Journal of Linear and Topological Algebra
Vol. 13*, No.* 01*,* 2024*,* 19- 28
DOR:
DOI: 10.30495/JLTA.2024.2004057.1608

Graphical cyclic $\mathcal{J}$-integral Banach type mappings and the existence of their best proximity points

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Received 16 December 2023; Revised 13 January 2024; Accepted 24 January 2024.

Communicated by Hamidreza Rahimi

Abstract. The underlying aim of this paper is first to state the cyclic version of $\mathcal{J}$-integral Banach type contractive mappings introduced by Fallahi, Ghahramani and Soleimani Rad [Integral type contractions in partially ordered metric spaces and best proximity point, Iran. J. Sci. Technol. Trans. Sci. 44 (2020), 177-183] and second to show the existence of best proximity points for such contractive mappings in a metric space with a graph, which can entail a large number of former best proximity point results. One fundamental issue that can be distinguished between this work and previous researches is that it can also involve all of results stated by taking comparable and ϑ-close elements.

Keywords: $\mathcal{J}$-quasi-contraction, orbitally $\mathcal{J}$-continuous, graphical metric spaces, best proximity point.

2010 AMS Subject Classification: 47H10.

1. Preliminaries

Since 1922, metric fixed point (fp) theory and contractions have became an important tools in nonlinear analysis and many researchers have applied them in many nonlinear functions problems and engineering (see [4, 16] and their references). For instance, in 2004, Ran and Reurings [15] considered a partial order set (POS) on a metric space (MS) and discussed the existence of fp(s) of contractive mappings and their uniqueness for comparable elements.

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

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Theorem 1.1 [15] Consider a POS $(\mathcal{W}, \preceq)$, a complete MS $(\mathcal{W}, \mathcal{D})$ and a nondecreasing mapping $\mathcal{K} : \mathcal{W} \to \mathcal{W}$ so that $\mathcal{D}(\mathcal{K}\mathfrak{p}, \mathcal{K}\mathfrak{q}) \leqslant \theta\mathcal{D}(\mathfrak{p}, \mathfrak{q})$ for any $\mathfrak{p}, \mathfrak{q} \in \mathcal{W}$ with $\mathfrak{p} \preceq \mathfrak{q}$, where $\theta \in [0, 1)$. Also, assume

- either $\mathcal{K}$ is continuous;
- or if a nondecreasing sequence $\mathfrak{p}_n$ converges to a $\mathfrak{p} \in \mathcal{W}$, then $\mathfrak{p}_n \preceq \mathfrak{p}$.

If there is $\mathfrak{p}_0 \in \mathcal{W}$ satisfying $\mathfrak{p}_0 \preceq \mathcal{K}\mathfrak{p}_0$, then $\mathcal{K}$ has a fp. Further, if each two fp(s) are comparable, then the fp is unique.

Note that we say $\mathcal{K}$ in Theorem 1.1 is nondecreasing when $\mathfrak{p} \preceq \mathfrak{q}$ implies $\mathcal{K}\mathfrak{p} \preceq \mathcal{K}\mathfrak{q}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathcal{W}$. Also, we say $\mathfrak{p}$ and $\mathfrak{q}$ are comparable whenever $\mathfrak{p} \preceq \mathfrak{q}$ or $\mathfrak{q} \preceq \mathfrak{p}$. In 2005, Nieto and Rodrıguez-López [13] used this definition and fp result to solve some differential equations. Moreover, in 2011, Abkar and Gabeleh [1] fused Theorems 1.1 and the definition of cyclic mappings introduced by Kirk et al. [12] and established an fp result.

Theorem 1.2 [1] Take a POS $(\mathcal{W}, \preceq)$, two subsets $\mathcal{H}, \mathcal{F} \neq \emptyset$ of a complete MS $(\mathcal{W}, \mathcal{D})$ and a cyclic mapping $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ so that $\mathcal{D}(\mathcal{K}\mathfrak{p}, \mathcal{K}^2\mathfrak{q}) \leqslant \theta\mathcal{D}(\mathfrak{p}, \mathcal{K}\mathfrak{q})$ for each $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{H} \times \mathcal{H}$ with $\mathfrak{q} \preceq \mathfrak{p}$, where $\theta \in (0, 1)$ and $\mathcal{K}^2$ is nondecreasing on $\mathcal{H}$. Also, presume that

- either $\mathcal{K}$ is continuous;
- or if a nondecreasing sequence $\mathfrak{p}_n$ converges to a $\mathfrak{p} \in \mathcal{W}$, then $\mathfrak{p}_n \preceq \mathfrak{p}$.

