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Hybrid extragradient-type algorithm for zeros and fixed point problems in Banach spaces

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Abstract. In this paper, we introduce a new hybrid extragradient-type algorithm for approximating an element in the set of common solutions of equilibrium problems and common fixed points of family of Bregman demigeneralized mappings which is also a common zero of the sum of maximal monotone and Bregman inverse strongly monotone operators in the setting of reflexie Banach space. Strong convergence of the proposed algorithm to a solutions of the said problems is established which improves and generalizes many recently announced results in the literature.

Keywords: Equilibrium problem, maximal monotone operator, Bregman inverse strongly monotone operator, Bregman demigeneralized mapping.

2010 AMS Subject Classification: 47H09, 47J25.

1. Introduction

Let E be a reflexive real Banach space and E^* be its dual space. An operator $A : E \to 2^{E^*}$ is called $\alpha-$inverse strongly monotone if there exists a positive real number α such that for any $x, y \in E$, $u \in Ax,\ v \in Ay$ we have

$$\langle u - v,\ x - y\rangle \geqslant \alpha\|u - v\|^2. \tag{1}$$

For $\alpha = 0$ in (1) then the operator A is known to be monotone. Let $G(A) := \{(x, u) \in E \times E^* : u \in E^*\}$ be the graph of a monotone operator A, then A is maximal monotone

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if we can not find any other monotone operator say $\hat{A}$ such that $G(A) \subset G(\hat{A})$. Monotone operator theory which was originally studied independently by Ka$\check{c}$urovsk$\tilde{ii}$ [14], Minty [18] and Zarantonello [32] plays a vital role in such areas as semigroup theory, optimization and variational inequality problems among others.

The problem of finding the zeros of the sum of two monotone operators say A and B is to find $x \in E$ such that

$$0 \in (A + B)x. \tag{2}$$

We denote by $(A+B)^{-1}(0)$ the solution set of (2). This inclusion problem, which includes other important problems such as minimization problems, equilibrium problems, variational inequality problems, fixed point problems as special cases, has recently received the attention of many authors due to its several applications. Indeed, many nonlinear problems arising in such areas as signal processing, machine learning, and image recovery can be mathematically modeled as problem (2) (see for example [13]) and the references therein. Notable efforts have been recorded, by several authors, to approximation methods of solution for a sum of two monotone mappings, see [21].

One of the well known method for solving problem (2) is the forward-backward splitting method due to Passty [21] in the setting of Hilbert space which is presented as for $x_1 \in E$,

$$x_{n+1} = (I + \gamma B)^{-1}(x_n - \gamma A x_n) \qquad (n \geqslant 1), \tag{3}$$

where $\gamma > 0$. Other method includes Douglas-Rachford splitting algorithm [16] presented as $x_1 \in E$ and

$$x_{n+1} = 2J_{\gamma A}(2J_{\gamma B} - I)x_n + (I - 2J_{\gamma B})x_n \qquad (n \geqslant 1), \tag{4}$$

where A and B are two maximal monotone operators.

We remark here that algorithms (3) and (4) mentioned above do not guarantee strong convergence to the solution of problem (2).

Let $h : C \times C \to \mathbb{R}$ be a bifunction with C a nonempty closed convex subset of a real Banach space E. Then the equilibrium problem (EP) for a bifunction h is to find a point

$$z \in C \text{ for which } h(z, y) \geqslant 0, \ \forall y \in C \text{ is satisfied.} \tag{5}$$

Problem (5) was originally studied by Bluem and Otli [4] in the setting of Hilbert space. It includes, as a special cases, many other important problems such as variational inequality problem, minimization problem, fixed point problem to mention but a few. Various techniques have been used to study the problems, one of such techniques is the so-called extragradient method which was introduced in [23] by Quoc et al. in the frame work of Hilbert spaces. They studied the following iterative scheme:

$$\begin{cases} z_n \in \operatorname{Argmin}_{z \in C}\{h(x_n, z) + \frac{1}{2\lambda_n}\|z - x_n\|^2\}, \\ x_{n+1} \in \operatorname{Argmin}_{z \in C}\{h(z_n, z) + \frac{1}{2\lambda_n}\|z - x_n\|^2\}. \end{cases} \tag{6}$$

Under some certain assumptions, the sequence $\{x_n\}$ generated by (6) was shown to converge weakly to a solution of problem (5).

Let $dom(f)$ denote the domain of a proper, convex and lower semicontinuous function $f : E \to (-\infty, +\infty]$. Then $dom(f) := \{x \in E : f(x) < +\infty\}$. Now, for any $u \in int(domf)$

and $y \in E$, we denote by $f^{'}(u, y)$ the right-hand derivative of f at u in the direction of y, which is defined as

$$f^{'}(u, y) = \lim_{t \to 0} \frac{f(u + ty) - f(u)}{t}. \tag{7}$$

The function f is known to be Gâteaux differentiable at u if, for each y, the limit in (7) exists. In this regard, the gradient of f at u is a function $\nabla f(u) : E \to (-\infty, +\infty]$ given by $\langle \nabla f(u), y \rangle = f^{'}(u, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable on $int(domf)$ if it is Gâteaux differentiable at every point $u \in int(domf)$. In addition, f is said to be *Fréchet* differentiable at u provided the limit in (7) is attained uniformly for any $y \in E$ with $\|y\| = 1$ and it is uniformly *Fréchet* differentiable on a subset Ω of E if the limit in (7) is attained uniformly for $u \in E$ and $\|y\| = 1$. Let $u \in int(domf)$, the subdifferential of f at u, $\partial f(u)$, is a convex set defined as

$$\partial f(u) = \{u^* \in E^* : f(u) + \langle u^*, y - u \rangle \leqslant f(y), \ \forall \, y \in E\},$$

and the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(u^*) = \sup\{\langle u^*, u \rangle - f(u) : u \in E\}, \ \forall \, u^* \in E^* \tag{8}$$

Observe that f^* defined by (8) above is proper, convex and lower semicontinuous as f is. In addition, $(u, u^*) \in \partial f$ if and only if $f(u) + f^*(u^*) = \langle u^*, u \rangle$, see [15].

Definition 1.1 [3] The function $f : E \to (-\infty, +\infty]$ is known to be:

(1) Essentially smooth if ∂f is locally bounded and single-valued on its domain;
(2) Essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $domf$;
(3) Legendre when it is both essentially smooth and essentially strictly convex.

For a Legendre function f, we have the following properties:

(i) f is Legendre if and only if f^* is Legendre (see [5, Corollary 5.5]);
(ii) $(\partial f)^{-1} = \partial f^*$ (see [5, p.83]);
(iii) ∇f is a bijection and it satisfies

$$\nabla f = (\nabla f^*)^{-1}, \ ran \nabla f = dom \nabla f^* = int(domf^*) \text{ and}$$
$$\nabla f^* = dom \nabla f = int(domf).$$

Let $f : E \to (-\infty, +\infty]$ be convex and Gâteaux differentiable function. The function $D_f : domf \times int(domf) \to [0, +\infty)$ defined as

$$D_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle, \ \forall x \in domf, \ y \in int(domf) \tag{9}$$

is called the Bregman distance with respect to f [10].

Observe that D_f here is not a distance function in the usual sense. In general, D_f neither satisfies symmetric nor triangular inequality. However, for all $x \in domf$ and $y, z \in int(domf)$, D_f satisfies the so-called three point identity

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle x - y, \nabla f(z) - \nabla f(y) \rangle.$$

Let $T: C \to C$ be a map with C a nonempty subset of a Banach space E. A point $\hat{x} \in C$ is called a fixed point of T if $T\hat{x} = \hat{x}$. The set of fixed point of T is denoted by $Fix(T)$. If C contains a sequence $\{x_n\}$ which converges weakly to $\hat{x}$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $\hat{x}$ is called an asymptotic fixed point of the map T [24]. The set of asymptotic fixed point of T is denoted by $\hat{F}(T)$.

Definition 1.2 [2] Let C be a nonempty closed convex subset of E. A mapping $T : C \to int(domf)$ is called

(i) Bregman firmly nonexpansive if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leqslant \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle,\ \forall x, y \in C.$$

(ii) Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$ if $D_f(p, Tx) \leqslant D_f(p, x)$ for all $p \in \hat{F}(T)$.

(iii) Bregman quasi-nonexpansive if $D_f(p, Tx) \leqslant D_f(p, x)$ for all $x \in C$ and for all $p \in Fix(T)$.

Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $\lambda > 0$. An operator $Res^f_{\lambda B} : E \to 2^E$ defined by $Res^f_{\lambda B} := (\nabla f + \lambda B)^{-1} \circ \nabla f$ is called the resolvent operator of B. It is known that $Res^f_{\lambda B}$ is a Bregman firmly nonexpansive operator, it is also single-valued and $Fix(Res^f_{\lambda B}) = B^{-1}(0)$ [27]. Also, if $f : E \to \mathbb{R}$ is a Legendre function which is bounded and uniformly Fréchet differentiable on bounded subsets of E, then $Res^f_{\lambda B}$ is Bregman strongly nonexpansive and $\hat{F}(Res^f_{\lambda B}) = F(Res^f_{\lambda B})$ [26].

