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New lower bound for numerical radius for off-diagonal 2×2 matrices

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Abstract. New norm and numerical radius inequalities for operators on Hilbert space are given. Among other inequalities, we prove that if $A, B \in B(H)$, then

$$\|A\| - \frac{3\|A - B^*\|}{2} \leqslant \omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

Moreover, $\omega(AB) \leqslant \frac{3}{2}\|Im(A)\|\|B\| + D_B\ \omega(A)$. In particular, if A is self-adjointable, then $\omega(AB) \leqslant D_B\|A\|$, where $D_B = \inf\limits_{\lambda \in \mathbb{C}} \|B - \lambda I\|$.

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1. Introduction and preliminaries

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. The numerical radius of $A \in B(H)$ is defined by $\omega(A) = \sup\{\ |\langle Ax, x\rangle| : \|x\| = 1\ \}$. In [11], Yamazaki proved for any $A \in B(H)$ that $\omega(A) = \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}A)\|$. It is well known that $\omega(\cdot)$ is a norm on $B(H)$ which is equivalent to the usual operator norm $\|.\|$. In fact, for all $A \in B(H)$,

$$\frac{\|A\|}{2} \leqslant \omega(A) \leqslant \|A\|. \tag{1}$$

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The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal. Several numerical radius inequalities improving the inequalities in (1) have been recently given in [2, 6, 8, 9].

If A and B are operators in $B(H)$, we write the direct sum $A \oplus B$ for the 2×2 operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $H \oplus H$. Thus, $\omega(A \oplus B) = \max(\omega(A), \omega(B))$. Also,

$$\|A \oplus B\| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| = \max(\|A\|, \|B\|).$$

Some numerical radius inequalities for certain 2×2 operator matrices is obtained in [4]. More precisely,

$$\sqrt[2n]{\max(\omega((AB)^n), \omega((BA)^n)} \leqslant \omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leqslant \frac{\|A\| + \|B\|}{2}$$

for $n = 1, 2, ...$ In [5], Holbrook showed for any $A, B \in B(H)$ that $\omega(AB) \leqslant 4\omega(A)\omega(B)$. In the case $AB = BA$, then $\omega(AB) \leqslant 2\omega(A)\omega(B)$. In [3], it is shown for any $A, B \in B(H)$ that

$$\omega(A^*B \pm BA) \leqslant 2\|A\|\omega(B). \tag{2}$$

Let $D_A = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$ and let R_A denote the radius of the smallest disk in the complex plane containing $\sigma(A)$ (the spectrum of A). Stampfli in [10] proved that if $A \in B(H)$ and A is normal, then $D_A = R_A$.

The question about the best constant k such that the inequality

$$w\,(AB) \leq k\|A\|\omega(B) \tag{3}$$

holds for all operators $A, B \in B(H)$ is still open.

Concerning the inequality (3), it is shown in [1] that if $A, B \in B(H)$, then $\omega(AB) \leqslant (\|A\| + D_A)\omega(B)$ and

$$\omega(AB) \leqslant \|A\|\omega(B) + \frac{1}{2}\omega(B^*A^* - AB^*). \tag{4}$$

Also, if $A > 0$, then $\omega(AB) \leqslant \frac{3}{2}\|A\|\omega(B)$.

In Section 2, we introduce some inequalities between the operator norm and the numerical radius of operators on Hilbert spaces. Particularly, we establish lower bound for the numerical radius of the off-diagonal parts of 2×2 operator matrices.

2. Main results

In order to derive our main results, we need the following lemma. The lemma, which can be found in [7], gives new numerical radius inequalities for products of two Hilbert space operators.

Lemma 2.1 Let $A, B \in \mathbb{B}(\mathbb{H})$. Then

$$w\,(AB) \leq \omega(A)\omega(B) + D_A D_B. \tag{5}$$

The following result may be as well.