If there is $\mathfrak{p}_0 \in \mathcal{W}$ satisfying $\mathfrak{p}_0 \preceq \mathcal{K}^2\mathfrak{p}_0$, then $\mathcal{H} \cap \mathcal{F} \neq \emptyset$ and $\mathcal{K}$ has a fp in $\mathcal{H} \cap \mathcal{F}$. Further, if $\mathfrak{p}_{n+1} = \mathcal{K}(\mathfrak{p}_n)$, then $\mathfrak{p}_{2n} \to \mathfrak{p}$.

It should be noted that a mapping $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ is named cyclic if $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{F}$ and $\mathcal{K}(\mathcal{F}) \subseteq \mathcal{H}$.

Presume $\mathcal{H}, \mathcal{F} \neq \emptyset$ are subsets of a MS, $\mathsf{dist}(\mathcal{H}, \mathcal{F}) = \inf\{\mathcal{D}(\mathfrak{p}, \mathfrak{q}) : \mathfrak{p} \in \mathcal{H}, \mathfrak{q} \in \mathcal{F}\}$ and $\mathcal{K} : \mathcal{H} \to \mathcal{F}$ is a non-self mapping. The bpp(s) of $\mathcal{K}$ is all $\mathfrak{p} \in \mathcal{H}$ with $\mathcal{D}(\mathfrak{p}, \mathcal{K}\mathfrak{p}) = \mathsf{dist}(\mathcal{H}, \mathcal{F})$. In the sequel, Eldred and Veeremani [6] and Suzuki et al. [18] presented the existence of bpp(s) of cyclic contractive mappings on various metric spaces regarding some properties like unconditionally Cauchy (UC) property.

Definition 1.3 [18] Taking $\mathcal{H}, \mathcal{F} \neq \emptyset$ two subsets of a MS $(\mathcal{W}, \mathcal{D})$, we say the pair $(\mathcal{H}, \mathcal{F})$ has UC property whenever for two sequences $\{\mathfrak{p}_n\}$ and $\{\mathfrak{p}'_n\}$ in $\mathcal{H}$ and a sequence $\{\mathfrak{q}_n\}$ in $\mathcal{F}$, $\lim\limits_{n\to\infty} \mathcal{D}(\mathfrak{p}_n, \mathfrak{q}_n) = \lim\limits_{n\to\infty} \mathcal{D}(\mathfrak{p}'_n, \mathfrak{q}_n) = \mathsf{dist}(\mathcal{H}, \mathcal{F})$ implies $\lim\limits_{n\to\infty} \mathcal{D}(\mathfrak{p}, \mathfrak{p}'_n) = 0$.

Lemma 1.4 [18] Let $\mathcal{H}, \mathcal{F} \neq \emptyset$ be subsets of a MS $(\mathcal{W}, \mathcal{D})$ and the pair $(\mathcal{H}, \mathcal{F})$ has the UC property. Also, presume that $\{\mathfrak{p}_n\}$ and $\{\mathfrak{q}_n\}$ are sequences in $\mathcal{H}$ and $\mathcal{F}$, respectively, provided that

$$\text{either} \quad \lim_{m\to\infty} \sup_{n\geqslant m} \mathcal{D}(\mathfrak{p}_m, \mathfrak{q}_n) = \mathsf{dist}(\mathcal{H}, \mathcal{F}) \quad \text{or} \quad \lim_{n\to\infty} \sup_{m\geqslant n} \mathcal{D}(\mathfrak{p}_m, \mathfrak{q}_n) = \mathsf{dist}(\mathcal{H}, \mathcal{F}).$$

Then $\{\mathfrak{p}_n\}$ is a Cauchy sequence.

The theory of bpp of various mappings in different type of $MS(s)$ has been continued by many researchers (see also [8, 10, 14, 17] and references therein). On the other hand, if $\mathcal{H} \cap \mathcal{F} = \emptyset$ in Theorem 1.2, then $\mathcal{K}\mathfrak{p} = \mathfrak{p}$ has no solution. Hence, we may think about an approximate solution $\mathfrak{p} \in \mathcal{H} \cup \mathcal{F}$ so that the error $\mathsf{dist}(\mathfrak{p}, \mathcal{K}\mathfrak{p})$ is minimum. As $\mathcal{K}$ is cyclic on $\mathcal{H} \cup \mathcal{F}$, we obtain $\mathcal{D}(\mathfrak{p}, \mathcal{K}\mathfrak{p}) \geqslant \mathsf{dist}(\mathcal{H}, \mathcal{F})$. Hence, Abkar and Gabele introduced some useful tools for finding bpp of cyclic contractive and cyclic φ-contractive mapping, respectively.