A multivalued operator $A : E \to 2^{E^*}$ is called Bregman inverse strongly monotone [20] if for any $x, y \in int(domf)$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geqslant 0,\ \forall u \in Ax,\ v \in Ay.$$

Define $A^f : E \to 2^E$ by $A^f := \nabla f^* \circ (\nabla f - A)$. Then A^f here is called the antiresolvent operator of A. It was shown in [8] that A is Bregman inverse strongly monotone if and only if A^f is single-valued Bregman firmly nonexpansive and $F(A^f) = A^{-1}(0)$.

In [20], the problem of finding zero of sum of maximal monotone and Bregman inverse strongly monotone operators involving fixed point of Bregman nonspreading mapping have been studied. Tuyen, Promkam and Sunthrayuth [30] also studied the following iterative algorithm for approximating common zero of the sum of maximal monotone and Bregman inverse strongly monotone operators in the setting of reflexive Banach space:

$$\begin{cases} x_1, u \in C, \\ y_n = \nabla f^*(\beta_0 \nabla f(x_n) + \sum_{i=1}^N \beta_i \nabla f(Res^f_{\lambda B_i} \circ A^f_i) x_n), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(y_n)),\ n \geqslant 1. \end{cases} \tag{10}$$

They proved strong convergence theorem of the sequence $\{x_n\}$ generated by Algorithm (10).

In [1], on the other hand, a class of map called Bregman demigenerelized mapping was studied.

Definition 1.3 [1] Let E be a reflexive Banach space, C be a nonempty closed convex subset of E and $\eta \in (-\infty, 1)$. Then a map $T : C \to E$ with $F(T) \neq \emptyset$ is called

$(\eta, 0)-$Bregman demigeneralized map if for any $x \in C$ and $q \in F(T)$

$$\langle x - q, \nabla f(x) - \nabla f(Tx)\rangle \geqslant (1-\eta)D_f(x, Tx). \tag{11}$$

Ali et al. [1], using Bregman distance, proposed and studied an iterative scheme for finding a common element in the set of common fixed points for finite families of Bregman demigenerelized mappings and the set of solutions of generalized mixed equilibrium problems. They proved strong convergence theorem of the sequence generated by the following algorithm:

$$\begin{cases} u_0, x_1 \in X \quad \text{chosen arbitrarily}, \\ y_n = \nabla f^*(\lambda_n \nabla f(x_n) + (1-\lambda_n)\nabla f(T_i x_n)), \\ z_n = Res^f_{\varphi_m,\phi_m,\Phi_m} \circ \cdots \circ Res^f_{\varphi_2,\phi_2,\Phi_2} \circ Res^f_{\varphi_1,\phi_1,\Phi_1}(y_n), \\ w_n = \nabla f^*(\alpha_n \nabla f(x_n) + \beta_n \nabla f(z_n) + \gamma_n \nabla f(y_n)), \\ x_{n+1} = P^f_C(\nabla f^*(\sigma_n \nabla f(u_0) + (1-\sigma_n)\nabla f(w_n))), n \geqslant 1. \end{cases}$$

In this paper, motivated by the above mentioned researches, we propose and study a new hybrid extragradient-type iterative algorithm for finding a common solution in the set of common fixed point of finite families of Bregman demigeneralized mappings and a set of solution of equilibrium problems which is a common zero of the sum of maximal monotone and Bregman inverse strongly monotone operators in the setting of reflexive Banach spaces. Our results complement and extends some results announced recently by some authors in the literature.

2. Preliminaries

We shall, throughout this paper, use " $\rightharpoonup$ " and " $\to$ " for weak and strong convergence respectively. The following concepts and Lemmas are also very essential in the proof of our main results.

Lemma 2.1 [29] Let C be a nonempty convex subset of a reflexive Banach space E and $f : C \to \mathbb{R}$ be a convex and subdifferentiable function. Then f attains its minimum at $x \in C$ if and only if $0 \in \partial f(x) + N_C(x)$, where $N_C(x)$ is a normal cone of C at x; that is,

$$N_C(x) := \{x^* \in E^* : \langle x - z, x^*\rangle \geqslant 0, \ \forall\, z \in C\}.$$

Lemma 2.2 [11] Let E be a reflexive Banach space. Suppose $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are two convex functions such that $dom f \cap dom g \neq \emptyset$ and f is continuous. Then, for all $x \in E$, $\partial(f+g) = \partial f(x) + \partial g(x)$.

Lemma 2.3 [1] Let E be a reflexive Banach space and C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive and Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E. Suppose η is a real number satisfying $\eta \in (-\infty, 1)$ and T is an $(\eta, 0)-$Bregman demigeneralized mapping of C onto E. Then $F(T)$ is closed and convex.

Lemma 2.4 [1] Let C be a nonempty closed convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ be a Fréchet differentiable convex function. For $\eta \in (-\infty, 0]$, let

$T : C \to E$ be $(\eta, 0)-$Bregman demigeneralized map with $F(T) \neq \emptyset$. Let α be real number in $[0, 1)$ and let $S = \nabla f^*(\alpha \nabla f + (1-\alpha)\nabla f(T))$. Then $S : C \to E$ is quasi-Bregman nonexpansive map.

Lemma 2.5 [30] Let $A : E \to E^*$ be bregman inverse strongly monotone map and $B : E \to 2^{E^*}$ be maximal monotone operator. Let $T_\lambda x := Res^f_{\lambda B} \circ A^f(x)$ for $x \in E$ and $\lambda > 0$. Then $F(T_\lambda) = (A+B)^{-1}(0)$.

Lemma 2.6 [30] Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping and $B : E \to 2^{E^*}$ be a maximal monotone operator. Then the following hold:

$$D_f(z, Res^f_{\lambda B} \circ A^f(x)) + D_f(Res^f_{\lambda B} \circ A^f(x), x) \leqslant D_f(z, x)$$

for all $z \in (A+B)^{-1}(0)$, $x \in E$ and $\lambda > 0$.

Lemma 2.7 [26] Let $f : E \to \mathbb{R}$ be a Legendre function and C be a nonempty closed convex subset of E. If $T : C \to E$ is Bregman quasi-nonexpansive operator, then $F(T)$ is closed and convex.

Lemma 2.8 [31] Suppose f is convex and bounded on bounded subset of E. Then f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^*.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of f at $x \in domf$ is the function $v_f(x, .) : [0, +\infty] \to [0, +\infty]$ such that $v_f(x, t) := \inf\{D_f(x, y) : y \in int(domf), \|y - x\| = t\}$.

The function f is called totally convex at x if $v_f(x, t) > 0$ whenever $t > 0$. It is convex if it is totally convex at any point $x \in int(domf)$. This notion was first studied by Butnariu and Iusem [7]. Let B be a nonempty bounded subset of E. The modulus of total convexity of f on the set B is the function $v_f : int(domf) \times [0, +\infty) \to [0, +\infty)$ defined by $v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap domf\}$.

The function f is called totally convex on bounded subset if $v_f(B, t) > 0$ for any nonempty and bounded subset B of E and for any $t > 0$.

Lemma 2.9 [25] If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

Lemma 2.10 [28] Let $f : E \to \mathbb{R}$ be Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

A function $f : E \to \mathbb{R}$ is called sequentially consistent [9] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E so that one is bounded and $\lim_{n\to\infty} D_f(y_n, x_n) = 0$ implies $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.11 [7] If $domf$ contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.

Recall that the Bregman projection [6] with respect to f, $P^f_C : int(domf) \to C$, of $x \in int(domf)$ onto nonempty closed convex set $C \subset domf$ is defined as a unique vector $P^f_C(x) \in C$ satisfying $D_f(P^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}$.

The following properties concerning the Bregman projection were studied in [9].

Lemma 2.12 [9] Let C be a nonempty closed convex subset of a reflexive Banach space E, $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and $x \in E$. Then

(i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leqslant 0$, $\forall y \in C$;
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leqslant D_f(y, x)$, $\forall x \in E,\ y \in C$.

Lemma 2.13 [9] Let E be a reflexive Banach space, $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and $V_f : E \times E^* \to [0, +\infty]$ be defined by $V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*)$ for all $x \in E$ and $x^* \in E^*$. Then the following hold true

$$D_f(x, \nabla f^*(x^*)) = V_f(x, x^*),\ \forall x \in E,\ x^* \in E^* \tag{12}$$

and

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \leqslant V_f(x, x^* + y^*), \forall x \in E, \forall x^*, y^* \in E^*. \tag{13}$$

Lemma 2.14 [22] Let E be a reflexive Banach space, $f : E \to (-\infty, +\infty]$ be a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function. Thus, for all $z \in E$, we have

$$D_f\big(z, \nabla f^*\big(\sum_{i=1}^{N} t_i \nabla f(x_i)\big)\big) \leqslant \sum_{i=1}^{N} t_i D_f(z, x_i),$$

where $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^{N} t_i = 1$.

