\overline{\lambda_2} \le \underline{\alpha_2}$, $-\overline{\lambda_3} \le \underline{\alpha_3}$ and $\overline{\alpha_2} + \overline{\alpha_3} \le \underline{\lambda_1}$, then the matrix C^I is nonnegative. To continue the proof, we follow the above process. Consider

$$A^I = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & [\underline{\lambda_n}, \overline{\lambda_n}] \end{pmatrix}.$$

Similarly, $\alpha_2 = [\underline{\alpha_2}, \overline{\alpha_2}]$, $\alpha_3 = [\underline{\alpha_3}, \overline{\alpha_3}]$, $\cdots$, $\alpha_n = [\underline{\alpha_n}, \overline{\alpha_n}]$ are interval numbers and

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \\ 1 & 0 & 0 & \ddots & 0 \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{3}$$

Then the matrix

$$C^I = LA^IL^{-1} = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ [\underline{\lambda_1}, \overline{\lambda_1}] - [\underline{\lambda_2}, \overline{\lambda_2}] - t & \alpha_2 + [\underline{\lambda_2}, \overline{\lambda_2}] & \alpha_3 & \cdots & \alpha_n \\ & & \ddots & & \\ [\underline{\lambda_1}, \overline{\lambda_1}] - [\underline{\lambda_{n-1}}, \overline{\lambda_{n-1}}] - t & \alpha_2 & \alpha_3 & \ddots & \alpha_n \\ [\underline{\lambda_1}, \overline{\lambda_1}] - [\underline{\lambda_n}, \overline{\lambda_n}] - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n + [\underline{\lambda_n}, \overline{\lambda_n}] \end{pmatrix} \tag{4}$$

with $t = \sum_{i=2}^{n} \alpha_i$ is similar to the matrix A^I, and if

$$\begin{aligned} -\underline{\lambda_i} &\le \underline{\alpha_i}, \qquad i = 2, 3, \cdots, n, \\ \overline{\alpha_2} + \overline{\alpha_3} + \cdots + \overline{\alpha_n} &\le \underline{\lambda_1}, \end{aligned} \tag{5}$$

then the interval matrix C^I is nonnegative and by similarity transformation for every point set σ of σ^I, we can select the point matrix C of C^I such that σ is its spectrum. The system of inequalities (5) always has a feasible solution. For example, one of these solutions is to put $\alpha_i = -[\underline{\lambda_i}, \overline{\lambda_i}]$ for $i = 2, 3, \cdots, n$. ■

***Example* 2.2** For the following interval set of eigenvalues find an interval matrix C^I such that realize this set.

$$\sigma^I = \{[14, 17], [-4, -3], [-3, -2], [-2, -1], [-1, 0]\}$$

All of necessary conditions satisfy. At first we find the characteristic polynomial

$$P(\lambda) = \lambda^5 - [4, 11]\lambda^4 - [49, 159]\lambda^3 - [104, 589]\lambda^2 - [60, 850]\lambda - [0, 408].$$

Because all coefficients of the above polynomial except λ^5 are negative, the following nonnegative interval companion matrix realizes the subset of σ

$$C^I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ [0, 408] & [60, 850] & [104, 589] & [49, 159] & [4, 11] \end{pmatrix}.$$

Also, by choosing the interval matrix A^I as

$$A^I = \begin{pmatrix} [14, 17] & [4, 5] & [3, 4] & [2, 3] & [1, 2] \\ 0 & [-4, -3] & 0 & 0 & 0 \\ 0 & 0 & [-3, -2] & 0 & 0 \\ 0 & 0 & 0 & [-2, -1] & 0 \\ 0 & 0 & 0 & 0 & [-1, 0] \end{pmatrix}$$

and the point matrix L from (3) for $n = 6$, we can find the nonnegative interval matrix $C^I = LA^I L^{-1}$ as follows:

$$C^I = \begin{pmatrix} [0,7] & [4,5] & [3,4] & [2,3] & [1,2] \\ [3,11] & [0,2] & [3,4] & [2,3] & [1,2] \\ [2,10] & [4,5] & [3,4] & [2,3] & [1,2] \\ [1,9] & [4,5] & [3,4] & [2,3] & [1,2] \\ [0,8] & [4,5] & [3,4] & [2,3] & [0,2] \end{pmatrix},$$

that σ is its spectrum.