Theorem 1.5 [2] Let $(\mathcal{W}, \preceq)$ be a PO set, $\mathcal{H}, \mathcal{F} \neq \emptyset$ be two closed subsets of a complete MS $(\mathcal{W}, \mathcal{D})$ and $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ be a cyclic mapping fulfilling

$$\mathcal{D}(\mathcal{K}\mathfrak{p}, \mathcal{K}^2\mathfrak{q}) \leqslant \mathcal{D}(\mathfrak{p}, \mathcal{K}\mathfrak{q}) - \varphi\big(\mathcal{D}(\mathfrak{p}, \mathcal{K}\mathfrak{q})\big) + \varphi\big(\mathsf{dist}(\mathcal{H}, \mathcal{F})\big)$$

for each $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{H} \times \mathcal{H}$ with $\mathfrak{q} \preceq \mathfrak{p}$, where $\varphi : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ is a strictly increasing function and $\mathcal{K}^2$ is nondecreasing on $\mathcal{H}$. Also, presume that the following condition is held:

- If a nondecreasing sequence $\mathfrak{p}_n$ converges to a $\mathfrak{p}$ in $\mathcal{W}$, then $\mathfrak{p}_n \preceq \mathfrak{p}$.

If there is $\mathfrak{p}_0 \in \mathcal{W}$ satisfying $\mathfrak{p}_0 \preceq \mathcal{K}^2\mathfrak{p}_0$, $\mathfrak{p}_{n+1} = \mathcal{K}\mathfrak{p}_n$ for $n \geqslant 0$ and $\{\mathfrak{p}_{2n}\}$ possesses a convergent subsequence in $\mathcal{H}$, then $\mathcal{K}$ has a bpp in $\mathcal{H}$.

To follow POS and fp subjects, in 2008, Jachymski [11] stated a graphical MS and introduced several concepts and fp theorems. After that, many researchers working on both fP theory and bpp theorems extended Jachymski's idea in different directions regarding different spaces and various contraction (also, see [7]). Note that the results of these references can well expand the results regarding a PO relationship. Presume $\mathcal{J}$ is a graph. A link is an edge of $\mathcal{J}$ in which its ends is different. Also, a loop is an edge of $\mathcal{J}$, where its ends is identical. Parallel edges of $\mathcal{J}$ are two or more links of $\mathcal{J}$ with same pairs of ends. Suppose $(\mathcal{W}, \mathcal{D})$ is a MS and $\mathcal{J}$ is a directed graph, where $\mathsf{V}(\mathcal{J})$ is vertex set coinciding with $\mathcal{W}$ and $\mathcal{E}(\mathcal{J})$ is edge set containing all loops and $\mathcal{J}$ has no parallel edges. Then, $(\mathcal{W}, \mathcal{D})$ is named a MS with the graph $\mathcal{J}$ (or GMS). Additionally, suppose $\mathcal{J}^{-1}$ is a directed graph obtained from $\mathcal{J}$ by reversing the directions of the edges of $\mathcal{J}$ and $\tilde{\mathcal{J}}$ is the undirected graph gotten from $\mathcal{J}$ by removing the directions of the edges $\mathcal{J}$. It's clear that $\mathsf{V}(\mathcal{J}^{-1}) = \mathsf{V}(\tilde{\mathcal{J}}) = \mathsf{V}(\mathcal{J}) = \mathcal{W}$, $\mathcal{E}(\mathcal{J}^{-1}) = \{(\mathfrak{p}, \mathfrak{q}) \in \mathcal{W} \times \mathcal{W} : (\mathfrak{q}, \mathfrak{p}) \in \mathcal{E}(\mathcal{J})\}$ and $\mathcal{E}(\tilde{\mathcal{J}}) = \mathcal{E}(\mathcal{J}) \cup \mathcal{E}(\mathcal{J}^{-1})$.

