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Equivalent characterization of right (left) centralizers or centralizers on Banach algebras

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Abstract. Let $\mathcal{A}$ be a unital Banach algebra, $w \in \mathcal{A}$, and $\gamma : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. We show that γ satisfies $a\gamma(b) = \gamma(w)$ $(\gamma(a)b = \gamma(w))$ whenever $a, b \in \mathcal{A}$ with $ab = w$ and w is a left (right) separating point in $\mathcal{A}$ if and only if γ is a right (left) centralizer. Also, we prove that γ satisfies $a\gamma(b) = \gamma(a)b = \gamma(w)$ whenever $a, b \in \mathcal{A}$ with $ab = w$ and w is a left or right separating point in $\mathcal{A}$ if and only if γ is a centralizer. We also provide some applications of the obtained results for characterization of a continuous linear map $\gamma : \mathcal{A} \to \mathcal{A}$ on a unital Banach $*$-algebra $\mathcal{A}$ satisfying $a\gamma(b)^* = \gamma(w^*)^*$ $(\gamma(a)^*b = \gamma(w^*)^*)$ whenever $a, b \in \mathcal{A}$ with $ab^* = w$ $(a^*b = w)$ and w is a left (right) separating point, or γ satisfying $a\gamma(b)^* = \gamma(c)^*d = \gamma(w^*)^*$ whenever $a, b, c, d \in \mathcal{A}$ with $ab^* = c^*d = w$ and w is a left or right separating point.

Keywords: Left centralizer, right centralizer, centralizer, Banach algebra, Banach $\star$-algebra.

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1. Introduction

Throughout this paper all algebras and vector spaces will be over the complex field $\mathbb{C}$. Let $\mathcal{A}$ be an algebra. One of the interesting issues in mathematics is to characterize the structure of a linear (additive) map $\gamma : \mathcal{A} \to \mathcal{A}$ satisfying

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow a\gamma(b) = \gamma(w) \quad (\mathbb{R}_w),$$

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$$a, b \in \mathcal{A},\ ab = w \Longrightarrow \gamma(a)b = \gamma(w) \quad (\mathbb{L}_w),$$

or

$$a, b \in \mathcal{A},\ ab = w \Longrightarrow a\gamma(b) = \gamma(a)b = \gamma(w) \quad (\mathbb{C}_w),$$

where $w \in \mathcal{A}$ is fixed. Recall that a linear (additive) map $\gamma : \mathcal{A} \to \mathcal{A}$ is said to be a *right* (*left*) *centralizer* if $\gamma(ab) = a\gamma(b)(\gamma(ab) = \gamma(a)b)$ for each $a, b \in \mathcal{A}$. The map γ is called a *centralizer* if it is both a left centralizer and a right centralizer. In case $\mathcal{A}$ has a unity 1, $\gamma : \mathcal{A} \to \mathcal{A}$ is a right (left) centralizer if and only if γ is of the form $\gamma(a) = a\gamma(1)(\gamma(a) = \gamma(1)a)$ for all $a \in \mathcal{A}$. Also, γ is a centralizer if and only if $\gamma(a) = a\gamma(1) = \gamma(1)a$ for each $a \in \mathcal{A}$. The concept appears naturally in C^*-algebras. In ring theory it is more common to work with module homomorphisms. Clearly, each right (left) centralizer or centralizer satisfies $\mathbb{R}_w$ ($\mathbb{L}_w$) or $\mathbb{C}_w$ but in general, the converse is not true. In fact, the characterization of a linear (additive) map $\gamma : \mathcal{A} \to \mathcal{A}$ satisfying $\mathbb{R}_w$ ($\mathbb{L}_w$) or $\mathbb{C}_w$, one of the main questions is whether the γ is expressed in terms of a right (left) centralizer or centralizer? In some of the results, the linear (additive) map γ satisfying $\mathbb{R}_w$ ($\mathbb{L}_w$) or $\mathbb{C}_w$ for specific points is described. In [2], Brešar proves that if $\mathcal{R}$ is a prime ring with a nontrival idempotent, then every additive mapping satisfying $\mathbb{C}_0$ is a centralizer. In [12], X. Qi and J. Hou characterize linear mappings satisfying $\mathbb{C}_0$ in triangular algebras. In [13], the authors study additive mappings satisfying $\mathbb{R}_w$ ($\mathbb{L}_w$) or $\mathbb{C}_w$ for various types of elements w in $B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. For more information on mappings satisfying $\mathbb{R}_w$ ($\mathbb{L}_w$) or $\mathbb{C}_w$, we refer to [1, 5–10] and references therein. Also, using fixed point techniques, (left, right) centralizers and their stability have been studied, see [3, 4, 11]. Motivated by the above studies, in this paper we consider the conditions $\mathbb{R}_w$ ($\mathbb{L}_w$) or $\mathbb{C}_w$ for a continuous linear map $\gamma : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is a Banach algebra with unity 1 and $w \neq 0$ is a left (right) separating point of $\mathcal{A}$. We say that $w \in \mathcal{A}$ is a *left* (*right*) *separating point* of $\mathcal{A}$ if the condition $wx = 0$ (or $xw = 0$) for $x \in \mathcal{A}$ implies $x = 0$. In fact, under these conditions we prove that γ is a right (left) centralizer or centralizer. We also provide some applications of the obtained results for characterization of a continuous linear map $\gamma : \mathcal{A} \to \mathcal{A}$ satisfying

