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Topological vector space derived from a (Tallini) topological hypervector space

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Abstract. In this paper, we consider a hypervector space (in the sense of Tallini) V over a field K. We use the fundamental relation ε^* over V, as the smallest equivalence relation on V, to derived the fundamental vector space V/ε^*. In this regards, we prove that if V is a (resp. quasi) topological hypervector space, then the fundamental vector space V/ε^* with the property that each open subset of it is a complete part, then its fundamental vector space V/ε^* is a topological vector space. Finally, we prove that for a topological vector space $(V, +, \cdot, K)$ and every subspace W of V, the hypervector space $(\overline{V}, +, \circ, K)$ is a topological hypervector space and we will prove $\overline{V}/\varepsilon^*$ and V/W are homeomorphic, where $\overline{V} = V$.

Keywords: Topological hypervector space, fundamental relation, complete part, homeomorphic.

2010 AMS Subject Classification: 47A12.

1. Introduction and preliminaries

Marty [25] introduced the concept of hypergroups as a generalization of groups and used it in different contexts like algebraic functions, rational fractions and non commutative groups. In classical algebraic structures, the synthetic result of two elements is an element, while in the hyper algebraic system, the synthetic result of two elements is a set of elements, therefore it can be said that the notion of hyperstructures is a generalization of classical algebraic structures, from this point of view. Hyperstructures have many applications to several sectors of both pure and applied sciences as geometry, graphs and

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hypergraphs, fuzzy sets and rough sets, automata, cryptography, codes, relation algebras, $\mathbb{C}$-algebras, artificial intelligence, probabilities, chemistry, physics, especially in atomic physics and in harmonic analysis (for more detail, see [10, 11]). This subject has been studied and extended by many researchers. A hypervector space is an special kind of hyperstructure, which is a generalization of classical vector spaces. There are various kinds of hypervector spaces, but in this paper we deal with to hypervector spaces in the sense of Tallini (for more details, see [31, 32]). The haypervector spaces and one combination to fuzzy sets appeared by many authors (for examples see [2–8, 15]).

In theory of algebraic hyperstructures many equivalence relations has been studied by many authors. First the fundamental relation on a hypergroup denoted by β^* defined and introduced by Koskas [24] and was mainly studied by Corsini [10] and Vougiouklis [33]. Later on, Freni [16] introduced a relation γ^* on a hypergroup, as the smallest equivalence relation on a hypergroup, such that its corresponding quotient structure is an abelain group. Then, Davvaz et al. [1], Ameri et al. [8] and Hamidi et al. [18] introduced the ν^*-relation, ξ^*-relation and τ^*-relation, respectively. In [33], Vougiouklis introduced the fundamental relation ε^* on H_v-vector space (as a general class of hypervector spaces) and in [7], Ameri et al. used the fundamental relation ε^* on a given hypervector space V over a classical field K (in the sense of Tallini) to study the relationship between dimension of V over K ($dimV$) and the ($dimV/\varepsilon^*$) (for more detail, see [7]).

The notion of a topological (transposition) hypergroup was introduced and studied by Ameri [2], the notions of a (resp. pseudo, strong pseudo) topological (transposition) hypergroup was introduced and some basic properties of such topological hypergroups was studied. Later on, many researchers has been worked in this field (for more details, see [9, 13, 14, 17, 19–22, 26–30, 34]).

As it is well known a topological hypervector space is a generalization of a topological vector space. In this paper we follow [2], and use the upper topology over $P^*(V)$ (the family of all nonempty subsets of V) and introduce the topological hypervector spaces with respect to this topology. Also, we compare some different properties between a topological hypervector space to a classical topological vector space. In particular, we prove if in a topological hypervector space $(V, +, \circ, K, \mathcal{T})$ every open subset is a complete part, then quotient space V/ε^*, is a topological vector space. Finally, we consider a topological vector space $(V, +, \cdot, K, \mathcal{T})$ and a subspace W of V, to construct a topological hypervector space $(\overline{V}, +, \circ, K, \mathcal{T})$, and prove $\overline{V}/\varepsilon^*$ and V/W are homeomorphic.