Remark 1 *We should pay attention to this that the interval matrix C^I that we obtained in the previous example is not such that the set of its eigenvalues is the σ^I, but for each point set that we choose from the σ^I, we can choose a ponit matrix from the interval matrix C^I, such that the selected point set be the spectrum of the selected point matrix. For example the following point matrix*

$$\begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 3 & 2 & 4 & 3 & 2 \\ 10 & 5 & 4 & 3 & 2 \\ 9 & 5 & 4 & 3 & 2 \\ 8 & 5 & 4 & 3 & 2 \end{bmatrix} \in C^I,$$

but its eigenvalues, $\left[0.0\ 0.0\ 18.7223\ -3.3612 + 0.7484\,i\ -3.3612 - 0.7484\,i\right]$, *don't belong to the σ^I.*

3. Spectrum with two positive interval eigenvalues

First, for a set of interval eigenvalues $\sigma^I = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}], [\underline{\lambda_4}, \overline{\lambda_4}]\}$ such that $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant 0 > [\underline{\lambda_4}, \overline{\lambda_4}] \geqslant [\underline{\lambda_3}, \overline{\lambda_3}]$ and $\sum_{i=1}^{4} [\underline{\lambda_i}, \overline{\lambda_i}] \geqslant 0$ and $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant \left|[\underline{\lambda_i}, \overline{\lambda_i}]\right|, i = 3, 4$, we find an interval nonnegative 4×4 matrix C^I such that realizes σ and then for a given interval set σ with two positive interval eigenvalues and three negative interval eigenvalues that have necessary conditions (1)-(4) solve the problem and finally for two positive interval eigenvalues and more than three interval negative eigenvalues with necessary conditions again study the problem.

Theorem 3.1 Let $\sigma^I = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}], [\underline{\lambda_4}, \overline{\lambda_4}]\}$ such that $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant 0 > [\underline{\lambda_4}, \overline{\lambda_4}] \geqslant [\underline{\lambda_3}, \overline{\lambda_3}]$ and $\sum_{i=1}^{4} [\underline{\lambda_i}, \overline{\lambda_i}] \geqslant 0$ and $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant \left|[\underline{\lambda_i}, \overline{\lambda_i}]\right|, i = 3, 4$. Then there exists an interval nonnegative matrix that realizes σ^I.

Proof. If $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_4}, \overline{\lambda_4}] + [\underline{\lambda_3}, \overline{\lambda_3}]$ then by (2) we can construct an interval nonnegative matrix C_1^I with spectrum $\{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_4}, \overline{\lambda_4}], [\underline{\lambda_3}, \overline{\lambda_3}]\}$ and the nonnegative interval matrix $C^I = \begin{pmatrix} C_1^I & 0 \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] \end{pmatrix}$ realizes σ. Otherwise consider the interval matrix A^I and point matrix L as follows:

$$A^I = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] & \alpha_2 + \alpha_4 & \alpha_3 & 0 \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] & \alpha & \alpha_4 \\ 0 & 0 & [\underline{\lambda_3}, \overline{\lambda_3}] & 0 \\ 0 & 0 & 0 & [\underline{\lambda_4}, \overline{\lambda_4}] \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

where $\alpha = \beta_3 = [\underline{\beta_3}, \overline{\beta_3}]$ and $\alpha_i = [\underline{\alpha_i}, \overline{\alpha_i}]$ for $i = 2, 3, 4$ are real interval numbers. For convenience, we denote $\lambda_i = \left[\underline{\lambda_i}, \overline{\lambda_i}\right]$ for $i = 1, \cdots, 4$. Then the following interval matrix

$$C^I = LA^IL^{-1} = \begin{pmatrix} \lambda_1 - t & \alpha_2 + \alpha_4 & \alpha_3 & 0 \\ \lambda_1 - \lambda_2 - t - \alpha & \alpha_2 + \lambda_2 & \alpha_3 + \beta_3 & \alpha_4 \\ \lambda_1 - \lambda_3 - t & \alpha_2 + \alpha_4 & \alpha_3 + \lambda_3 & 0 \\ \lambda_1 - \lambda_2 - t - \alpha & \alpha_2 + \lambda_2 - \lambda_4 & \alpha_3 + \alpha & \alpha_4 + \lambda_4 \end{pmatrix},$$

is similar to the matrix A, where $t = \sum_{i=2}^{4} \alpha_i$. If

$$\begin{aligned} -\underline{\lambda_i} &\le \underline{\alpha_i}, \qquad i = 2, 3, 4, \\ \sum_{i=2}^{4} \overline{\alpha_i} &\le \underline{\lambda_1} \\ -\underline{\alpha_3} &\le \underline{\beta_3} \le \overline{\beta_3} \le \underline{\lambda_1} - \overline{\lambda_2} - \sum_{i=2}^{4} \overline{\alpha_i}, \end{aligned} \tag{6}$$

then the matrix C is nonnegative. ■

Now we consider the set of σ^I with two positive eigenvalues and three negative eigenvalues with special conditions.