Let E be Banach space, and B and S be a closed unit ball and a unit sphere of E, respectively. Let $rB = \{z \in E : \|z\| \leqslant r\}$ for all $r > 0$. Then the function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subset (see [31]) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by

$$\rho_r(t) := \inf_{x,y\in rB, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},\ \forall t \geqslant 0.$$

The function ρ_r here is called the gauge function of uniform convexity of f which is known to be nondecreasing. However, if f is uniformly convex then the following result is well known.

Lemma 2.15 [19] Let E be a Banach space, $r > 0$ be a constant and $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then

$$f\big(\sum_{k=0}^{n} \alpha_k x_k\big) \leqslant \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i\alpha_j \rho_r(\|x_i - x_j\|),\ \forall i, j \in \{0, 1, 2, \cdots, n\},$$

where $x_k \in rB$, $\alpha_k \in (0, 1)$ and $k = 0, 1, 2, \cdots, n$ with $\sum_{k=0}^{n} = 1$ and ρ_r a gauge function of uniform convexity of f.

The function f is also said to be uniformly smooth on bounded subsets [31] if

$\lim\limits_{t\to 0^-}\frac{\rho_r(t)}{t}=0$ for all $r>0$, where $\rho_r:[0,\infty)\to[0,\infty]$ here is defined by

$$\rho_r(t)=\sup_{x\in rB,y\in S,\alpha\in(0,1)}\frac{\alpha f(x+(1-\alpha)ty)+(1-\alpha)f(x-\alpha ty)-f(x)}{\alpha(1-\alpha)},\forall t\geqslant 0.$$

A function f is called strongly coercive if $\lim\limits_{\|x\|\to\infty}\frac{f(x)}{\|x\|}=+\infty$.

Definition 2.16 [12] Let C be a nonempty closed convex subset of a Banach space E. A bifunction $h:C\times C\to\mathbb{R}$ is called

- Monotone if $h(x,y)+h(y,x)\leqslant 0$ for all $x,y\in C$,
- Pseudomonotone if $h(x,y)\geqslant 0$ implies $h(y,x)\leqslant 0$.

Observe that every monotone bifunction is pseudomonotone but not the converse. We require, in this paper, the bifunction h satisfies the following properties:

$C1$. h is Pseudomonotone,
$C2$. h is Bregman-Lipschitz type continuous, i.e. $h(x,y)+h(y,z)\geqslant h(x,z)-c_1D_f(y,x)-c_2D_f(z,y),\forall z\in C,x,y\in intdom(f)$ and for some $c_1,c_2>0$, where $f:E\to(-\infty,+\infty]$ is a Legendre function,
$C3$. h is weakly continuous on $C\times C$, i.e. if $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and $y\in C$ respectively, then $h(x_n,y_n)\to h(x,y)$,
$C4$. $h(x,.):C\to\mathbb{R}$ is convex, lower semicontinuous and subdifferentiable,
$C5$. $\limsup\limits_{t\to 0^-}h(tx+(1-t)y,w)\leqslant h(y,w)$ for each $x,y,w\in C$.

Lemma 2.17 [12] Let h be a bifunction satisfying $(C1)$, $(C3)-(C5)$. Then $EP(h,C)$ is closed and convex.

Lemma 2.18 [17] Let $\{r_n\}$ be a sequence of real numbers such that there exists a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ satisfying $r_{n_j}<r_{n_j+1}\quad\forall j\geqslant 0$. Let $\{m_k\}\subset\mathbb{N}$ be defined by $m_k=max\{i\leqslant k:r_i<r_{i+1}\}$. Then $\{m_k\}$ is a nondecreasing sequence satisfying $\lim\limits_{k\to\infty}m_k=\infty$ and for all $k\geqslant n_0$, the following two estimates hold:

$$r_{m_k}\leqslant r_{m_k+1}\text{ and }r_k\leqslant r_{m_k+1}.$$

Lemma 2.19 [26] Let $\{r_n\}$ be a sequence of nonnegative real numbers satisfying

$$r_{n+1}\leqslant(1-\mu_n)r_n+\mu_n\gamma_n,n\geqslant 0\tag{14}$$

with $\{\mu_n\}\subset[0,1]$ such that $\sum_{n=0}^{\infty}\mu_n=\infty$ and $\limsup\limits_{n\to\infty}\gamma_n\leqslant 0$. Then $\lim\limits_{n\to\infty}r_n=0$.

3. Main results

We construct, in this section, an extragradient-type algorithm for approximation of a common element in the set of common fixed point of Bregman demigeneralized mappings, set of Bregman inverse strongly monotone and maximal monotone operators in the setting of reflexive Banach space as follows.

$$\begin{cases} v, x_1 \in E \text{ chosen arbitrarily}, \\ z_n^i = \underset{y\in C}{\operatorname{argmin}}\{\lambda_n h_i(x_n, y) + D_f(y, x_n),\ i = 1, 2, 3, \cdots, N\}, \\ y_n^i = \underset{y\in C}{\operatorname{argmin}}\{\lambda_n h_i(z_n^i, y) + D_f(y, x_n),\ i = 1, 2, 3, \cdots, N\}, \\ i_n = \operatorname{argmax}\{D_f(y_n^i, x_n),\ i = 1, 2, 3, \cdots, N\}, \overline{y}_n = y_n^{i_n}, \\ w_n = \nabla f^*(\beta_n \nabla f(\overline{y}_n) + (1-\beta_n)\nabla f(U_j \overline{y}_n)),\ j = 1, 2, \cdots, m, \\ u_n = \nabla f^*(\alpha_n \nabla f(\overline{y}_n) + \gamma_n \nabla f(x_n) + \eta_n \nabla f(w_n) \\ \qquad + \sum_{s=1}^{N} \delta_{n,s} \nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n))), \\ x_{n+1} = \nabla f^*(\mu_n \nabla f(v) + (1-\mu_n)\nabla f(u_n)), n \geqslant 1, \end{cases} \tag{15}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\delta_{n,s}\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences satisfying the following conditions:
$A1 : \{\mu_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \mu_n = 0$ and $\sum_{n=0}^{\infty} \mu_n = \infty$,
$A2 : 0 < \gamma \leqslant \alpha_n, \gamma_n, \eta_n, \delta_{n,s} \leqslant \beta < 1$ and $\alpha_n + \gamma_n + \eta_n + \sum_{s=1}^{N} \delta_{n,s} = 1$, $\forall n \in \mathbb{N}$,
$A3 : 0 < \gamma \leqslant \lambda_n \leqslant \beta < \min\{\frac{1}{c_1} \frac{1}{c_2}\}$, with $c_1 = \max_{1\leqslant i\leqslant N} c_{1,i}$, $c_2 = \max_{1\leqslant i\leqslant N} c_{2,i}$ such that $c_{1,i}$ and $c_{2,i}$ are Bregman Lipschitz coefficients of h_i for all $i = 1, 2, \cdots, N$,
$A4 : 0 < a \leqslant \beta_n \leqslant \min\{1 - k_1, 1 - k_2, \cdots, 1 - k_m\}$ $\forall n$.

Lemma 3.1 Let C be a nonempty closed convex subset of a reflexive Banach space E, and let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function. For $i = 1, 2, 3, \cdots, N$, let $h_i : C \times C \to \mathbb{R}$ be bifunction satisfying assumptions $(C1) - (C5)$. For $\{\lambda_n\} \subset (0, +\infty)$, let $\{x_n\}$ be a sequence generated by Algorithm (15). Then for any $q \in \Omega = \{x^* \in \cap_{s=1}^{N}(A_s + B_s)^{-1}(0) \cap (\cap_{j=1}^{m} F(U_j)) \cap (\cap_{i=1}^{N} EP(h_i, C))\}$

$$D_f(q, \overline{y}_n) \leqslant D_f(q, x_n) - (1 - c_{1,i}\lambda_n) D_f(z_n^i, x_n) - (1 - c_{2,i}\lambda_n) D_f(\overline{y}_n, z_n^i). \tag{16}$$

Proof. Let $q \in \Omega$. Then it follows from Algorithm (15), Lemma 2.1, Lemma 2.2 that for each $i = 1, 2, 3, \cdots, N$, $y_n^i = \underset{y\in C}{\operatorname{argmin}}\{\lambda_n h_i(z_n^i, y) + D_f(y, x_n)\}$ if and only if

$$0 \in \lambda_n \partial_2 h_i(z_n^i, y_n^i) + \nabla_1 D_f(y_n^i, x_n) + N_C(y_n^i).$$