Theorem 2.2 If $A, B, C \in \mathbb{B}(\mathbb{H})$, then

$$\|Re(CA)\| \leqslant \frac{3\|A - B^*\|\|C\|}{4} + \frac{1}{2}\omega(CA + BC^*).$$

Proof. Clearly, $\|Re(AB)\| = \omega(Re(AB))$. Then

$$\begin{aligned}
\|Re(AB)\| &= \omega(Re(AB)) \\
&= \omega(\frac{AB + B^*A^*}{2}) \\
&\leqslant \frac{1}{2}\omega(A(B + B^*)) + \frac{1}{2}\omega(AB - BA^*) \\
&\leqslant \frac{\|B + B^*\|\omega(A)}{2} + \frac{1}{2}D_A D_{B+B^*} + \frac{1}{2}\omega(AB - BA^*) \qquad \text{(by (5))} \\
&\leqslant \frac{\|B + B^*\|}{2}(\omega(A) + \|A\|) + \frac{1}{2}\omega(AB - BA^*).
\end{aligned}$$

Hence,

$$\|Re(AB)\| \leqslant \frac{\|B + B^*\|}{2}(\omega(A) + \|A\|) + \frac{1}{2}\omega(AB - BA^*). \tag{6}$$

Now, let $A_1 = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$ and $B_1 = \begin{bmatrix} 0 & -B \\ A & 0 \end{bmatrix}$. By (6), we have

$$\|Re(A_1B_1)\| \leqslant \frac{\|B_1 + {B_1}^*\|}{2}(\omega(A_1) + \|A_1\|) + \frac{1}{2}\omega(A_1B_1 - B_1{A_1}^*)$$

and so

$$\begin{aligned}
\|Re(CA)\| &= \left\| Re\left(\begin{bmatrix} CA & 0 \\ 0 & 0 \end{bmatrix}\right)\right\| \\
&= \|Re(A_1B_1)\| \\
&\leqslant \frac{1}{2}\left\|\begin{bmatrix} 0 & A^* - B \\ A - B^* & 0 \end{bmatrix}\right\|(\omega(A_1) + \|A_1\|) + \frac{1}{2}\omega\left(\begin{bmatrix} CA + BC^* & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\|A^* - B\|(\omega(A_1) + \|C\|) + \frac{1}{2}\omega(CA + BC^*).
\end{aligned}$$

Consequently,

$$\|Re(CA)\| \leqslant \frac{1}{2}\|A^* - B\|(\omega(A_1) + \|C\|) + \frac{1}{2}\omega(CA + BC^*). \tag{7}$$

Since ${A_1}^2 = 0$ and $\omega(A_1) = \frac{\|C\|}{2}$, the result follows from (7). ■

The following result may be as well.

Corollary 2.3 If $A, B \in \mathbb{B}(\mathbb{H})$, then

$$\|A\| - \frac{3\|A - B^*\|}{2} \leqslant \omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

Proof. Replacing C by I in Theorem 2.2 gives

$$\|Re(A)\| \leqslant \frac{3\|A - B^*\|}{4} + \frac{1}{2}\omega(A + B). \tag{8}$$

Now, let

$$A_1 = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \text{ and } B_1 = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}.$$

By (8), we have

$$\begin{aligned} \frac{\|A\|}{2} &= \|Re(A_1)\| \\ &\leqslant \frac{3}{4}\|A_1 - {B_1}^*\| + \frac{1}{2}\omega(A_1 + B_1) \\ &\leqslant \frac{3}{4}\left\|\begin{bmatrix} 0 & A - B^* \\ 0 & 0 \end{bmatrix}\right\| + \frac{1}{2}\omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \end{aligned}$$

and so

$$\|A\| \leqslant \frac{3\|A - B^*\|}{2} + \omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right).$$

This completes the proof. ■

As a natural application of the above Corollary in providing upper bounds for the nonnegative quantity $\|A\| - \omega(A), A \in \mathbb{B}(\mathbb{H})$, we can state the following result:

Corollary 2.4 If $A, B \in \mathbb{B}(\mathbb{H})$, then $\|A\| - \omega(A) \leqslant 3\|Im(A)\|$.

Proof. Replacing B by A in Theorem 2.3, we deduce the desired result. ■

The following result may be as well.

Theorem 2.5 If $A, C \in \mathbb{B}(\mathbb{H})$, then $\omega(CA) \leqslant \frac{3}{2}\|Im(A)\|\|C\| + D_C\ \omega(A)$.