Theorem 3.2 Consider spectrum $\sigma^I = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}], [\underline{\lambda_4}, \overline{\lambda_4}], [\underline{\lambda_5}, \overline{\lambda_5}]\}$ such that $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant 0 > [\underline{\lambda_5}, \overline{\lambda_5}] \geqslant [\underline{\lambda_4}, \overline{\lambda_4}] \geqslant [\underline{\lambda_3}, \overline{\lambda_3}]$, and $\sum_{i=1}^{5} [\underline{\lambda_i}, \overline{\lambda_i}] \geqslant 0$ and $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant \left|[\underline{\lambda_i}, \overline{\lambda_i}]\right|, i = 3, 4, 5$. If there exist real interval numbers $\beta_3 = [\underline{\beta_3}, \overline{\beta_3}]$ and $\alpha_i = [\underline{\alpha_i}, \overline{\alpha_i}]$ for $i = 2, 3, 4, 5$ such that

$$\begin{aligned} -\underline{\lambda_i} &\le \underline{\alpha_i}, \qquad i = 2, 3, 4, 5, \\ \sum_{i=2}^{5} \overline{\alpha_i} &\le \underline{\lambda_1}, \\ \underline{\alpha_2} + \underline{\alpha_4} + \underline{\alpha_5} &\geqslant 0, \\ -\underline{\alpha_3} &\le \underline{\beta_3} \le \overline{\beta_3} \le \underline{\lambda_1} - \overline{\lambda_2} - \sum_{i=2}^{5} \overline{\alpha_i}, \end{aligned} \tag{7}$$

then there exists a nonnegative interval 5×5 matrix that realizes σ.

Proof. Let $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_5}, \overline{\lambda_5}] + [\underline{\lambda_4}, \overline{\lambda_4}] + [\underline{\lambda_3}, \overline{\lambda_3}]$. Then, by (4), we can construct an interval nonnegative matrix C_1^I with spectrum $\left\{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_5}, \overline{\lambda_5}], [\underline{\lambda_4}, \overline{\lambda_4}], [\underline{\lambda_3}, \overline{\lambda_3}]\right\}$ and the nonnegative interval matrix $C^I = \begin{pmatrix} C_1^I & 0 \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] \end{pmatrix}$ has spectrum σ. Otherwise in this case we consider

$$A^I = \begin{pmatrix} [\underline{\lambda_1}, \overline{\lambda_1}] & \alpha_2 + \alpha_4 + \alpha_5 & \alpha_3 & 0 & 0 \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] & \beta_3 & \alpha_4 & \alpha_5 \\ 0 & 0 & [\underline{\lambda_3}, \overline{\lambda_3}] & 0 & 0 \\ 0 & 0 & 0 & [\underline{\lambda_4}, \overline{\lambda_4}] & 0 \\ 0 & 0 & 0 & 0 & [\underline{\lambda_5}, \overline{\lambda_5}] \end{pmatrix},$$

and

$$L = \begin{pmatrix} 1\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0 \\ 1\,0\,1\,0\,0 \\ 1\,1\,0\,1\,0 \\ 1\,1\,0\,0\,1 \end{pmatrix}, \tag{8}$$

where $\beta_3 = [\underline{\beta_3}, \overline{\beta_3}]$ and $\alpha_i = [\underline{\alpha_i}, \overline{\alpha_i}]$ for $i = 2, 3, 4, 5$ are real interval numbers and similar above theorem, we denote $\lambda_i = [\underline{\lambda_i}, \overline{\lambda_i}]$ for $i = 1, \cdots, 4$. Then the following interval matrix