Therefore, there exist $z \in \partial_2 h_i(z_n^i, y_n^i)$ and $\overline{z} \in N_C(y_n^i)$ such that

$$0 = \lambda_n z + \nabla f(y_n^i) - \nabla f(x_n) + \overline{z}. \tag{17}$$

Since $z \in \partial_2 h_i(z_n^i, y_n^i)$ for each $i \in \{1, 2, \cdots, N\}$, we obtain $h_i(z_n^i, y) - h_i(z_n^i, y_n^i) \geqslant \langle y - y_n^i, z\rangle$ for all $y \in C$. Replacing y by q in the above inequality, we have

$$h_i(z_n^i, q) - h_i(z_n^i, y_n^i) \geqslant \langle q - y_n^i, z\rangle,\ \forall i = 1, 2, 3, \cdots, N. \tag{18}$$

Using (17) and definition of $N_C(y_n^i)$, we also have $\langle y - y_n^i,\ -\lambda_n z - \nabla f(y_n^i) + \nabla f(x_n)\rangle \leqslant 0$ so that

$$\langle y - y_n^i, \nabla f(y_n^i) - \nabla f(x_n)\rangle \geqslant \lambda_n \langle y_n^i - y, z\rangle,\ \forall y \in C. \tag{19}$$

Replacing y by q in (19), we have

$$\langle q - y_n^i, \nabla f(y_n^i) - \nabla f(x_n)\rangle \geqslant \lambda_n \langle y_n^i - q, z\rangle,\ \forall i = 1, 2, 3, \cdots, N. \tag{20}$$

Using (18), (20) and pseudomonotonicity of $h_i^{'s}$, we obtain $\forall i = 1, 2, \cdots, N$ that

$$\langle q - y_n^i, \nabla f(y_n^i) - \nabla f(x_n)\rangle \geqslant \lambda_n \big(h_i(z_n^i, y_n^i) - h_i(z_n^i, q)\big) \geqslant \lambda_n h_i(z_n^i, y_n^i). \tag{21}$$

On the other hand, using $(C2)$ with $x = x_n$, $y = z_n^i$ and $z = y_n^i$, we get

$$h_i(z_n^i, y_n^i) \geqslant h_i(x_n, y_n^i) - h_i(x_n, z_n^i) - c_1 D_f(z_n^i, x_n) - c_2 D_f(y_n^i, z_n^i). \tag{22}$$

Inequality (21) together with (22) give

$$\langle q - y_n^i, \nabla f(y_n^i) - \nabla f(x_n)\rangle \geqslant \lambda_n \big(h_i(x_n, y_n^i) - h_i(x_n, z_n^i) - c_1 D_f(z_n^i, x_n) - c_2 D_f(y_n^i, z_n^i)\big). \tag{23}$$

In a similar manner, since $z_n^i = \underset{y\in C}{\operatorname{argmin}}\{\lambda_n h_i(x_n, y) + D_f(y, x_n),\ i = 1, 2, 3, \cdots, N\}$, it follows as in (21) that

$$\langle z_n^i - y, \nabla f(z_n^i) - \nabla f(x_n)\rangle \leqslant \lambda_n \big(h_i(x_n, y) - h_i(x_n, z_n^i)\big), \forall y \in C$$

so that for $y = y_n^i$, we have

$$\langle z_n^i - y_n^i, \nabla f(z_n^i) - \nabla f(x_n)\rangle \leqslant \lambda_n \big(h_i(x_n, y_n^i) - h_i(x_n, z_n^i)\big). \tag{24}$$

Using (23) and (24), we obtain

$$\begin{aligned}\langle q - y_n^i, \nabla f(y_n^i) - \nabla f(x_n)\rangle \geqslant \langle z_n^i - y_n^i, \nabla f(z_n^i) - \nabla f(x_n)\rangle - \lambda_n c_1 D_f(z_n^i, x_n)\\ - \lambda_n c_2 D_f(y_n^i, z_n^i)\big).\end{aligned}$$

Using the three point identity, this implies that

$$\begin{aligned}0 &\geqslant \langle y_n^i - z_n^i, \nabla f(x_n) - \nabla f(z_n^i)\rangle + \langle q - y_n^i, \nabla f(x_n) - \nabla f(y_n^i)\rangle\\ &\quad - \lambda_n c_1 D_f(z_n^i, x_n) - \lambda_n c_2 D_f(y_n^i, z_n^i)\big)\\ &\geqslant D_f(q, y_n^i) - D_f(q, x_n) + (1 - \lambda_n c_1) D_f(z_n^i, x_n) + (1 - \lambda_n c_2) D_f(y_n^i, z_n^i)\end{aligned}$$

from which we obtain

$$D_f(q, y_n^i) \leqslant D_f(q, x_n) - (1 - \lambda_n c_1) D_f(z_n^i, x_n) - (1 - \lambda_n c_2) D_f(y_n^i, z_n^i)$$

for each $i \in \{1, 2, \cdots, N\}$. Thus,

$$D_f(q, \overline{y}_n) \leqslant D_f(q, x_n) - (1 - \lambda_n c_1) D_f(z_n^i, x_n) - (1 - \lambda_n c_2) D_f(y_n^i, z_n^i) \tag{25}$$

for each $i \in \{1, 2, \cdots, N\}$. ■

Theorem 3.2 Let C be a nonempty closed convex subset of a reflexive Banach space E, and let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. For $\overline{N} \in \mathbb{N}$, $s \in \{1, 2, \cdots, \overline{N}\}$, let $A_s : E \to 2^{E^*}$ and $B_s : E \to 2^{E^*}$ be finite families of Bregman inverse strongly monotone and maximal monotone operators respectively. Suppose $U_j : C \to C$, $j = 1, 2, \cdots, m$, is finite family of $(k_j, 0)-$Bregman demigeneralized mappings such that $(I-U_j)$ is demiclosed at origin and $k_j \in (-\infty, 0)$ for each $j = 1, 2, 3, \cdots, m$. For $i = 1, 2, 3, \cdots, N$, let $h_i : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions $(C1)-(C5)$, such that $\Omega := (\cap_{s=1}^{\overline{N}} F(Res^f_{\lambda B_s} \circ A^f_s)) \cap (\cap_{j=1}^{m} F(U_j)) \cap (\cap_{i=1}^{N} EP(h_i, C)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm (15) converges strongly to some $q \in \Omega$.

Proof. Set $T^\lambda_s := Res^f_{\lambda B_s} \circ A^f_s$, then for each $s \in \{1, 2, \cdots, \overline{N}\}$, it follows from Lemma 2.6 that T^λ_s is Bregman quasi nonexpansive operator. Thus, from Lemma 2.3, Lemma 2.7 and Lemma 2.17, we obtain that Ω is closed and convex. Let $q \in \Omega$. Then, from Algorithm (15), Lemma 2.4 and Lemma 3.1, we have

$$\begin{aligned} D_f(q, w_n) &= D_f(q, \nabla f^*(\beta_n \nabla f(\overline{y}_n) + (1 - \beta_n)\nabla f(U_j\overline{y}_n))) \\ &= D_f(q, S\overline{y}_n) \\ &\leqslant D_f(q, \overline{y}_n) \\ &\leqslant D_f(q, x_n). \end{aligned} \tag{26}$$

Using (26), Lemma 2.14 and Lemma 2.6, we get

$$\begin{aligned} D_f(q, u_n) &= D_f(q, \nabla f^*(\alpha_n \nabla f(\overline{y}_n) + \gamma_n \nabla f(x_n) + \eta_n \nabla f(w_n) \\ &\quad + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n)))) \\ &\leqslant \alpha_n D_f(q, \overline{y}_n) + \gamma_n D_f(q, x_n) + \eta_n D_f(q, w_n) + \sum_{s=1}^{\overline{N}} \delta_{n,s} D_f(q, T^\lambda_s w_n) \\ &\leqslant D_f(q, x_n). \end{aligned}$$

Now, it follows from Algorithm (15) and Lemma 2.14 that

$$\begin{aligned} D_f(q, x_{n+1}) &= D_f(q, \nabla f^*(\mu_n \nabla f(v) + (1 - \mu_n)\nabla f(u_n))) \\ &\leqslant \mu_n D_f(q, v) + (1 - \mu_n) D_f(q, u_n) \\ &\leqslant \mu_n D_f(q, v) + (1 - \mu_n) D_f(q, x_n) \\ &\leqslant \text{Max}\{D_f(q, v), D_f(q, x_n)\} \\ &\;\;\vdots \\ &\leqslant \text{Max}\{D_f(q, v), D_f(q, x_1)\}. \end{aligned}$$

Thus, the sequence $\{D_f(q, x_{n+1})\}$ is bounded. Hence, by Lemma 2.10, the sequence $\{x_n\}$ is also bounded. Consequently, $\{w_n\}$, $\{u_n\}$, $\{\overline{y}_n\}$, $\{T^\lambda_s w_n\}$ and $\{U_j\overline{y}_n\}$ for $j = 1, 2, \cdots, m$ are all bounded. Also, since f is bounded on bounded subset of E we have that ∇f

is bounded on bounded subset of E^* which implies $\{\nabla f(x_n)\}$, $\{\nabla f(w_n)\}$, $\{\nabla f(u_n)\}$, $\{\nabla f(\overline{y}_n)\}$, $\{\nabla T_s^\lambda w_n\}$ and $\{\nabla f(U_j\overline{y}_n)\}$ are all bounded, too.