$$a, b \in \mathcal{A},\ ab^* = w \Longrightarrow a\gamma(b)^* = \gamma(w^*)^* \quad (\mathbb{R}^*_w),$$

$$a, b \in \mathcal{A},\ a^*b = w \Longrightarrow \gamma(a)^*b = \gamma(w^*)^* \quad (\mathbb{L}^*_w),$$

or

$$a, b, c, d \in \mathcal{A},\ ab^* = c^*d = w \Longrightarrow a\gamma(b)^* = \gamma(c)^*d = \gamma(w^*)^* \quad (\mathbb{C}^*_w),$$

where $\mathcal{A}$ is a Banach $*$-algebras and $w \in \mathcal{A}$ is a left (right) separating point.

2. Conditions $\mathbb{R}_w$ and $\mathbb{R}^*_w$

This section is devoted to a continuous linear map with property $\mathbb{R}_w$ or $\mathbb{R}^*_w$ on a unital Banach algebra, in which $w \neq 0$ is a left separating point.

Theorem 2.1 Let $\mathcal{A}$ be a Banach algebra with unity 1. Suppose that w in $\mathcal{A}$ is a left separating point, and $\gamma : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{R}_w$;
(ii) γ is a right centralizer.

Proof. (i) $\Rightarrow$ (ii): Since $w1 = w$, it follows that $\gamma(w) = w\gamma(1)$. Let $a \in \mathcal{A}$ be an arbitrary element and $\lambda \in \mathbb{C}$ (the complex field). We have $w\exp(\lambda a)\exp(-\lambda a) = w$, where exp is the exponential function in $\mathcal{A}$. Hence,

$$
\begin{aligned}
\gamma(w) &= \gamma(w\exp(\lambda a)(\exp(-\lambda a)) \\
&= w\exp(\lambda a)\gamma(\exp(-\lambda a)) \\
&= w\exp(\lambda a)\gamma(\sum_{m=0}^{\infty}\frac{(-1)^m\lambda^m}{m!}a^m) \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m\lambda^m}{m!}w\exp(\lambda a)\gamma(a^m) \\
&= \sum_{m=0}^{\infty}\frac{(-1)^m\lambda^m}{m!}\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}wa^n\right)\gamma(a^m) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m\lambda^{m+n}}{m!n!}wa^n\gamma(a^m) \\
&= w\gamma(1) + \sum_{k=1}^{\infty}\lambda^k\left(\sum_{m+n=k}\frac{(-1)^m}{m!n!}wa^n\gamma(a^m)\right),
\end{aligned}
$$

since γ is a continuous linear map. Thus,

$$\sum_{k=1}^{\infty}\lambda^k\left(\sum_{m+n=k}\frac{(-1)^m}{m!n!}wa^n\gamma(a^m)\right) = 0$$

for any $\lambda \in \mathbb{C}$. Consequently,

$$\sum_{m+n=k}\frac{(-1)^m}{m!n!}wa^n\gamma(a^m) = 0$$

for all $a \in \mathcal{A}$ and $k \in \mathbb{N}$. Let $k = 1$. we find that $wa\gamma(1) - w\gamma(a) = 0$ for all $a \in \mathcal{A}$. By the fact that w is a left separating point, we get $\gamma(a) = a\gamma(1)$ and hence, γ is a right centralizer.
(ii) $\Rightarrow$ (i): is clear. ■

Since the unity 1 is a left separating point, we have the following corollary, which was proved in [7, Theorem 2.4].

Corollary 2.2 Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\gamma : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{R}_1$;
(ii) γ is a right centralizer.