$$C^I = LA^IL^{-1} = \begin{pmatrix} \lambda_1 - t & \alpha_2 + \alpha_4 + \alpha_5 & \alpha_3 & 0 & 0 \\ \lambda_1 - \lambda_2 - t - \beta_3 & \alpha_2 + \lambda_2 & \alpha_3 + \beta_3 & \alpha_4 & \alpha_5 \\ \lambda_1 - \lambda_3 - t & \alpha_2 + \alpha_4 + \alpha_5 & \alpha_3 + \lambda_3 & 0 & 0 \\ \lambda_1 - \lambda_2 - t - \beta_3 & \alpha_2 + \lambda_2 - \lambda_4 & \alpha_3 + \beta_3 & \alpha_4 + \lambda_4 & \alpha_5 \\ \lambda_1 - \lambda_2 - t - \beta_3 & \alpha_2 + \lambda_2 - \lambda_5 & \alpha_3 + \beta_3 & \alpha_4 & \alpha_5 + \lambda_5 \end{pmatrix}. \tag{9}$$

is similar to the matrix A, where $t = \sum_{i=2}^{4} \alpha_i$. Since hold all conditions (7), then this matrix is nonnegative and has spectrum σ. ■

***Example* 3.3** Let

$$\sigma^I = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}], [\underline{\lambda_4}, \overline{\lambda_4}], [\underline{\lambda_5}, \overline{\lambda_5}]\} = \{[11, 12], [3, 4], [-4, -3], [-3, -2], [-2, -1]\}$$

where $[11, 12] > [3, 4] > 0 \geq [-2, -1] \geq [-3, -2] \geq [-4, -3]$ and $\sum_{i=1}^{5} \lambda_i \geq 0$. We choose the interval matrix A^I as

$$A^I = \begin{pmatrix} [11, 12] & [1, 5] & [4, 5] & 0 & 0 \\ 0 & [3, 4] & [-4, -3] & [3, 4] & [2, 3] \\ 0 & 0 & [-4, -3] & 0 & 0 \\ 0 & 0 & 0 & [-3, -2] & 0 \\ 0 & 0 & 0 & 0 & [-2, -1] \end{pmatrix},$$

and choose the matrix (8). Then

$$C^I = L^{-1}A^I L = \begin{pmatrix} [1,6] & [1,5] & [4,5] & 0 & 0 \\ [0,7] & [0,2] & [0,2] & [3,4] & [2,3] \\ [4,10] & [1,5] & [0,2] & 0 & 0 \\ [0,7] & [2,5] & [0,2] & [0,2] & [2,3] \\ [0,7] & [1,4] & [0,2] & [3,4] & [0,2] \end{pmatrix}.$$

Now, we consider the general case of two positive interval eigenvalues.

Theorem 3.4 Take spectrum $\sigma^I = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \cdots, [\underline{\lambda_n}, \overline{\lambda_n}]\}$ such that $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant 0 > [\underline{\lambda_n}, \overline{\lambda_n}] \geqslant [\underline{\lambda_{n-1}}, \overline{\lambda_{n-1}}] \geqslant \cdots \geqslant [\underline{\lambda_3}, \overline{\lambda_3}]$, and $\sum_{i=1}^{n}[\underline{\lambda_i}, \overline{\lambda_i}] \geqslant 0$ and $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant \left|[\underline{\lambda_i}, \overline{\lambda_i}]\right|, i = 3, 4, \cdots, n$. If there exist real interval numbers $\beta_i = [\underline{\beta_i}, \overline{\beta_i}]$ for $i = 3, 4, \cdots, r-1$ and $\alpha_i = [\underline{\alpha_i}, \overline{\alpha_i}]$ for $i = 2, 3, \cdots, n$ such that

$$\begin{aligned} -\underline{\lambda_i} \leq \underline{\alpha_i}, \qquad & i = 2, 3, \cdots, n, \\ \sum_{i=2}^{n} \overline{\alpha_i} \leq \underline{\lambda_1} & \\ \underline{\alpha_2} + \sum_{i=r}^{n} \underline{\alpha_i} \geqslant 0, & \\ -\underline{\alpha_j} \leq \underline{\beta_j} \leq \overline{\beta_j} \leq \underline{\lambda_1} - \overline{\lambda_2} - \sum_{i=2}^{n} \overline{\alpha_i}, \; & j = 3, 4, \cdots, r-1 \end{aligned} \tag{10}$$

then there exists a nonnegative interval $n \times n$ matrix that σ is its spectrum.