In what follow, we shall show that $x_n \to q$ as $n \to \infty$. Now, let $\rho_r^* : E^* \to \mathbb{R}$ be a gauge function of uniform convexity of the conjugate function f^* with $r := \sup_n\{\|\overline{y}_n\|, \|\nabla f(x_n)\|\}$. We then have by Lemma 2.13 and Lemma 2.15 that

$$\begin{aligned}
D_f(q,u_n) &= D_f(q, \nabla f^*(\alpha_n \nabla f(\overline{y}_n) + \gamma_n \nabla f(x_n) + \eta_n \nabla f(w_n) + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n)))) \\
&= V_f(q, \alpha_n \nabla f(\overline{y}_n) + \gamma_n \nabla f(x_n) + \eta_n \nabla f(w_n) + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n))) \\
&= f(q) - \langle q, \alpha_n \nabla f(\overline{y}_n) + \gamma_n \nabla f(x_n) + \eta_n \nabla f(w_n) + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n))\rangle \\
&\quad + f^*(\alpha_n \nabla f(\overline{y}_n) + \gamma_n \nabla f(x_n) + \eta_n \nabla f(w_n) + \sum_{s=1}^{\overline{N}} \delta_{n,s} \nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n))) \\
&\leqslant \alpha_n(f(q) - \langle q, \nabla f(\overline{y}_n)\rangle + f^*(\nabla f(\overline{y}_n))) + \gamma_n(f(q) - \langle q, \nabla f(x_n)\rangle + f^*(\nabla f(x_n))) \\
&\quad + \eta_n(f(q) - \langle q, \nabla f(w_n)\rangle + f^*(\nabla f(w_n))) + \sum_{s=1}^{\overline{N}} \delta_{n,s}(f(q) - \langle q, \nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n))\rangle \\
&\quad + f^*(\nabla f(Res^f_{\lambda B_s} \circ A^f_s(w_n)))) - \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|) \\
&= \alpha_n D_f(q, \overline{y}_n) + \gamma_n D_f(q, x_n) + \eta_n D_f(q, w_n) + \sum_{s=1}^{\overline{N}} \delta_{n,s} D_f(q, T_s^\lambda w_n) \\
&\quad - \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|) \\
&\leqslant D_f(q, x_n) - \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|);
\end{aligned}$$

that is,

$$D_f(q,u_n) \leqslant D_f(q,x_n) - \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|). \tag{27}$$

Similarly,

$$D_f(q,u_n) \leqslant D_f(q,x_n) - \alpha_n \gamma_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(x_n)\|) \tag{28}$$

and

$$\begin{aligned}
D_f(q,u_n) &\leqslant D_f(q,x_n) - \sum_{s=1}^{\overline{N}} \alpha_n \delta_{n,s} \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n)\|) \\
&\leqslant D_f(q,x_n) - \alpha_n \delta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n)\|) \qquad (29)
\end{aligned}$$

where $\delta_n := \min_{1 \leqslant s \leqslant \overline{N}} \delta_{n,s}$.

Using (27), (28) and (29), we respectively obtain

$$\begin{aligned} D_f(q, x_{n+1}) &\leqslant \mu_n D_f(q, v) + (1-\mu_n) D_f(q, u_n) \\ &\leqslant D_f(q, x_n) - \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|) - \mu_n[D_f(q, x_n) \\ &\quad - D_f(q, v) - \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|)] \end{aligned} \tag{30}$$

and

$$\begin{aligned} D_f(q, x_{n+1}) &\leqslant D_f(q, x_n) - \alpha_n \gamma_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(x_n)\|) - \mu_n[D_f(q, x_n) \\ &\quad - D_f(q, v) - \alpha_n \gamma_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(x_n)\|)] \end{aligned} \tag{31}$$

and

$$\begin{aligned} D_f(q,x_{n+1}) &\leqslant D_f(q, x_n) - \alpha_n \delta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n)\|) \\ &\quad - \mu_n[D_f(q, x_n) - D_f(q, v) - \alpha_n \delta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n))\|)]. \end{aligned} \tag{32}$$

Also, (30), (31) and (32) imply

$$\begin{aligned} \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|) &\leqslant D_f(q, x_n) - D_f(q, x_{n+1}) - \mu_n[D_f(q, x_n) \\ &\quad - D_f(q, v) - \alpha_n \eta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|)] \end{aligned} \tag{33}$$

and

$$\begin{aligned} \alpha_n \gamma_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(x_n)\|) &\leqslant D_f(q, x_n) - D_f(q, x_{n+1}) - \mu_n[D_f(q, x_n) \\ &\quad - D_f(q, v) - \alpha_n \gamma_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(x_n)\|)] \end{aligned} \tag{34}$$

and

$$\begin{aligned} \alpha_n \delta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n)\|) &\leqslant D_f(q, x_n) - D_f(q, x_{n+1}) \\ &\quad - \mu_n[D_f(q, x_n) - D_f(q, v) - \alpha_n \delta_n \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n)\|)], \end{aligned} \tag{35}$$

respectively. Set $t_n = \nabla f^*(\mu_n \nabla f(v) + (1-\mu_n)\nabla f(u_n))$. Then, using Lemma 2.13, we have

$$\begin{aligned} D_f(q, x_{n+1}) &= D_f(q,\ \nabla f^*(\mu_n \nabla f(v) + (1-\mu_n)\nabla f(u_n))) \\ &= V_f(q,\ \mu_n \nabla f(v) + (1-\mu_n)\nabla f(u_n)) \\ &\leqslant V_f(q,\ \mu_n \nabla f(v) + (1-\mu_n)\nabla f(u_n) - \mu_n(\nabla f(v) - \nabla f(q))) \\ &\quad + \mu_n \langle t_n - q,\ \nabla f(v) - \nabla f(q)\rangle \\ &= V_f(q,\ \mu_n \nabla f(q) + (1-\mu_n)\nabla f(u_n)) + \mu_n \langle t_n - q,\ \nabla f(v) - \nabla f(q)\rangle \\ &= \mu_n V_f(q,\ \nabla f(q)) + (1-\mu_n) V_f(q,\ \nabla f(u_n)) + \mu_n \langle t_n - q,\ \nabla f(v) - \nabla f(q)\rangle \\ &= \mu_n D_f(q,\ q) + (1-\mu_n) D_f(q,\ u_n) + \mu_n \langle t_n - q,\ \nabla f(v) - \nabla f(q)\rangle \\ &\leqslant (1-\mu_n) D_f(q,\ x_n) + \mu_n \langle t_n - q,\ \nabla f(v) - \nabla f(q)\rangle. \end{aligned} \tag{36}$$

Now, as $\{D_f(q, x_{n+1})\}$ is bounded from Lemma 3.1, we proceed by the following two cases.

Case 1: Suppose $\{D_f(q, x_n)\}$ is monotone decreasing sequence, then $\lim_{n\to\infty} D_f(q, x_n)$ exists. Therefore, using this and the conditions on α_n, γ_n, η_n, δ_n, it follows from (33), (34) and (35) that

$$\lim_{n\to\infty} \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(w_n)\|) = 0, \tag{37}$$

$$\lim_{n\to\infty} \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(x_n)\|) = 0, \tag{38}$$

$$\lim_{n\to\infty} \rho_r^*(\|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n)\|) = 0. \tag{39}$$

Hence, by the property of ρ_r^*, we obtain from (37), (38) and (39) that

$$\begin{aligned} &\lim_{n\to\infty} \|\nabla f(\overline{y}_n) - \nabla f(w_n)\| = 0, \ \lim_{n\to\infty} \|\nabla f(\overline{y}_n) - \nabla f(x_n)\| = 0 \text{ and} \\ &\lim_{n\to\infty} \|\nabla f(\overline{y}_n) - \nabla f(T_s^\lambda w_n)\| = \lim_{n\to\infty} \|\nabla f(\overline{y}_n) - \nabla f(Res_{\lambda B_s}^f \circ A_s^f(w_n))\| = 0. \end{aligned} \tag{40}$$

As ∇f^* is norm-to-norm uniformly continuous on bounded subset of E^*, we have from (40) that

$$\begin{cases} (i) \lim_{n\to\infty} \|\overline{y}_n - w_n\| = 0, \\ (ii) \lim_{n\to\infty} \|\overline{y}_n - x_n\| = 0, \\ (iii) \lim_{n\to\infty} \|\overline{y}_n - T_s^\lambda w_n\| = \lim_{n\to\infty} \|\overline{y}_n - Res_{\lambda B_s}^f \circ A_s^f(w_n)\| = 0. \end{cases} \tag{41}$$