We review some definitions and results from [33], which we need in what follows. A topological group is a group G which is also a topological space such that the multiplication map $(g, h) \to gh$ from $G \times G$ to G, and the inverse map $g \to -g$ from G to G, are both continuous. Similarly, we can define topological rings and topological fields. A topological vector space is a vector space X over a topological field K (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that vector addition $+ : X \times X \to X$ and scalar multiplication. $: K \times X \to X$ are continuous functions with respect to product topologies on $X \times X$, $K \times X$ and X, respectively, for then the mapping $x \mapsto -x = (-1)x$, is continuous and the topology of X is compatible with its additive group structure.

Let G be a nonempty set and $P^*(G)$ be the family of all nonempty subsets of G and $n \in \mathbb{N}$. Every functions $\cdot_i : G \times G \longrightarrow P^*(G)$, where $i \in \{1, 2, \ldots, n\}$ are called hyperoperations and for all x, y of G, $\cdot_i(x, y)$ is called the hyperproduct of x, y. An algebraic system $(G, \cdot_1, \cdot_2, \ldots, \cdot_n)$ is called a hyperstructure and binary structure $(G, \cdot)$ endowed with only hyperoperation is called a hypergroupoid. For every two nonempty subsets A and B of $G, A \cdot B$ means $\bigcup_{a\in A, b\in B} a \cdot b$.

Definition 1.1 [31] Let K be a field and $(V,+)$ be an abelian group. A hypervector space over K is a quadruple $(V,+,\circ,K)$, where "$\circ$" is a mapping $\circ : K \times V \to P^*(V)$ such that for all $a,b \in K$ and $x,y \in V$ the following conditions hold:

(H_1) $a \circ (x+y) \subseteq a \circ x + a \circ y$,
(H_2) $(a+b) \circ x \subseteq a \circ x + b \circ x$,
(H_3) $a \circ (b \circ x) = (ab) \circ x$,
(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
(H_5) $x \in 1 \circ x$,

where for all $A,B \in P^*(V), A+B = \{a+b \mid a \in A, b \in B\}$.

If in (H_1) the equality holds, the hypervector space is called strongly right distributive. If in (H_2) the equality holds, the hypervector space is called strongly left distributive. We will call a hypervector space is a strongly distributive hypervector space if it is both strongly left and strongly right distributive.

Every classical vector space over the field K is a strongly distributive hypervector space over K.

A nonempty subset W of V is called a subhyperspace if W is itself a hypervector space with the external hyperoperation on V, i.e. for all $a \in K$ and $x,y \in W, x-y \in W$and $a \circ x \subseteq W$. Let $\Omega = 0 \times 0_V$ where 0_V is the zero of $(V,+)$. If V is either strongly right distributive, or left, then Ω is a subgroup of $(V,+)$. A strongly right distributive hypervector space is strongly left distributive.

Lemma 1.2 Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent:

(i) f is continuous;
(ii) for all open subset U of $Y, f^{-1}(U)$ is open in X;
(iii) for all $x \in X$ and all open subset V of X containig $f(x)$, thetre exists an open subset U of X containig x such that $f(U) \subseteq V$.

Lemma 1.3 [2] Let $(X,\mathcal{T})$ be a topological space. Then the family $\mathcal{B}$ consisting of all $S_U = \{W \in P^*(X) : W \subseteq U,\ U \in \mathcal{T}\}$ is a base for a topology on $P^*(X)$. This topology is denoted by $\mathcal{T}^*$.