Proof. Let $\alpha_i; \alpha_2, \ldots, \alpha_n$ or $i = 2, 3, \ldots, n$ and $\alpha_i \geqslant -\lambda_i$ $(i = 3, \ldots, n)$; be real interval numbers. Let r be the smallest positive integer with $0 \geqslant \lambda_2 + \sum_{i=r}^{n} \lambda_i$. Set $t = \sum_{i=2}^{n} \alpha_i$. As $\lambda_1 + \lambda_2 = -\lambda_3 - \cdots - \lambda_n$, we have $3 \leqslant r \leqslant n$. Consider the matrices

$$A^I = \begin{pmatrix} \lambda_1 & \alpha_2 + (\alpha_r + \cdots + \alpha_n) & \alpha_3 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\ 0 & \lambda_2 & \beta_3 & \cdots & \beta_{r-1} & \alpha_r & \cdots & \alpha_n \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_{r-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_r & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where the second column of L has $n-r+1$ ones. Then the matrix $C^I = LA^IL^{-1}$ is given by

$$C^I = \begin{pmatrix} c_{11} & c_{12} & \alpha_3 & \alpha_4 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\ c_{21} & \alpha_2+\lambda_2 & \alpha_3+\beta_3 & \alpha_4+\beta_4 & \cdots & \alpha_{r-1}+\beta_{r-1} & \alpha_r & \cdots & \alpha_n \\ c_{31} & c_{32} & \lambda_3+\alpha_3 & \alpha_4+\beta_4 & \cdots & \alpha_{r-1}+\beta_{r-1} & 0 & \cdots & 0 \\ c_{4,1} & c_{42} & \alpha_3+\beta_3 & \lambda_4+\alpha_4 & \cdots & \alpha_{r-1}+\beta_{r-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ c_{r-1,1} & c_{r-1,2} & \alpha_3+\beta_3 & \alpha_4+\beta_4 & \cdots & \lambda_{r-1}+\alpha_{r-1} & 0 & \cdots & 0 \\ c_{r1} & c_{r2} & \alpha_3+\beta_3 & \alpha_4+\beta_4 & \cdots & \alpha_{r-1}+\beta_{r-1} & \lambda_r+\alpha_r & \cdots & \alpha_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \alpha_3+\beta_3 & \alpha_4+\beta_4 & \cdots & \alpha_{r-1}+\beta_{r-1} & 0 & \cdots & \lambda_n+\alpha_n \end{pmatrix},$$

where

$$\begin{aligned} c_{11} &= \lambda_1 - t, \\ c_{31} &= c_{41} = \cdots = c_{r-1,1} = \lambda_1 - \lambda_2 - t, \\ c_{21} &= c_{r1} = \cdots = c_{n1} = \lambda_1 - \lambda_2 - t - (\beta_3 + \cdots + \beta_{r-1}), \\ c_{12} &= c_{32} = \alpha_2 + \alpha_r + \cdots + \alpha_n \text{ and} \\ c_{i2} &= \alpha_2 + \lambda_2 - \lambda_i, \qquad i = 4, \ldots, n. \end{aligned}$$

The interval matrix C^I is nonnegative (and hence a realization of σ^I) if and only if hold conditions (10). Setting $\alpha_i = -\lambda_i$ $(i = 2, \ldots, n)$ and $\beta_i = \lambda_1$ for $i = 3, \ldots, n$ results in a nonnegative C^I. Hence, σ is realizable. ■

4. Spectrum with three positive interval eigenvalues

Now we consider NIIEP of given σ^I with three positive interval eigenvalues. Furthermore if we have one negative interval eigenvalue in σ^I, since we assume that $\lambda_1 = [\underline{\lambda_1}, \overline{\lambda_1}]$ is Perron interval eigenvalue of σ^I then by (1) we can find 2×2 nonnegative interval matrix C_1^I that has eigenvaliues $\sigma_1^I = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_4}, \overline{\lambda_4}]\}$ and the following 4×4 nonnegative interval matrix has eigenvalues σ^I:

$$C^I = \begin{pmatrix} C_1^I & 0 & 0 \\ 0 & [\underline{\lambda_2}, \overline{\lambda_2}] & 0 \\ 0 & 0 & [\underline{\lambda_3}, \overline{\lambda_3}] \end{pmatrix}.$$

In the following theorem we assume that σ^I has five membrs, that three of which are nonnegative.