Also,

$$\|w_n - Res_{\lambda B_s}^f \circ A_s^f(w_n)\| \leqslant \|w_n - \overline{y}_n\| + \|\overline{y}_n - Res_{\lambda B_s}^f \circ A_s^f(w_n)\|$$

from which it follows that

$$\lim_{n\to\infty} \|w_n - T_s^\lambda w_n\| = 0, \ \forall s \in \{1, 2, \cdots, \overline{N}\}. \tag{42}$$

Now, let $k = \operatorname*{Max}_{1\leqslant j\leqslant m} k_j$. Then, by inequality (11), we obtain

$$\begin{aligned} \langle \overline{y}_n - q, \ \nabla f(\overline{y}_n) - \nabla f(w_n)\rangle &= \langle \overline{y}_n - q, \ \nabla f(\overline{y}_n) - \beta_n \nabla f(\overline{y}_n) - (1-\beta_n)\nabla f(U_j\overline{y}_n)\rangle \\ &= \langle \overline{y}_n - q, \ (1-\beta_n)\nabla f(\overline{y}_n) - (1-\beta_n)\nabla f(U_j\overline{y}_n)\rangle \\ &= (1-\beta_n)\langle \overline{y}_n - q, \ \nabla f(\overline{y}_n) - \nabla f(U_j\overline{y}_n)\rangle \\ &\geqslant (1-\beta_n)(1-k)D_f(\overline{y}_n, \ U_j\overline{y}_n) \end{aligned}$$

so that

$$\begin{aligned} (1-\beta_n)(1-k)D_f(\overline{y}_n, \ U_j\overline{y}_n) &\leqslant \langle \overline{y}_n - q, \ \nabla f(\overline{y}_n) - \nabla f(w_n)\rangle \\ &\leqslant \|\overline{y}_n - q\|\|\nabla f(\overline{y}_n) - \nabla f(w_n)\| \end{aligned}$$

and hence,

$$\lim_{n\to\infty} D_f(\overline{y}_n,\ U_j\overline{y}_n) = 0,\ \forall j \in \{1, 2, \cdots, m\}. \tag{43}$$

Since f is totally convex, we have that f is sequentially consistent. Therefore, it follows from (43) that

$$\lim_{n\to\infty} \|\overline{y}_n - U_j\overline{y}_n\| = 0,\ \forall j \in \{1, 2, \cdots, m\}. \tag{44}$$

Since $\{\overline{y}_n\} \subseteq E$ is bounded and E is a reflexive Banach space, then there exists a subsequence $\{\overline{y}_{n_l}\}$ of $\{\overline{y}_n\}$ such that $\overline{y}_{n_l} \rightharpoonup p$ as $l \to \infty$. This together with (44) and the fact that $(I - U_j)$ is demiclosed at zero give $p \in \cap_{j=1}^m F(U_j)$. From (41)(i), we obtain $w_{n_l} \rightharpoonup p$ as $l \to \infty$ which also together with (42) give $p \in F(T_s^\lambda w_n))$ for each $s \in \{1, 2, \cdots, \overline{N}\}$ and hence,

$$p \in (\cap_{j=1}^m F(U_j)) \cap (\cap_{s=1}^{\overline{N}} F(T_s^\lambda w_n)).$$

Next, we show that $p \in \cap_{i=1}^N EP(h_i, p)$. From Lemma 3.1 and the three point identity, we have

$$\begin{aligned}(1 - c_{1,i}\lambda_n)D_f(z_n^i, x_n) &\leqslant D_f(q, x_n) - D_f(q, \overline{y}_n)\\ &\leqslant D_f(q, x_n) - D_f(q, \overline{y}_n) + D_f(x_n, \overline{y}_n)\\ &= \langle q - x_n, \nabla f(\overline{y}_n) - \nabla f(x_n)\rangle\\ &\leqslant \|q - x_n\|\|\nabla f(\overline{y}_n) - \nabla f(x_n)\|\end{aligned}$$

from which we obtain using $(A3)$ and (40) that $\lim\limits_{n\to\infty} D_f(z_n^i, x_n) = 0$ and hence,

$$\lim_{n\to\infty} \|z_n^i - x_n\| = 0. \tag{45}$$

On the other hand, since $z_n^i = \operatorname*{argmin}\limits_{y\in C}\{\lambda_n h_i(x_n, y) + D_f(y, x_n)\}$ for each $i = 1, 2, 3, \cdots, N$, we have from Lemma 2.1, Lemma 2.2 and assumption $C4$ that

$$0 \in \lambda_n \partial_2 h_i(x_n,\ z_n^i) + \nabla_1 D_f(z_n^i,\ x_n) + N_C(z_n^i).$$

Therefore, for each $i \in \{1, 2, \cdots, N\}$, there exist $\sigma_n^i \in \partial_2 h_i(x_n, z_n^i)$ and $\overline{\sigma}_n^i \in N_C(z_n^i)$ such that

$$\lambda_n\sigma_n^i + \nabla f(z_n^i) - \nabla f(x_n) + \overline{\sigma}_n^i = 0. \tag{46}$$

Also, $\overline{\sigma}_n^i \in N_C(z_n^i)$ implies $\langle w - z_n^i,\ \overline{\sigma}_n^i\rangle \leqslant 0\ \forall w \in C$. Combining this with (46), we obtain $\langle w - z_n^i, -\lambda_n\sigma_n^i - \nabla f(z_n^i) + \nabla f(x_n)\rangle \leqslant 0$ from which we get

$$\lambda_n\langle w - z_n^i,\ \sigma_n^i\rangle \geqslant \langle z_n^i - w,\ \nabla f(z_n^i) - \nabla f(x_n)\rangle. \tag{47}$$

Also, since $\sigma_n^i \in \partial_2 h_i(x_n,\ z_n^i)$, we have

$$h_i(x_n,\ w) - h_i(x_n,\ z_n^i) \geqslant \langle w - z_n^i,\ \sigma_n^i\rangle. \tag{48}$$

From (47) and (48), it follows that

$$\lambda_n(h_i(x_n,\ w) - h_i(x_n,\ z_n^i)) \geqslant \langle z_n^i - w,\ \nabla f(z_n^i) - \nabla f(x_n)\rangle,\ \forall w \in C.$$

From the above inequality, we obtain

$$(h_i(x_{n_l},\ w) - h_i(x_{n_l},\ z_{n_l}^i)) \geqslant \frac{1}{\lambda_{n_l}}\langle z_{n_l}^i - w,\ \nabla f(z_{n_l}^i) - \nabla f(x_{n_l})\rangle,\ \forall w \in C. \tag{49}$$

From (45) and the fact that $x_{n_l} \rightharpoonup p$ as $l \to \infty$, we get that $z_{n_l}^i \rightharpoonup p$ as $l \to \infty$. Allowing $l \to \infty$ in (49), we get by $(C3)$ and $(A3)$ that $h_i(p, w) \geqslant 0, \forall w \in C$ and so

$$p \in \cap_{i=1}^N EP(h_i, C).$$

Hence, $p \in \Omega$.

Claim 1: $\limsup_{n\to\infty}\langle t_n - q,\ \nabla f(v) - \nabla f(q)\rangle \leqslant 0$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - q,\ \nabla f(v) - \nabla f(q)\rangle = \lim_{k\to\infty}\langle x_{n_k} - q,\ \nabla f(v) - \nabla f(q)\rangle.$$

Since $\{x_{n_k}\}$ is a bounded sequence we have that there exist $\{x_{n_{k_j}}\}$, a subsequence of $\{x_{n_k}\}$, such that $x_{n_{k_j}} \rightharpoonup \hat{v} \in \Omega$ as $j \to \infty$. Assume w.l.o.g. $x_{n_k} \to \hat{v}$ as $k \to \infty$. Then it follows from Lemma 2.12(i) that

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - q,\ \nabla f(v) - \nabla f(q)\rangle &= \lim_{k\to\infty}\langle x_{n_k} - q,\ \nabla f(v) - \nabla f(q)\rangle \\ &= \langle \hat{v} - q,\ \nabla f(v) - \nabla f(q)\rangle \leqslant 0.\end{aligned} \tag{50}$$