To show main results, some symbols and definitions, which is introduced, are also required in next section. Presume that $\mathcal{H}, \mathcal{F} \neq \emptyset$ are two subset of a GMS $(\mathcal{W}, \mathcal{D})$.

$$\mathsf{dist}(\mathcal{H}, \mathcal{F}) = \inf\big\{\mathcal{D}(\mathfrak{p}, \mathfrak{q}) : \mathfrak{p} \in \mathcal{H}, \mathfrak{q} \in \mathcal{F}\big\}.$$

- Assume that $\mathcal{K} : \mathcal{W} \to \mathcal{W}$ is a mapping. We mean $C_{\mathcal{K}}$ by the set of all points $\mathfrak{p} \in \mathcal{W}$ provided that $(\mathcal{K}^m\mathfrak{p}, \mathcal{K}^n\mathfrak{p})$ is an edge of $\tilde{\mathcal{J}}$ for each $m, n \in \mathbb{N} \cup \{0\}$; that is,

$$C_{\mathcal{K}} = \big\{\mathfrak{p} \in \mathcal{W} : (\mathcal{K}^m\mathfrak{p}, \mathcal{K}^n\mathfrak{p}) \in \mathcal{E}(\tilde{\mathcal{J}}) \quad m, n = 0, 1, \cdots\big\}.$$

  Notice that $C_{\mathcal{K}}$ may become an empty set. For example, take $\mathbb{R}$ along with the usual Euclidean metric and a graph G given by $\mathsf{V}(\mathcal{J}) = \mathbb{R}$ and $\mathcal{E}(\mathcal{J}) = \{(\mathfrak{p}, \mathfrak{p}) : \mathfrak{p} \in \mathbb{R}\}$. If $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ is defined by $\mathcal{K}\mathfrak{p} = \mathfrak{p} + 1$ for any $\mathfrak{p} \in \mathbb{R}$, clearly $C_{\mathcal{K}} = \emptyset$.

Definition 1.6 [11] Presume that $(\mathcal{W}, \mathcal{D})$ is a GMS. A mapping $\mathcal{K} : \mathcal{W} \to \mathcal{W}$ is known as an orbitally $\mathcal{J}$-continuous mapping on $\mathcal{W}$ whenever $\mathcal{K}^{\mathfrak{q}_n}\mathfrak{p} \to \mathfrak{q}$ implies $\mathcal{K}(\mathcal{K}^{\mathfrak{q}_n}\mathfrak{p}) \to \mathcal{K}\mathfrak{q}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathcal{W}$ and sequences $\{\mathfrak{q}_n\}$ of natural numbers so that $(\mathcal{K}^{\mathfrak{q}_n}\mathfrak{p}, \mathcal{K}^{\mathfrak{q}_n+1}\mathfrak{p}) \in \mathcal{E}(\mathcal{J})$ for every $n \in \mathbb{N}$.

Definition 1.7 [11] Taking $(\mathcal{W}, \mathcal{D})$ is a GMS, we say $\mathcal{J}$ is a C-graph on $\mathcal{W}$ if $\mathfrak{p} \in \mathcal{W}$ and $\{\mathfrak{p}_n\}$ is a sequence in $\mathcal{W}$ so that $\mathfrak{p}_n \to \mathfrak{p}$ and $(\mathfrak{p}_{n+1}, \mathfrak{p}_n) \in \mathcal{E}(\mathcal{J})$ for each $n \in \mathbb{N}$, then there is a subsequence $\{\mathfrak{p}_{2n_i}\}$ of $\{\mathfrak{p}_n\}$ such that $(\mathfrak{p}_{2n_i}, \mathfrak{p}) \in \mathcal{E}(\mathcal{J})$ for every $i \in \mathbb{N}$.

In the sequel, we assume that (X, d) is a metric space endowed with graph. We denote by λ the Lebesgue measure on the Borel σ-algebra of $[0,+\infty)$. For a Borel set $\mathcal{F} = [\mathfrak{p}, \mathfrak{q}]$, we will use the notation $\int_{\mathfrak{p}}^{\mathfrak{q}} \chi(t)dt$ to show the Lebesgue integral of a function χ on $\mathcal{F}$. We employ a class Υ consisting of all functions $\chi : [0,+\infty) \to [0,+\infty)$ satisfying the following properties:

(Υ1) χ is Lebesgue-integrable on $[0,+\infty)$;
(Υ2) The value of the Lebesgue integral $\int_0^{\nu} \chi(t)dt$ is positive and finite for all $\nu > 0$.