Theorem 4.1 Let $\sigma^I = \{\lambda_1, \lambda_2, \ldots, \lambda_5\} = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \cdots, [\underline{\lambda_5}, \overline{\lambda_5}]\}$, be a list of real interval numbers satisfying $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant [\underline{\lambda_3}, \overline{\geqslant}\lambda_3] \geqslant 0 > [\underline{\lambda_5}, \overline{\lambda_5}] \geqslant [\underline{\lambda_4}, \overline{\lambda_4}]$ and $\sum_{i=1}^{5} \lambda_i \geqslant 0$. If $\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geqslant 0$, $|\lambda_4| \geqslant |\lambda_3$, $|\lambda_5| \geqslant |\lambda_3|$ and $|\lambda_4| + |\lambda_5| \geqslant \lambda_2 + \lambda_2$, then σ^I is realizable.

Proof. We consider two matrices A and L as follows:

$$A^I = \begin{pmatrix} \lambda_1 & -\lambda_2-\lambda_3-\lambda_4-\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_2 & -\lambda_3-\lambda_5 & -\lambda_4 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 & -\lambda_5 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}, \ L = \begin{pmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&1&1&0&0 \\ 1&1&0&1&0 \\ 1&1&1&0&1 \end{pmatrix}, \ L^{-1} = \begin{pmatrix} 1&0&0&0&0 \\ -1&1&0&0&0 \\ 0&-1&1&0&0 \\ 0&-1&0&1&0 \\ 0&0&-1&0&1 \end{pmatrix}.$$

Then

$$C^I = LAL^{-1} = \begin{pmatrix} \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 & -\lambda_2-\lambda_3-\lambda_4-\lambda_5 & 0 & 0 & 0 \\ \lambda_1+\lambda_3+\lambda_4+\lambda_5 & 0 & -\lambda_3-\lambda_5 & -\lambda_4 & 0 \\ \lambda_1+\lambda_3+\lambda_4+\lambda_5 & -\lambda_3-\lambda_4 & 0 & 0 & -\lambda_5 \\ \lambda_1+\lambda_3+\lambda_4+\lambda_5 & -\lambda_4 & -\lambda_3-\lambda_5 & 0 & 0 \\ \lambda_1+\lambda_3+\lambda_4+\lambda_5 & -\lambda_3-\lambda_4 & -\lambda_5 & 0 & 0 \end{pmatrix}.$$

By the given conditions in the theorem, it is clear that the matrix C^I is a nonnegative interval matrix. ■

Theorem 4.2 Let $\sigma^I = \{\lambda_1, \lambda_2, \ldots, \lambda_6\} = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \cdots, [\underline{\lambda_6}, \overline{\lambda_6}]\}$ be a list of real interval numbers satisfying $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant [\underline{\lambda_3}, \overline{\lambda_3}] \geqslant 0 > [\underline{\lambda_6}, \overline{\lambda_6}] \geqslant [\underline{\lambda_5}, \overline{\lambda_5}] \geqslant [\underline{\lambda_4}, \overline{\lambda_4}]$, and $\sum_{i=1}^{6} \lambda_i \geqslant 0$. If there exist interval real numbers $\alpha_i = [\underline{\alpha_i}, \overline{\alpha_i}]$ (i=1, …, 6), $\beta_{24} = [\underline{\beta_{24}}, \overline{\beta_{24}}]$, $\beta_{34} = [\underline{\beta_{34}}, \overline{\beta_{34}}]$ and $\beta_{35} = [\underline{\beta_{35}}, \overline{\beta_{35}}]$ such that

$$\begin{aligned} &\sum_{i=2}^{6} \overline{\alpha_i} \leqslant \underline{\lambda_1} \\ &-\underline{\lambda_i} \leqslant \underline{\alpha_i} \qquad i = 2, \cdots, 6 \\ &-\underline{\alpha_5} \leqslant \underline{\beta_{35}} \leqslant \overline{\beta_{35}} \leqslant \underline{\lambda_2} - \overline{\lambda_3} + \overline{\alpha_2} \\ &-\underline{\alpha_4} \leqslant \underline{\beta_{24}} \leqslant \overline{\beta_{24}} \leqslant \underline{\lambda_1} - \overline{\lambda_2} - \sum_{i=2}^{6} \overline{\alpha_i} \\ &-\underline{\alpha_4} \leqslant \underline{\beta_{24}} + \underline{\beta_{34}} \leqslant \overline{\beta_{24}} + \overline{\beta_{34}} \leqslant \underline{\lambda_1} - \overline{\lambda_2} - \sum_{i=2}^{6} \overline{\alpha_i} \\ &\underline{\alpha_2} + \underline{\alpha_5} \geqslant 0 \\ &\underline{\alpha_3} + \underline{\alpha_6} \geqslant 0, \end{aligned} \tag{11}$$

then σ^I is realizable.