Lemma 1.4 [2, 12] Let $(X,\circ)$ be a hypergroupoid and $\mathcal{T}$ be a topology on X. Then the following assertions are equivalent:

(i) for any $U \in \mathcal{T}$, the set $\{(x,y) \in X \times X : x \circ y \subseteq U\}$ is open in $X \times X$;
(ii) for every $x,y \in X$ and $U \in \mathcal{T}$ such that $x \circ y \subseteq U$, there exist $U_x,\ U_y \in \mathcal{T}$ containing x,y respectively, such that $U_x \circ U_y \subseteq U$;
(iii) for every $x,y \in X$ and $U \in \mathcal{T}$ such that $x \circ y \subseteq U$, there exist $U_x,\ U_y \in \mathcal{T}$ containing x,y respectively, such that $a \circ b \subseteq U$ for any $a \in U_x$ and $b \in U_y$.

Let $(V,+,\circ,K)$ be hypervector space over a topological field K and $\mathcal{T}$ be a topology on V. In the following we use the topology $\mathcal{T}^*$ on $P^*(V)$ and the product topology on $V \times V$.

2. Topological hypervector spaces

In this section, we introduce topological hypervector spaces as a generalization of topological vector spaces and study some of their properties and make a comparison between them.

Definition 2.1 Let $(V, +, \circ, K)$ be a hypervector space over a topological filed K and $(V, \mathcal{T})$ be a topological space. Then $(V, +, \circ, K, \mathcal{T})$ is said to be a topological hypervector space ($thvs$) if the operations $+ : V \times V \to V, (x, y) \mapsto x + y$, $i : V \to V, x \mapsto -x$ and the hyperoperation $\circ : K \times V \to P^*(V), (a, x) \mapsto a \circ x$ are continuous.

***Example* 2.2** Every topological vector space $(V, +, \cdot, K, \mathcal{T})$ with hyperoperation $a \circ x = \{a \cdot x\}$ is a topological hypervector space over K.

***Example* 2.3** Evrey hypervector space $(V, +, \circ, K)$ with trivial topology $\mathcal{T}$ is a topological hypervector space. If we have $\mathcal{T} = \{\emptyset, V\}$, then $\mathcal{T}^* = \{\emptyset, S_V\} = \{\emptyset, P^*(V)\}$.

***Example* 2.4** Let $K = \mathbb{Z}_2$ and $V = \mathbb{Z}_2$. Then $(V, +, \circ, K)$ is a hypervector space as follows:

$\circ$	$\bar{0}$	$\bar{1}$
$\bar{0}$	$\{\bar{0}, \bar{1}\}$	$\{\bar{0}, \bar{1}\}$
$\bar{1}$	$\{\bar{0}, \bar{1}\}$	$\{\bar{0}, \bar{1}\}$

Let $\mathcal{T} = \{\emptyset, \{\bar{0}\}, \{\bar{1}\}, V\}$ be a toplogy on V and on K. We have

$$\mathcal{T}^* = \{\emptyset, \{\{\bar{0}\}\}, \{\{\bar{1}\}\}, \{\{\bar{0}\}, \{\bar{1}\}\}, P^*(V)\}.$$

It is clear that V is a topological hypervector space.

***Example* 2.5** By cosidering the external hyperoperation

$$\circ : \mathbb{R} \times \mathbb{R}^2 \to P^*(\mathbb{R}^2), a \circ (x, y) = a \cdot x \times \mathbb{R},$$

$(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly distributive hypervector space. The family $\mathcal{B} = \{(x, y) : a < x < b, y \in \mathbb{R}\}$ is a base for a topology on $\mathbb{R}^2$ therefore, $(\mathbb{R}^2, +, \circ, \mathbb{R}, \mathcal{T})$ is a topological hypervector space.

***Example* 2.6** Let $\circ : \mathbb{R} \times \mathbb{R} \to P^*(\mathbb{R})$, $a \circ x = \{a \cdot x, -a \cdot x\}$ be a external hyperoperation on $\mathbb{R}$. Then $(\mathbb{R}, +, \circ, \mathbb{R})$ is a hypervector space, but it is neither the right distriutive nor the left distributive. With standard topology on $\mathbb{R}$, $(\mathbb{R}, +, \circ, \mathbb{R}, \mathcal{T})$ is a topological hypervector space.