In the following theorem, we examine condition $\mathbb{R}_w^*$.

Theorem 2.3 Let $\mathcal{A}$ be a Banach $*$-algebra with unity 1. Suppose that w in $\mathcal{A}$ is a left separating point, and $\gamma : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{R}_w^*$;
(ii) γ is a left centralizer.

Proof. (i) $\Rightarrow$ (ii): Consider $a, b \in \mathcal{A}$ with $ab = w$. So $a(b^*)^* = w$, and by assumption, $a\gamma(b^*)^* = \gamma(w^*)^*$. Define the continuous linear map $\xi : \mathcal{A} \to \mathcal{A}$ by $\xi(a) = \gamma(a^*)^*$. So, $a\xi(b) = \xi(w)$. Consequently, ξ satisfies $\mathbb{R}_w$, and by Theorem 2.1, $\xi(a) = a\xi(1)$ for all $a \in \mathcal{A}$. By definition of ξ, it results that $\gamma(a) = \gamma(1)a$ for all $a \in \mathcal{A}$.
(ii) $\Rightarrow$ (i): Suppose that $\gamma(a) = \gamma(1)a$ for all $a \in \mathcal{A}$, and $a, b \in \mathcal{A}$ with $ab^* = w$. Hence,

$$a\gamma(b)^* = a(\gamma(1)b)^* = ab^*\gamma(1)^* = w\gamma(1)^* = (\gamma(1)w^*)^* = \gamma(w^*)^*.$$

■

The result below is straightforward.

Corollary 2.4 Let $\mathcal{A}$ be a Banach algebra with unity 1 and $\gamma : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{R}_1^*$;
(ii) γ is a left centralizer.

3. Conditions $\mathbb{L}_w$ and $\mathbb{L}_w^*$

In this section, we study conditions $\mathbb{L}_w$ and $\mathbb{L}_w^*$ for a continuous linear map on a unital Banach algebra, in which $w \neq 0$ is a right separating point.

Theorem 3.1 Suppose that $\mathcal{A}$ is a Banach algebra with unity 1. Let w in $\mathcal{A}$ be a right separating point, and $\gamma : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{L}_w$;
(ii) γ is a left centralizer.

Proof. (i) $\Rightarrow$ (ii): Since $1w = w$, it follows that $\gamma(w) = \gamma(1)w$. Let $a \in \mathcal{A}$ be an arbitrary element and $\lambda \in \mathbb{C}$. We have $\exp(-\lambda a)\exp(\lambda a)w = w$, where exp is the exponential function in $\mathcal{A}$. So,

$$\gamma(w) = \gamma(\exp(-\lambda a)(\exp(\lambda a)w) = \gamma(\exp(-\lambda a))\exp(\lambda a)w.$$

Now, according to these points and using a method similar to the proof of Theorem 2.1 on the above equation, the proof is obtained.
(ii) $\Rightarrow$ (i): is clear. ■

The following conclusion is obvious.

Corollary 3.2 Assume that $\mathcal{A}$ is a Banach algebra with unity 1, and $\gamma : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{L}_1$;
(ii) γ is a left centralizer.

In the following theorem, we check condition $\mathbb{L}_w^*$.

Theorem 3.3 Let $\mathcal{A}$ be a Banach $*$-algebra with unity 1, and w in $\mathcal{A}$ is a right separating point. Suppose that $\gamma : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{L}_w^*$;
(ii) γ is a right centralizer.

Proof. (i) $\Rightarrow$ (ii): Let $a, b \in \mathcal{A}$ with $ab = w$. So $(a^*)^*b = w$, and by assumption, $\gamma(a^*)^*b = \gamma(w^*)^*$. Define the continuous linear map $\xi : \mathcal{A} \to \mathcal{A}$ by $\xi(a) = \gamma(a^*)^*$. Hence, $\xi(a)b = \xi(w)$. Therefore, ξ satisfies $\mathbb{L}_w$, and by Theorem 3.1, $\xi(a) = \xi(1)a$ for all $a \in \mathcal{A}$. By definition of ξ it results that $\gamma(a) = a\gamma(1)$ for all $a \in \mathcal{A}$.
(ii) $\Rightarrow$ (i): Suppose that $\gamma(a) = a\gamma(1)$ for all $a \in \mathcal{A}$, and $a, b \in \mathcal{A}$ with $a^*b = w$. Hence,

$$\gamma(a)^*b = (a\gamma(1))^*b = \gamma(1)^*a^*b = \gamma(1)^*w = (w^*\gamma(1))^* = \gamma(w^*)^*.$$

■

The next result is obvious.