Proof. Let

$$A^I = \begin{pmatrix} \lambda_1 & \alpha_2+\alpha_5+\alpha_6+\alpha_3 & 0 & \alpha_4 & 0 & 0 \\ 0 & \lambda_2 & \alpha_6+\alpha_3 & \beta_{24} & \alpha_5 & 0 \\ 0 & 0 & \lambda_3 & \beta_{34} & \beta_{35} & \alpha_6 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 1&0&0&1&0&0 \\ 1&1&0&0&1&0 \\ 1&1&1&0&0&1 \end{pmatrix}.$$

Then the matrix $C^I = LA^IL^{-1}$ is

$$\begin{pmatrix} \lambda_1-t & \alpha_2+\alpha_3+\alpha_5+\alpha_6 & 0 & \alpha_4 & 0 & 0 \\ \lambda_1-t-\lambda_2-\beta_{24} & \alpha_2+\lambda_2 & \alpha_3+\alpha_6 & \alpha_4+\beta_{24} & \alpha_5 & 0 \\ \lambda_1-t-\lambda_2-\beta_{24}-\beta_{34} & \alpha_2+\lambda_2-\lambda_3-\beta_{35} & \alpha_3+\lambda_3 & \alpha_4+\beta_{24}+\beta_{34} & \alpha_5+\beta_{35} & \alpha_6 \\ \lambda_1-t-\lambda_4 & \alpha_2+\alpha_3+\alpha_5+\alpha_6 & 0 & \alpha_4+\lambda_4 & 0 & 0 \\ \lambda_1-t-\lambda_2-\beta_{24} & \alpha_2+\lambda_2-\lambda_5 & \alpha_3+\alpha_6 & \alpha_4+\beta_{24} & \alpha_5+\lambda_5 & 0 \\ \lambda_1-t-\lambda_2-\beta_{24}-\beta_{34} & \alpha_2+\lambda_2-\lambda_3-\beta_{35} & \alpha_3+\lambda_3-\lambda_6 & \alpha_4+\beta_{24}+\beta_{34} & \alpha_5+\beta_{35} & \alpha_6+\lambda_6 \end{pmatrix} \tag{12}$$

where $t = \sum_{i=2}^{6} \alpha_i$. The matrix C^I is nonnegative if and only if the claimed system of inequalities (11) is consistent. ■

***Example* 4.3** Consider the following interval spectrum

$$\sigma^I = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\} = \{[14, 15], [4, 5], [1, 2], [-4, -3], [-3, -2], [-2, -1]\},$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \geq \lambda_6 \geq \lambda_5 \geq \lambda_4$ and $\sum\limits_{i=1}^{5} \lambda_i \geq 0$. By Theorem 4.2 we construct the nonnegative interval matrix C^I that σ^I is its spectrum. Let $\alpha_2 = [-2, 0]$, $\alpha_3 = [-1, 0]$, $\alpha_4 = [6, 7]$, $\alpha_5 = [3, 4]$ and $\alpha_6 = [2, 3]$, then

$$A^I = \begin{pmatrix} [14,15] & [0,7] & 0 & [4,6] & 0 & 0 \\ 0 & [4,5] & [1,3] & [-4,-4] & [3,4] & 0 \\ 0 & 0 & [1,2] & 0 & [-3,2] & [2,3] \\ 0 & 0 & 0 & [-4,-3] & 0 & 0 \\ 0 & 0 & 0 & 0 & [-3,-2] & 0 \\ 0 & 0 & 0 & 0 & 0 & [-2,-1] \end{pmatrix}.$$

Now, by relation (12), we have

$$C^I = LAL^{-1} = \begin{pmatrix} [1,11] & [0,7] & 0 & [4,6] & 0 & 0 \\ [0,11] & [0,5] & [1,3] & [3,6] & [3,4] & 0 \\ [0,11] & [1,12] & [0,2] & [3,6] & [0,6] & [2,3] \\ [4,15] & [0,7] & 0 & [0,3] & 0 & 0 \\ [0,11] & [2,8] & [1,3] & [3,6] & [0,2] & 0 \\ [0,11] & [1,12] & [2,5] & [3,6] & [0,6] & [0,2] \end{pmatrix}.$$