***Example* 2.7** Let $\circ : \mathbb{R} \times \mathbb{R} \to P^*(\mathbb{R})$, $a \circ x = \{a \cdot x, -a \cdot x, 0\}$ be a external hyperoperation on $\mathbb{R}$. Then $(\mathbb{R}, +, \circ, \mathbb{R})$ is a hypervectorspace, but it is neither the right distributive nor the left distributive. With standard topology on $V = \mathbb{R}$ and discrete topology on $K = \mathbb{R}$, V is a topological hypervector space.

Topological hypervector spaces are a generaliation of topological vector spaces but some characteristics of topological vector spaces are not valid in topological hypervector spaces. If V is a $thvs$, $(V, +)$ is a topological group.

Lemma 2.8 Let V be a $thvs$. Then

- for fixed $x \in V$, the map $y \mapsto x + y$ is a homeomorphism of V onto V;
- if U is open and $x \in V$, then $x + U$ is open; if U is open and A is any subset of V, then $A + U$ is open;
- for fixed $a \in K$, the map $x \mapsto a \circ x$ is continuous, but not necessarily open. In Example 2.6, $U = (2, 3)$ is open and $2 \circ (2, 3) = (-6, -4) \cup (4, 6)$ is also open, but in the Example2.7, $U = (2, 3)$ is open and $2 \circ (2, 3) = (-6, -4) \cup \{0\} \cup (4, 6)$ is not open in $\mathbb{R}$.

The complete parts were introduced for the first time by Koskas [24]. Then, this concept was studied by many authors. Let $(V, +, \circ, K)$ be a hypervector space over K and A be a nonempty subset of V. We say that A is a complete part of V if for nonzero natural number n, for all $a_1, \ldots, a_n$ of K, and for all $x_1, \ldots, x_n$ of V, the following implication holds:

$$A \cap \sum_{i=1}^{n} a_i \circ x_i \neq \emptyset \Longrightarrow \sum_{i=1}^{n} a_i \circ x_i \subseteq A.$$

Theorem 2.9 Let V be a *thvs*, $A \subseteq V$ and U be an open subset of V, such that U is a complete part of V. Then $A \subseteq a^{-1} \circ U$ if and only if $a \circ A \subseteq U$ for all $a \in K$.

Proof. Suppose that $A \subseteq a^{-1} \circ U$ and $x \in A$. So $x \in a^{-1} \circ U$, and there exists $u \in U$ such that $x \in a^{-1} \circ u$ thus, $a \circ x \subseteq a \circ (a^{-1} \circ u) = 1 \circ u$. We have $u \in 1 \circ u, u \in U$, which emplies that $1 \circ u \subseteq U$ since U is complete part. Therefore $a \circ x \subseteq U$.

Conversely, suppose that $a \circ A \subseteq U$ and $a \in K$. Then, we have $A \subseteq a^{-1} \circ (a \circ A) \subseteq a^{-1} \circ U$. ■

Theorem 2.10 Let U be an open subset of a *thvs*, such that U is a complet part. Then

(i) $a \circ U$ is an open subset of V for every $a \in K, a \neq 0$;
(ii) for any subset A of K such that $\forall a \in A, a \neq 0,\ A \circ U$ is open.

Proof. (i) The map $P_a : V \to P^*(V),\ P_A : x \mapsto a \circ x$ is continuous. For $a \neq 0$, we have

$$P_{a^{-1}}^{-1}(S_U) = \{x \in V : a^{-1} \circ x \subseteq U\} = a \circ U.$$

Thus, $a \circ U$ is open. (ii) Since the union of open subsets is open, therefore $A \circ U = \bigcup_{a \in A} a \circ U$ is open. ■

3. Topological fundamental vector spaces

In this section, the concept of a topological fundamental vector space derived of a topological hypervector space is introduced. Let $(V, +, \circ, K)$ be a hypervector space over K. The fundamental relation ε^* of V was introduced by T. Vougiouklis in [33] as the smallest equivalence relation on H_v-vector space, a general class of hypervector spaces, such that the quotient V/ε^* is a vector space over K. In the following, we introduce the fundamental relation on hypervector spaces in the sence of Tallini, and study the relationship between V and V/ε^* in the way of [7].