Corollary 3.4 Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\gamma : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{L}_1^*$;
(ii) γ is a right centralizer.

4. Conditions $\mathbb{C}_w$ and $\mathbb{C}_w^*$

In this section, we examine conditions $\mathbb{C}_w$ and $\mathbb{C}_w^*$ for a continuous linear map on a unital Banach algebra, in which $w \neq 0$ is a left or right separating point.

Theorem 4.1 Let $\mathcal{A}$ be a Banach algebra with unity 1. Suppose that w in $\mathcal{A}$ is a left or right separating point, and $\gamma : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{C}_w$;
(ii) γ is a centralizer.

Proof. (i) $\Rightarrow$ (ii): Let w be a left separating point and $a, b \in \mathcal{A}$ with $ab = w$. By assumption $a\gamma(b) = \gamma(w)$. It follows from Theorem 2.1 that $\gamma(a) = a\gamma(1)$ for all $a \in \mathcal{A}$. Suppose that $a \in \mathcal{A}$ is an invertible element. So $waa^{-1} = w$, and by assumption $\gamma(wa)a^{-1} = \gamma(w)$. Thus, $wa\gamma(1) = w\gamma(1)a$; that is, $a\gamma(1) = \gamma(1)a$ for all invertible element $a \in \mathcal{A}$, because w is a left separating point. Let $a \in \mathcal{A}$ be arbitrary and $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant \|a\|$. So $\lambda 1 - a$ is invertible in $\mathcal{A}$ and $(\lambda 1 - a)\gamma(1) = \gamma(1)(\lambda 1 - a)$. Hence, $a\gamma(1) = \gamma(1)a$ for all $a \in \mathcal{A}$. Now, we have $\gamma(a) = a\gamma(1) = \gamma(1)a$ for all $a \in \mathcal{A}$.

Assume that w is a right separating point and $a, b \in \mathcal{A}$ with $ab = w$. By assumption $\gamma(a)b = \gamma(w)$. It follows from Theorem 3.1 that $\gamma(a) = \gamma(1)a$ for all $a \in \mathcal{A}$. Suppose that $a \in \mathcal{A}$ is an invertible element. So $a^{-1}aw = w$, and by assumption $a^{-1}\gamma(aw) = \gamma(w)$. Thus $\gamma(1)aw = a\gamma(1)w$. That is $a\gamma(1) = \gamma(1)a$ for all invertible element $a \in \mathcal{A}$, because w is a right separating point. Now, with a proof similar to the above, we get the result.
(ii) $\Rightarrow$ (i): is clear. ■

The following result is straightforward.

Corollary 4.2 Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\gamma : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{C}_1$;
(ii) γ is a centralizer.

In the following, we check condition $\mathbb{C}_w^*$.

Theorem 4.3 Let $\mathcal{A}$ be a Banach $*$-algebra with unity 1. Suppose that w in $\mathcal{A}$ is a left or right separating point, and $\gamma : \mathcal{A} \to \mathcal{A}$ is a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{C}_w^*$;
(ii) γ is a centralizer.

Proof. (i) $\Rightarrow$ (ii): Consider $a, b \in \mathcal{A}$ with $ab = w$. So $a(b^*)^* = (a^*)^*b = w$, and by assumption, $a\gamma(b^*)^* = \gamma(a^*)^*b = \gamma(w^*)^*$. Define the continuous linear map $\xi : \mathcal{A} \to \mathcal{A}$ by $\xi(a) = \gamma(a^*)^*$. So, $a\xi(b) = \xi(a)b = \xi(w)$. Consequently, ξ satisfies $\mathbb{C}_w$, and by Theorem 4.1, $\xi(a) = a\xi(1) = \xi(1)a$ for all $a \in \mathcal{A}$. By definition of ξ it results that $\gamma(a) = \gamma(1)a = a\gamma(1)$ for all $a \in \mathcal{A}$.
(ii) $\Rightarrow$ (i): Similar to the proof of (ii) $\Rightarrow$ (i) Theorems 2.3 and 3.3, it is obtained. ■

The next result is obvious.

Corollary 4.4 Let $\mathcal{A}$ be a Banach algebra with unity 1, and $\gamma : \mathcal{A} \to \mathcal{A}$ be a continuous linear map. The following are equivalent:

(i) γ satisfies $\mathbb{C}_1^*$;
(ii) γ is a centralizer.

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