Theorem 4.4 Let $\sigma^I = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \cdots, [\underline{\lambda_6}, \overline{\lambda_6}]\}$ be a list of real interval numbers satisfying $[\underline{\lambda_1}, \overline{\lambda_1}] \geqslant [\underline{\lambda_2}, \overline{\lambda_2}] \geqslant [\underline{\lambda_3}, \overline{\lambda_3}] \geqslant 0 > [\underline{\lambda_n}, \overline{\lambda_n}] \geqslant [\underline{\lambda_{n-1}}, \overline{\lambda_{n-1}}] \geqslant \cdots \geqslant [\underline{\lambda_4}, \overline{\lambda_4}]$, and $\sum_{i=1}^{n} \lambda_i \geqslant 0$. If there exist interval real numbers $\alpha_i = [\underline{\alpha_i}, \overline{\alpha_i}]$ (i=1,...,6), $\beta_{2i} = [\underline{\beta_{2i}}, \overline{\beta_{2i}}]$, $i = 1, 2, \cdots, r_1 - 1$ and $\beta_{3i} = [\underline{\beta_{3i}}, \overline{\beta_{3i}}]$, $i =$

$1, 2, \cdots, r_2 - 1$ such that

$$
\begin{aligned}
&\sum_{i=2}^{n} \overline{\alpha_i} \leqslant \underline{\lambda_1} \\
&-\underline{\lambda_i} \leqslant \underline{\alpha_i} \qquad i = 2, \cdots, n \\
&-\underline{\alpha_i} \leqslant \underline{\beta_{3i}} \leqslant \overline{\beta_{3i}} \leqslant \underline{\lambda_2} - \overline{\lambda_3} + \overline{\alpha_2}, \ i = r_1, \cdots, r_2 - 1, \\
&-\underline{\alpha_i} \leqslant \underline{\beta_{2i}} \leqslant \overline{\beta_{2i}} \leqslant \underline{\lambda_1} - \overline{\lambda_2} - \sum_{i=2}^{n} \overline{\alpha_i}, \ i = 4, \cdots, r_1 - 1, \\
&-\underline{\alpha_i} \leqslant \underline{\beta_{2i}} + \underline{\beta_{3i}} \leqslant \overline{\beta_{2i}} + \overline{\beta_{3i}} \leqslant \underline{\lambda_1} - \overline{\lambda_2} - \sum_{i=2}^{n} \overline{\alpha_i} \\
&\underline{\alpha_2} + \sum_{i=r_1}^{n} \underline{\alpha_i} \geqslant 0 \\
&\underline{\alpha_3} + \sum_{i=r_2}^{n} \underline{\alpha_i} \geqslant 0,
\end{aligned}
\tag{13}
$$

then σ^I is realizable.

Proof. Let

$$
A^I = \begin{pmatrix}
\lambda_1 & \alpha_2 + \sum_{i=r_1}^{n} \alpha_i & 0 & \alpha_4 & \cdots & \alpha_{r_1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \lambda_2 & \alpha_3 + \sum_{i=r_2}^{n} \alpha_i & \beta_{34} & \cdots & \beta_{2,r_1} & \alpha_{r_1+1} & \cdots & \alpha_{r_2} & 0 & \cdots & 0 \\
0 & 0 & \lambda_3 & \beta_{34} & \cdots & \cdots & \cdots & \cdots & \beta_{3,r_2} & \alpha_{r_2+1} & \cdots & \alpha_n \\
0 & 0 & 0 & 0 & \lambda_4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & \ddots & & & & & & & \vdots \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\vdots & & & & & & & & & & \ddots & \vdots \\
0 & & & & & & & & & & \cdots & \lambda_n
\end{pmatrix}
$$

and

$$
L = \begin{pmatrix}
1 & 0 & \cdots & & & & & & & & \cdots & 0 \\
1 & 1 & 0 & \cdots & & & & & & & \cdots & 0 \\
1 & 1 & 1 & 0 & \cdots & & & & & & \cdots & 0 \\
1 & 0 & 0 & 1 & 0 & \cdots & & & & & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & & & & \cdots & \vdots \\
1 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & & & \cdots & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \ddots & \cdots & & \vdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & 0 \\
\vdots & \vdots & \vdots & & & & & & & & \ddots & 0 \\
1 & 1 & 1 & 0 & \cdots & & & & & \cdots & 0 & 1
\end{pmatrix}.
$$

Similar to [7], we can find $C^I = L^{-1} A^I L$ and according to the conditions of the theorem, the obtained interval matrix will be nonnegative. ■

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