Let U be the set of all finite linear combinations of elements of V with coefficent in K, that follows

$$U = \left\{ \sum_{i=1}^{n} a_i \circ x_i : a_i \in K, x_i \in V, n \in \mathbb{N} \right\}.$$

Now, consider the ε-relation over V by

$$x \varepsilon y \Longleftrightarrow \exists u \in U : \{x, y\} \subseteq u,\ \forall x, y \in V.$$

Let ε^* be the transitive closure of ε. We define addition operation and scalar multiplication on V/ε^* by

$$\begin{cases} \oplus : V/\varepsilon^* \times V/\varepsilon^* \to V/\varepsilon^* \\ \varepsilon^*(x) \oplus \varepsilon^*(y) = \{\varepsilon^*(t) : t \in \varepsilon^*(x) + \varepsilon^*(y)\}, \end{cases}$$

and

$$\begin{cases} \odot : K \times V/\varepsilon^* \to V/\varepsilon^* \\ a \odot \varepsilon^*(x) = \{\varepsilon^*(z) : z \in a \circ \varepsilon^*(x)\}. \end{cases}$$

Theorem 3.1 [7] Let $(V, +, \circ, K)$ be a hypervector space over K. Then,

(i) $\varepsilon^*(a \circ x) = \varepsilon^*(y)$ for all $y \in a \circ x, \forall a \in K, \forall x \in V$, where $\varepsilon^*(a \circ x) = \bigcup_{b \in a \circ x} \varepsilon^*(b)$.
(ii) $\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(x + y)$.
(iii) $\varepsilon^*(0)$ is the identity element of $(V/\varepsilon^*, \oplus)$.
(iv) $(V/\varepsilon^*, \oplus, \odot, K)$ is a vector space over K.

The vector space $(V/\varepsilon^*, \oplus, \odot, K)$ is called fundamental vector space of V.

Theorem 3.2 Let $(V, +, \circ, K)$ be a hypervector space over K and $(V/\varepsilon^*, \oplus, \odot, K)$ be the fundamental vector space of V. Then the canonical map $\pi : V \to V/\varepsilon^*$ such that $\pi(x) = \varepsilon^*(x)$, is an epimorphism.

Proof. Let $x, y \in V$ and $a \in K$, we see that $\pi(x + y) = \pi(x) \oplus \pi(y)$. Now we show that $\pi(a \circ x) = a \odot \pi(x)$. We have $\pi(a \circ x) = \varepsilon^*(a \circ x) = \varepsilon^*(y)$ for all $y \in a \circ x$.
On the other hand, we have $y \in a \circ x, x \in \varepsilon^*(x)$ that implies $y \in a \circ \varepsilon^*(x)$ thus, $a \odot \pi(x) = a \odot \varepsilon^*(x) = \{\varepsilon^*(z) : z \in a \circ \varepsilon^*(x)\} = \varepsilon^*(y)$. ■

Let X be a topological space and $\sim$ be any equivalence relation on X. The quotient set of all equivalence classes is given by the $X/\sim = \{[x] : x \in X\}$. We have the canonical map or quotient map $\pi : X \to X/\sim,\ x \mapsto [x]$, and we define a topology on $X/\sim$ by setting that: $U \subseteq X/\sim$ is open iff $\pi^{-1}(U)$ is open in X. Then it is easy to verify that:

- the canonical map π is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ s.t. π is continuous.
- the canonical map π is not neccessarily open or closed.