On the other hand, from Algorithm (15), we have

$$\begin{aligned}\|\nabla f(u_n) - \nabla f(\overline{y}_n)\| &\leqslant \alpha_n\|\nabla f(\overline{y}_n) - \nabla f(\overline{y}_n)\| + \gamma_n\|\nabla f(x_n) - \nabla f(\overline{y}_n)\| \\ &+ \eta_n\|\nabla f(w_n) - \nabla f(\overline{y}_n)\| + \sum_{s=1}^{\overline{N}}\delta_{n,s}\|\nabla f(Res_{B_s}^f \circ A_s^f(w_n)) - \nabla f(\overline{y}_n)\|\end{aligned}$$

which by (40) implies that $\lim_{n\to\infty}\|\nabla f(u_n) - \nabla f(\overline{y}_n)\| = 0$. Since ∇f^* is norm to norm uniformly continuous on bounded subset of E^*, we get that

$$\lim_{n\to\infty}\|u_n - \overline{y}_n\| = 0. \tag{51}$$

By definition of t_n, we have

$$D_f(u_n, t_n) \leqslant \mu_n D_f(u_n, v) + (1 - \mu_n)D_f(u_n, u_n)$$

from which it follows by $(A1)$ that $\lim_{n\to\infty} D_f(u_n, t_n) = 0$. Since f is totally convex on bounded subset of E, it implies f is sequentially consistent and hence

$$\lim_{n\to\infty} \|u_n - t_n\| = 0. \tag{52}$$

Also, $\|t_n - \overline{y}_n\| \leqslant \|t_n - u_n\| + \|u_n - \overline{y}_n\|$. Thus, by (51) and (52), we obtain

$$\lim_{n\to\infty} \|t_n - \overline{y}_n\| = 0. \tag{53}$$

Similarly, $\|t_n - x_n\| \leqslant \|t_n - \overline{y}_n\| + \|\overline{y}_n - x_n\|$, which implies by (45) and (53) that

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{54}$$

From (50) and (54), we obtain

$$\limsup_{n\to\infty} \langle t_n - q,\ \nabla f(v) - \nabla f(q)\rangle = \limsup_{n\to\infty} \langle x_n - q,\ \nabla f(v) - \nabla f(q)\rangle \leqslant 0, \tag{55}$$

proving claim 1. Thus, using (36) and (55) we conclude from Lemma 2.19 that $x_n \to q$ as $n \to \infty$, completing the proof of Case 1.

Case 2: Suppose $\{D_f(q, x_n)\}$ is not monotone decreasing sequence, then there exists a subsequence $\{D_f(q, x_{n_j})\}$ of $\{D_f(q, x_n)\}$ such that $D_f(q, x_{n_j}) \leqslant D_f(q, x_{n_{j+1}})\ \forall\ j \geqslant 1$. Also, for a large N satisfying $k \geqslant N$, define $\alpha : \mathbb{N} \to \mathbb{N}$ by

$$\alpha(k) = max\{j \leqslant k : D_f(q, x_j) \leqslant D_f(q, x_{j+1})\}.$$

Then, by Lemma 2.18, $\{\alpha(k)\}$ is nondecreasing sequence satisfying $\alpha(k) \to \infty$ as $k \to \infty$ and

$$D_f(q, x_{\alpha(k)}) \leqslant D_f(q, x_{\alpha(k)+1}) \text{ and } D_f(q, x_k) \leqslant D_f(q, x_{\alpha(k)+1}), \forall k \in N.$$

This together with (33), (34), (35) give as in Case 1 that

$$\begin{aligned} \alpha_{\alpha(k)}\eta_{\alpha(k)}\rho_r^*(\|\nabla f(\overline{y}_{\alpha(k)}) - \nabla f(w_{\alpha(k)})\|) \leqslant -u_{\alpha(k)}[D_f(q, x_{\alpha(k)}) - D_f(q, v) \\ - \alpha_{\alpha(k)}\eta_{\alpha(k)}\rho_r^*(\|\nabla f(\overline{y}_{\alpha(k)}) - \nabla f(w_{\alpha(k)})\|)] \end{aligned} \tag{56}$$

and

$$\begin{aligned} \alpha_{\alpha(k)}\gamma_{\alpha(k)}\rho_r^*(\|\nabla f(\overline{y}_{\alpha(k)}) - \nabla f(x_{\alpha(k)})\|) \leqslant -u_{\alpha(k)}[D_f(q, x_{\alpha(k)}) - D_f(q, v) \\ - \alpha_{\alpha(k)}\gamma_{\alpha(k)}\rho_r^*(\|\nabla f(\overline{y}_{\alpha(k)}) - \nabla f(x_{\alpha(k)})\|)] \end{aligned} \tag{57}$$

and

$$\begin{aligned} \alpha_{\alpha(k)}\delta_{\alpha(k)}\rho_r^*(\|\nabla f(\overline{y}_{\alpha(k)}) - \nabla f(T_s^\lambda w_{\alpha(k)})\|) \leqslant -u_{\alpha(k)}[D_f(q, x_{\alpha(k)}) \\ - D_f(q, v) - \alpha_{\alpha(k)}\delta_{\alpha(k)}\rho_r^*(\|\nabla f(\overline{y}_{\alpha(k)}) - \nabla f(T_s^\lambda w_{\alpha(k)})\|)], \end{aligned} \tag{58}$$

respectively.

Utilizing the property of ρ_r^*, conditions $(A1)$, $(A2)$ and the fact that ∇f^* is norm-to-norm uniformly continuous on bounded subset of E^* we obtain in a similar way as in Case 1 that

$$\lim_{k\to\infty}\|\overline{y}_{\alpha(k)} - w_{\alpha(k)}\| = 0,\ \lim_{k\to\infty}\|\overline{y}_{\alpha(k)} - x_{\alpha(k)}\| = 0,\ \text{and}\ \lim_{k\to\infty}\|\overline{y}_{\alpha(k)} - T_s^\lambda w_{\alpha(k)}\| = 0.$$

We also get by the same argument as in Case 1 that

$$\limsup_{k\to\infty}\langle t_{\alpha(k)} - q,\ \nabla f(v) - \nabla f(q)\rangle \leqslant 0. \tag{59}$$

Thus, from (36), we get

$$D_f(q, x_{\alpha(k)+1}) \leqslant (1-\mu_{\alpha(k)})D_f(q,\ x_{\alpha(k)}) + \mu_{\alpha(k)}\langle t_{\alpha(k)} - q,\ \nabla f(v) - \nabla f(q)\rangle. \tag{60}$$

Since $D_f(x_{\alpha(k)}, q) \leqslant D_f(x_{\alpha(k)+1}, q)$, we obtain from (60) that

$$D_f(q,\ x_{\alpha(k)}) \leqslant \langle t_{\alpha(k)} - q,\ \nabla f(v) - \nabla f(q)\rangle.$$

This together with (59) give

$$\lim_{k\to\infty} D_f(q,\ x_{\alpha(k)}) = 0. \tag{61}$$

Furthermore, since $D_f(q, x_k) \leqslant D_f(q, x_{\alpha(k)+1})$ for all $k \in \mathbb{N}$, it follows from (61) that $\lim_{k\to\infty} D_f(q,\ x_k) = 0$, which complete the proof of Case 2.

It is therefore concluded from the two cases, Case 1 and Case 2, that $x_n \to q$ as $n \to \infty$. This completes the proof. ■

As a consequences to our results we have from the following under-listed setting that:

(i) Setting in our scheme (15) for each i, $h_i(z,\ y) = 0\ \forall z \in E$, $U_j = I$ for each j, $\gamma_n + \eta_n = \delta_{n,0}$ and $\alpha_n = 0$ we deduced the following result which is clearly the result of Tuyen, Promkan and Sunthrayuth [30].

Corollary 3.3 Let E, $f : E \to \mathbb{R}$, $A_s : E \to 2^{E^*}$ and $B_s : E \to 2^{E^*}$ be as in Theorem (3.2). Suppose $\Omega := \big((\cap_{s=1}^{\overline{N}}(A_s + B_s)^{-1}(0)\big) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by

$$\begin{cases} v, x_1 \in E \ \text{ chosen arbitrarily,} \\ u_n = \nabla f^*(\delta_{n,0}\nabla f(x_n) + \sum_{s=1}^{\overline{N}} \delta_{n,s}\nabla f(Res_{\lambda B_s}^f \circ A_s^f(x_n))), \\ x_{n+1} = \nabla f^*(\mu_n \nabla f(v) + (1-\mu_n)\nabla f(u_n)), n \geqslant 1, \end{cases} \tag{62}$$

where $\{\delta_{n,s}\}$ and $\{\mu_n\}$ are sequences satisfying the following conditions:
$D1 : \{\mu_n\} \subset (0,1)$ such that $\lim_{n\ \to\infty} \mu_n = 0$ and $\sum_{n=0}^{\infty} \mu_n = \infty$,
$D2 : 0 < \gamma \leqslant \delta_{n,s} \leqslant \beta < 1$ and $\delta_{n,0} + \sum_{s=1}^{\overline{N}} \delta_{n,s} = 1$, $\forall n \in \mathbb{N}$,
converges strongly to some $q \in \Omega$.

(ii) Setting $\overline{N} = m = 1$ in theorem (3.2) we equally obtain the following result.