Proof. By Theorem 2.2,

$$\|Re(CA)\| \leqslant \frac{3\|A - B^*\|\|C\|}{4} + \frac{1}{2}\omega(CA + BC^*).$$

Replacing B by A in the last inequality gives

$$\begin{aligned}\|Re(CA)\| &\leqslant \frac{3\|A-A^*\|\|C\|}{4}+\frac{1}{2}\omega(CA+AC^*)\\ &=\frac{3}{2}\|Im(A)\|\|C\|+\frac{1}{2}\omega(CA+AC^*).\end{aligned}$$

Therefore,

$$\|Re(CA)\| \leqslant \frac{3}{2}\|Im(A)\|\|C\|+\frac{1}{2}\omega(CA+AC^*).$$

Let $\alpha_0=\frac{\bar{z}_0}{|z_0|}$, where $z_0\in\mathbb{C}$ is such that $\|C-z_0I\|=D_C$. Replacing C by $\alpha_0 C$ in the inequality (6) gives

$$\begin{aligned}\|Re(\alpha_0 CA)\| &\leqslant \frac{3}{2}\|Im(A)\|\|C\|+\frac{1}{2}\omega(\alpha_0 CA-\bar{\alpha}_0 CA^*)\\ &\leqslant \frac{3}{2}\|Im(A)\|\|C\|+\frac{1}{2}\omega(\alpha_0(C-z_0I)A-\bar{\alpha}_0 A(C-z_0I)^*)\\ &\leqslant \frac{3}{2}\|Im(A)\|\|C\|+\|C-z_0I\|\omega(A). \qquad \text{(by (2))}\end{aligned}$$

Thus,

$$\|Re(\alpha_0 CA)\| \leqslant \frac{3}{2}\|Im(A)\|\|C\|+D_C\ \omega(A). \tag{9}$$

On the other hand, there exists $\theta_0\in\mathbb{R}$ such that $\alpha_0=e^{i\theta_0}$. Let $\theta\in R$ and replacing C by $e^{i\theta}C$ in the inequality (9) gives

$$\|Re(e^{i(\theta+\theta_0)}CA)\| \leqslant \frac{3}{2}\|Im(A)\|\|C\|+D_C\ \omega(A).$$

Taking the supremum over $\theta\in R$ gives

$$\omega(CA) \leqslant \frac{3}{2}\|Im(A)\|\|C\|+D_C\ \omega(A),$$

which is exactly the desired result. ■

The following corollary are immediate consequences of Theorem 2.5.

Corollary 2.6 If $A,C\in\mathbb{B}(\mathbb{H})$ and A is self-adjointable, then $\omega(CA)\leqslant D_C\|A\|$.

The following theorem is a considerable improvement of the inequality (4).

Theorem 2.7 If $A,B\in\mathbb{B}(\mathbb{H})$, then $\omega(AB)\leqslant D_A\omega(B)+\frac{1}{2}\omega(AB-BA^*)$.

Proof. Let $\theta \in \mathbb{R}$. We have

$$\begin{aligned}
\|Re(e^{i\theta}AB)\| &= \omega(Re(e^{i\theta}AB)) \\
&= \omega(\frac{e^{i\theta}AB + e^{-i\theta}B^*A^*}{2}) \\
&= \omega(\frac{A(e^{i\theta}B + e^{-i\theta}B^*)}{2} + \frac{e^{-i\theta}(B^*A^* - AB^*)}{2}) \\
&\leqslant \frac{1}{2}D_A\|e^{i\theta}B + e^{-i\theta}B^*\| + \frac{1}{2}\omega(B^*A^* - AB^*) \qquad \text{(by Corollary(2.6))} \\
&= D_A\|Re(e^{i\theta}B)\| + \frac{1}{2}\omega(B^*A^* - AB^*).
\end{aligned}$$

Consequently,

$$\|Re(e^{i\theta}AB)\| \leqslant D_A\|Re(e^{i\theta}B)\| + \frac{1}{2}\omega(B^*A^* - AB^*).$$

Taking the supremum over $\theta \in R$ gives $\omega(AB) \leqslant D_A\omega(B) + \frac{1}{2}\omega(B^*A^* - AB^*)$, which is exactly the desired result. ■

The following corollary are immediate consequences of Theorem 2.7.

Corollary 2.8 If $A, B \in \mathbb{B}(\mathbb{H})$ and $AB = BA^*$, then $\omega(AB) \leqslant D_A\omega(B)$.

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