Theorem 3.3 Let $(V, +, \circ, K)$ be a topological hypervector space over K, such that every open subset of V is a complete part. Then the canonical map $\pi : V \to V/\varepsilon^*$ is open.

Proof. Let W be an open subset of V and $x \in \pi^{-1}(\pi(W))$, we have $\pi(x) \in \pi(W)$ thus there exists $v \in W$ such that $\pi(x) = \pi(v)$ and $x \in \varepsilon^*(v)$. Hence, there exist $a_1, \ldots, a_n \in K$ and $x_1, \ldots, x_n \in V$ such that $\{x, v\} \subseteq \sum_{i=1}^n a_i \circ x_i$. Since W is open so there exists an open subset U of V such that $v \in U \subseteq W$. Hence we have $v \in U \cap \sum_{i=1}^n a_i \circ x_i$ and U is complete part so $x \in \sum_{i=1}^n a_i \circ x_i \subseteq U \subseteq W$. Thus, $x \in U \subseteq \pi^{-1}(\pi(W))$. Therefore, $\pi(W)$ is open in V/ε^*. ■

Theorem 3.4 Let $(V, +, \circ, K, \mathcal{T})$ be a topological hypervector space over K such that every open subset of V is a complete part. Then, $(V/\varepsilon^*, \oplus, \odot, \mathcal{T}^*)$ is a topological vector space over K, where $\mathcal{T}^*$ is the quotient topology on V/ε^*.

Proof. By Theorem 3.1, $(V/\varepsilon^*, \oplus, \odot)$ is a vector space. We show that the mappings

$$\oplus : (\pi(x), \pi(y)(\mapsto \pi(x) \oplus \pi(y) \text{ and } \odot : (a, \pi(x)) \mapsto a \odot \pi(x)$$

are continuous, where $\oplus = \oplus_{\varepsilon^*}$ and $\odot = \odot_{\varepsilon^*}$.

(i) Let U be an open subset of V/ε^* and $x, y \in U$, such that $\pi(x) \oplus \pi(y) \in U$. So we have $\pi(x+y) \in U$ or $x+y \in \pi^{-1}(U)$. Since $\pi^{-1}(U)$ is open in V and V is $thvs$, it follows that there exist open subsets U_1, U_2 of V such that $x \in U_1, y \in U_2$ and $U_1 + U_2 \subseteq \pi^{-1}(U)$ or $\pi(U_+U_2) \subseteq U$, thus $\pi(U_1) \oplus \pi(U_2) \subseteq U$.

(ii) Let U be an open subset in V/ε^* and $a \in K, x \in V$ such that $a \odot \pi(x) \in U$. There exists $z \in a \circ \pi(x)$ and we have $\pi(z) \in U$ so $z \in \pi^{-1}(U)$. Since $a \circ x \subseteq a \circ \pi(x)$, so $a \circ x \subseteq \pi^{-1}(U)$. Thus there exixt open subsets U_1 and U_2 containing a and x from K and V, respectively such that $U_1 \circ U_2 \subseteq \pi^{-1}(U)$ hence $U_1 \odot \pi(U_2) \subseteq U$. Since we have

$$\pi(U_1 \circ U_2) = \pi(\bigcup_{a \in U_1} a \circ U_2) = \bigcup_{a \in U_1} \pi(a \circ U_2) = \bigcup_{a \in U_1} (a \odot \pi(U_2)) = U_1 \odot \pi(U_2).$$

■

4. Homeomorphism

A topological vector space (tvs) is a vector space V over a topological field K equipped with a topology such that the maps $(x, y) \mapsto x + y$ and $(a, x) \mapsto a \cdot x$ are continuous from $X \times X \to X$ and $K \times X \to X$, respectively.