Corollary 3.4 Assume Theorem (3.2) with $\overline{N} = m = 1$ such that $\Omega := (F(Res_{\lambda B}^f \circ A^f)) \cap (F(U)) \cap (\cap_{i=1}^{N} EP(h_i, C)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $q \in \Omega$.

$$\begin{cases} v, x_1 \in E \ \text{ chosen arbitrarily,} \\ z_n^i = \underset{y \in C}{\text{argmin}}\{\lambda_n h_i(x_n, y) + D_f(y, x_n),\ i = 1, 2, 3, \cdots, N\}, \\ y_n^i = \underset{y \in C}{\text{argmin}}\{\lambda_n h_i(z_n^i, y) + D_f(y, x_n),\ i = 1, 2, 3, \cdots, N\}, \\ i_n = \text{argmax}\{D_f(y_n^i, x_n),\ i = 1, 2, 3, \cdots, N\}, \overline{y}_n = y_n^{i_n}, \\ w_n = \nabla f^*(\beta_n \nabla f(\overline{y}_n) + (1 - \beta_n)\nabla f(U\overline{y}_n)), \\ u_n = \nabla f^*(\alpha_n \nabla f(\overline{y}_n) + \gamma_n \nabla f(x_n) + \eta_n \nabla f(w_n) + \delta_n \nabla f(Res_{\lambda B}^f \circ A^f(w_n))), \\ x_{n+1} = \nabla f^*(\mu_n \nabla f(v) + (1 - \mu_n)\nabla f(u_n)), n \geqslant 1, \end{cases} \tag{63}$$

where $0 < a \leqslant \beta_n \leqslant k$, $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\eta_n\}$, $\{\delta_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences as in Theorem (3.2).

Remark 1 *Theorem 3.2 improve some recent results in the literature. In particular, Theorem* 3.1 *of [30] is a corollary of Theorem 3.2 as indicated above. Also Theorem 3.2 complement Theorem* 3.1 *of [1].*

4. Example

Numerical example validating Theorem 3.2 of this paper is presented in this section.

***Example* 4.1** Let $E = \mathbb{R}$ with $\|.\| = |.|$, $C = [0\ 1]$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{3}x^2$, $\forall x \in \mathbb{R}$. Then C is a closed convex subset of a reflexive Banach space $\mathbb{R}$ and that f satisfies all the requirement of Theorem 3.2. Following (8), we have that $f^*(x^*) := \sup_{x \in \mathbb{R}}\{x^* x - f(x)\}\ \forall x^* \in \mathbb{R}$, thus $f^*(w) = \frac{2}{3}w^2$ and $\nabla f^*(w) = \frac{4}{3}w$. For $s \in \{1, 2\}$, let $A_s, B_s : \mathbb{R} \to \mathbb{R}$ be defined respectively by $A_s(x) = 2x$ and $B_s(x) = \frac{1}{2}x\ \forall x \in \mathbb{R}$. Then, for each $s = 1, 2$, A_s is maximal monotone and B_s is Bregman inverse strongly monotone with respective resolvent associated with f obtained as follows:

$$\begin{aligned} z = Res_{\lambda A_1}^f(x) &\Leftrightarrow z = (\nabla f + \lambda A_1)^{-1} \circ \nabla f(x) \\ &\Leftrightarrow (\nabla f + \lambda A_1)z = \nabla f(x) \\ &\Leftrightarrow 2\lambda z = \nabla f(x) - \nabla f(z) \\ &\Leftrightarrow z = \frac{2x}{6\lambda + 2} \end{aligned}$$

for each $x \in \mathbb{R}$; that is, $Res_{\lambda A_1}^f(x) = \frac{2x}{6\lambda+2}$, which is the resolvent of A_1. Also,

$$\begin{aligned} \hat{z} = B_1^f(x) &\Leftrightarrow \hat{z} = (\nabla f^* \circ (\nabla f - B_1))x \\ &\Leftrightarrow \hat{z} = \nabla f^*((\nabla f - B_1)(x)) \\ &\Leftrightarrow \hat{z} = \nabla f^*(\nabla f(x) - B_1(x)) \\ &\Leftrightarrow \hat{z} = \nabla f^*(\frac{1}{6}x) \\ &\Leftrightarrow \hat{z} = \frac{2}{9}x; \end{aligned}$$

that is, $B_1^f(x) = \frac{2}{9}x$, which is the resolvent of B_1. Thus,

$$Res^f_{\lambda A_1} \circ B_1^f(x) = Res^f_{\lambda A_1}(\frac{2}{9}x) = \frac{4x}{54\lambda + 18}.$$

Similarly,

$$Res^f_{\lambda A_2} \circ B_2^f(x) = Res^f_{\lambda A_2}(\frac{2}{9}x) = \frac{4x}{54\lambda + 18}.$$

Next, for $i = 1, 2$, define $h_i : C \times C \to \mathbb{R}$ by $h_i(x,\ y) = 2y^2 + 12xy - 14x^2$. It is then easy to verify that $0 \in \cap_{i=1}^2 EP(h_i,\ C)$ and that each h_i's satisfy assumptions $(C1)$, $(C3)$ and $(C5)$. In addition, h_i's satisfy assumptions $(C2)$ and $(C4)$ with $c_1 = c_2 = 6$ and $\partial_2 h_i(x,\ y) = 4y + 12x$ respectively. Indeed, for $z \in C$, $x, y \in int(domf)$ and $D_f(x,\ y) = (x-y)^2$, we have

$$\begin{aligned} h_i(x,y) + h_i(y,z) &= 2y^2 + 12xy - 14x^2 + 2z^2 + 12yz - 14y^2 \\ &= 2z^2 + 12xz - 14x^2 + 12xy + 12yz - 12xz - 12y^2 \\ &= h_i(x,y) - 6D_f(y,\ x) - 6D_f(z,y) + 6D_f(z,x) \\ &\geqslant h_i(x,y) - 6D_f(y,\ x) - 6D_f(z,y). \end{aligned}$$

Let $U_j : C \to C$ be define by $U_j(x) = \frac{x}{2}$ for all $x \in C$ and $j = 1, 2$. Obviously, $0 \in \cap_{j=1}^2 F(U_j)$ and U_j is Bregman demigeneralized maps for each $j \in \{1, 2\}$. Now,

$$\Omega := (\cap_{s=1}^2 F(Res^f_{\lambda A_s} \circ B_s^f)) \cap (\cap_{j=1}^2 F(U_j)) \cap (\cap_{i=1}^2 EP(h_i,\ C)) = \{0\} \neq \emptyset.$$

Thus, our Algorithm (15) takes the form

$$\begin{cases} z_n^i = \frac{1-6\lambda_n}{1+2\lambda_n} x_n,\ i = 1,2 \\ y_n^i = \frac{x_n - 6\lambda_n}{1+2\lambda_n} z_n^i,\ i = 1,2 \\ \overline{y}_n = y_n^i,\ i = 1,2 \\ w_n = (\frac{4}{9} + \frac{2}{9n})\overline{y}_n \\ u_n = \frac{8}{45}\big(x_n + \overline{y}_n + (1 + \frac{8}{27(3\lambda+1)})w_n\big) \\ x_{n+1} = \frac{4}{9(5n+2)} v + \frac{40n+12}{9(5n+2)} u_n,\ n \in \mathbb{N} \end{cases} \tag{64}$$

for $\mu_n = \frac{1}{2(5n+2)}$, $\alpha_n = \gamma_n = \eta_n = \delta_{n,1} = \delta_{n,2} = \frac{1}{5}$ and $\beta_n = \frac{1}{2n}$. Consider $\lambda_n = \frac{1}{n}$ and let $\{x_n\}$ be a sequence defined by Algorithm (64), then $x_n \to 0 \in \Omega = \{0\}$ as $n \to \infty$ under the following cases.

Case I: Set $x_1 = -7.4$, $v = -7.0$ and $\lambda = 100$.

Case II: Set $x_1 = 0.85$, $v = 0.25$ and $\lambda = 0.01$.

R2014a MATLAB version is utilized to obtain the graphs of the sequence $\{x_n\}$ against number of iterations for different given initial values as indicated above.

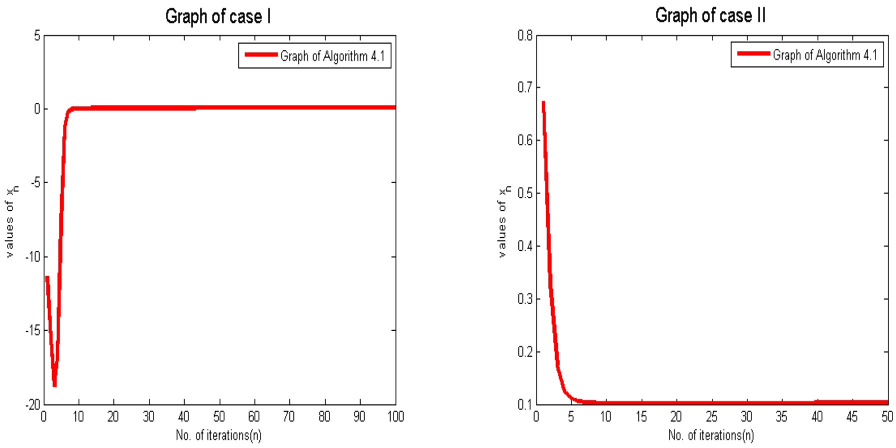

Figure 1. Case I and Case II graphs of a sequence $\{x_n\}$ generated by Algorithm (64) versus number of iterations.

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