Let X, Y be two vector space over K. A mapping $f : X \to Y$ is called homomorphism if we have

$$f(x + y) = f(x) + f(y), f(\lambda x) = \lambda f(x), \forall x, y \in X, \forall \lambda \in K.$$

A bijective homomorphism between two vector spaces X and Y over K is called algebraic isomorphism and we say that X and Y are algebraically isomorphic $X \cong Y$. Let X and Y be two tvs on K. A topological isomorphism (homeomorphism) from X to Y is a algebraic isomorphism which is also continuous and open.

Let V be a tvs and $W \subseteq V$ be a linear subspace of V. The quotient space V/W consists of cosets $x + W = [x]$ and the quotient map $\pi : V \to V/W$ is defined by $\pi(x) = x + W$.

In this section, we construct a topological hypervector space such as $\overline{V}$ using a classical topological vector space V and its linear subspace W and prove that $\overline{V}/\varepsilon^*$ and V/W are homeomorphic.

Theorem 4.1 For a linear subspace W of a tvs V, the quotient map $\pi : V \to V/W$ is a continuous and open map, when V/W is equipped with the quotient topology.

Proof. The mapping "π" is continuous by the definition of the quotient topology. Let U be open in V. Then we have

$$\pi^{-1}(\pi(U)) = U + W = \bigcup_{v \in W} (U + v),$$

since $U + v$ is open for any $v \in W$, hence $\pi^{-1}(\pi(U))$ is open in V as a union of open sets. Therefore $\pi(U)$ is open in V/W. ■

Theorem 4.2 [23] Let W be a linear subspace of a *tvs* V. Then the quotient space V/W equipped with the quotient topology is a *tvs*.

Let $(V, +, \cdot, K)$ be a classical vector space and W be a linear subspace of V and $\overline{V} = V$. Then $(\overline{V}, +, \circ, K)$ is a strongly distributive hypervector space where

$$\circ : K \times V \to P^*(V), \ a \circ x = a \cdot x + W,$$

$\overline{V}$ is said to be the associated hypervector space concerning the vector space V.

Theorem 4.3 Let $(V, +, \cdot, K)$ be a classical vector space and W be a linear subspace of V. Then $\overline{V}/\varepsilon^* \cong V/W$.

Proof. We define a mapping $f : \overline{V}/\varepsilon^* \to V/W$ by $f(\varepsilon^*(x)) = x + W$.

- the mapping f is well-defined. Let $\varepsilon^*(x) = \varepsilon^*(y)$, it follows that $x\varepsilon^*y$ and we have $x \in 1 \circ x = x + W, \ y \in 1 \circ y = y + W$, since the two sets $x + W$ and $y + W$ are equal or disjoint subset of V/W, thus $x + W = y + W$ and so $f(\varepsilon^*(x)) = f(\varepsilon^*(y))$.
- f is linear. Since,

$$f(\varepsilon^*(x) + \varepsilon^*(y)) = f(\varepsilon^*(x + y)) = x + y + W = x + W + y + W = f(\varepsilon^*(x)) + f(\varepsilon^*(y)).$$

and $f(a \odot \varepsilon^*(x)) = f(\varepsilon^*(z)), \ z \in a \circ \varepsilon^*(x)$, on the other hand, $a \cdot x \in 1 \circ (a \cdot x) = a \circ x \subseteq a \circ \varepsilon^*(x)$ which implies that

$$f(\varepsilon^*(z)) = f(\varepsilon^*(a \cdot x)) = a \cdot x + W = a \circ (x + W) = a \circ f(\varepsilon^*(x)).$$

- The mapping f is surjective. For one-to-one property of f, let $\varepsilon^*(x) \in Ker(f)$. Then $f(\varepsilon^*(x)) = x + W = W$ and thus $x \in W$. Therefore, $\varepsilon^*(W) = \varepsilon^*(0) = 0_{\overline{V}/\varepsilon^*}$, which implies that f is one-to-one.

Consequently f is an algebraic isomorphism. ■

Theorem 4.4 Let $(V, +, \cdot, K, \mathcal{T})$ be a *tvs*. Then $(\overline{V}, +, \circ, K, \mathcal{T})$ is a topological hypervector space.

Proof. It is enough to show that the mapping $\circ : K \times \overline{V} \to P^*(V), \ a \circ x = a \cdot x + W$ is continuous. Let U be an open subset of V. the mapping "$\circ$" is continuous if and only if $\{(a, x) \in K \times \overline{V} : a \circ x \subseteq U\}$ is an open subset of $K \times \overline{V}$ for all $U \in \mathcal{T}$. We have $a \circ x \subseteq U \Rightarrow a \cdot x + W \subseteq U$. Since $a \cdot x \in a \cdot x + W \subseteq U$ and the mapping "$\cdot$" is continuous, there exist U_1 and U_2 containing a and x respectively, such that $U_1 \cdot U_2 \subseteq U$. ■

Theorem 4.5 Let $(V, +, \cdot, K, \mathcal{T})$ be a *tvs* and W be a linear subspace of V. Then $\overline{V}/\varepsilon^*$ and V/W are topologically isomorphic.

Proof. By Theorem 4.3, the map

$$f : \overline{V}/\varepsilon^* \to V/W, \ f(\varepsilon^*(x)) = x + W$$

is algebraic isomorphism. It is enough to show that f is contiuous and open. Suppose that A is open in V/W. We show that $\pi^{-1}(f^{-1}(A))$ is open in V/W. Let $x \in \pi^{-1}(f^{-1}(A))$. Then $\pi(x) \in f^{-1}(A)$ and so $f(\pi(x)) \in A$, thus $x + W \in A$. Since the canonical map $q : V \to V/W$ is continuous, there exists an open subset U_x containing x of V such that $U_x \subseteq q^{-1}(A)$. We show that $U_x \subseteq \pi^{-1}(f^{-1}(A))$. If $t \in U_x$, then $t + W \in A$, and so

$t \in \pi^{-1}(f^{-1}(A))$. Therefore $\pi^{-1}(f^{-1}(A))$ is open in $\overline{V}$, and f is continuous. Now suppose that A is an open subset of $\overline{V}/\varepsilon^*$. We show that $f(A)$ is an open subset of V/W. Let $x+W \in f(A)$. Then $\varepsilon^*(x) \in A$. Since the canonical mapping $\pi : \overline{V} \to \overline{V}/\varepsilon^*$ is continuous, there exists an open subset U_x containing x of V such that $U_x \subseteq \pi^{-1}(A)$. We show that $\{z + W : z \in U_x\} \subseteq f(A)$. If $z \in U_x$, then $z + W = f(\varepsilon^*(x)) \in f(A)$, thus $f(A)$ is open in V/W. Therefore f is open. ■

5. Conclusion

In this paper, the notion of topological hypervector spaces (in the sense of Tallini) was introduced and some of the basic properties of these spaces were investigated, and by useing the fundamental relation ε^* on a hypervector space, a connection between topological hypervector spaces and topological vector spaces was established. In fact, by considering the notion of fundamental relation ε^* on a hypervector space, its fundamental vector space, V/ε^* has been constructed. In particular it was proved that if in a topological hypervector space V each open subset is a complete part, then the canonical map $\pi : V \to V/\varepsilon^*$ is open and the fundamental vector space V/ε^* is a topological vector space.

Also,for a topological vector space $(V, +, \cdot, K, \mathcal{T})$ on topological field K and its subspace W, we defined topological hypervector space $(\overline{V} = V, +, \circ, K, \mathcal{T})$ on topological field K and proved $\overline{V}/\varepsilon^* \cong V/W$.

In our future studies, we hope to define the topological Krasner hypervector space (H_V-vector space) V on the topological hyperfield K and use the fundamental relation ε^* on V and the fundamental relation γ^* on K, defined in [33], to investigate the conditions for topological vector space V/ε^* on the topological field K/γ^*.

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