The next lemma embodies some important properties of functions of the class Υ which we need in the sequel.

Lemma 1.8 [3] Let $\chi : [0,+\infty) \to [0,+\infty)$ be a function in the class Υ and $\{\mathfrak{p}_n\}$ be a sequence of nonnegative real numbers. Then the following statements hold:

1. If $\int_0^{\mathfrak{p}_n} \chi(t)dt \to 0$ as $n \to \infty$, then $\mathfrak{p}_n \to 0$ as $n \to \infty$;
2. If $\{\mathfrak{p}_n\}$ is monotone and converges to some $\mathfrak{p} \geqslant 0$, then $\int_0^{\mathfrak{p}_n} \chi(t)dt \to \int_0^{\mathfrak{p}} \chi(t)dt$ as $n \to \infty$.

2. Best proximity points

In the sequel, note that $(\mathcal{H}, \mathcal{F})$ will be a pair of nonempty subsets of $\mathcal{W}$. Now, we are ready to give the definition of $\mathcal{J}$-integral Banach type contractions in metric spaces with a graph which is motivated by [[5], Theorem 2.1].

Definition 2.1 Assume $(\mathcal{W}, \mathcal{D})$ is a GMS. A mapping $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ is known as cyclic $\mathcal{J}$-integral Banach type contractions on $\mathcal{H}$ if $\mathcal{K}$ is cyclic and there exists $\chi \in \Upsilon$ and constant $\eta \in (0,1)$ such that contractive condition

$$\int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}, \mathcal{K}^2\mathfrak{q})} \chi(t)dt \leqslant \eta \int_0^{\mathcal{D}(\mathfrak{p}, \mathcal{K}\mathfrak{q})} \chi(t)dt \tag{1}$$

is hold for all $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{H} \times \mathcal{H}$ with $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{E}(\mathcal{J})$.

Now, we are ready to state and prove first fundamental theorem of this section.

Theorem 2.2 Assume $(\mathcal{W}, \mathcal{D})$ is a GMS, $\mathcal{H}$ and $\mathcal{F}$ are closed subsets and $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ is a cyclic $\mathcal{J}$-integral Banach type contractions, where $\mathcal{K}^2$ preserves the edges of $\mathcal{J}$ on $\mathcal{H}$, $C_{\mathcal{K}}|_{\mathcal{H}} \neq \emptyset$ and $\mathfrak{p}_{n+1} = \mathcal{K}\mathfrak{p}_n$. If $\mathcal{J}$ is C-graph on $\mathcal{H}$ and $\{\mathfrak{p}_{2n}\}$ has a convergent subsequence in $\mathcal{H}$, then $\mathcal{K}$ has a bpp $\mathfrak{p}^* \in \mathcal{H}$.

Proof. As $C_{\mathcal{K}}|_{\mathcal{H}} \neq \emptyset$, assume $\mathfrak{p}_0 \in C_{\mathcal{K}}$ with $\mathfrak{p}_0 \in \mathcal{H}$. We have $(\mathfrak{p}_0, \mathcal{K}^2\mathfrak{p}_0) \in \mathcal{E}(\mathcal{J})$ and since $\mathcal{K}^2$ preserves the edges of $\mathcal{J}$ on $\mathcal{H}$, $(\mathfrak{p}_{2n}, \mathfrak{p}_{2n+2}) \in \mathcal{E}(\mathcal{J})$ for $n = 0, 1, \cdots$. Since $(\mathfrak{p}_{2n}, \mathfrak{p}_{2n+2}) \in \mathcal{E}(\mathcal{J})$ for every $n \in \mathbb{N} \cup \{0\}$ and by (1) on $\mathcal{H}$, we get

$$\begin{aligned}
\int_0^{\mathcal{D}(\mathfrak{p}_{2n}, \mathfrak{p}_{2n+1})} \chi(t)dt &= \int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}_{2n}, \mathcal{K}^2\mathfrak{p}_{2n-2})} \chi(t)dt \\
&= \eta \int_0^{\mathcal{D}(\mathfrak{p}_{2n}, \mathcal{K}\mathfrak{p}_{2n-2})} \chi(t)dt \\
&\leqslant \eta \int_0^{\mathcal{D}(\mathfrak{p}_{2n}, \mathfrak{p}_{2n-1})} \chi(t)dt.
\end{aligned}$$

Since integral is nondecreasing, then $\{\mathcal{D}(\mathfrak{p}_{2n-2}, \mathfrak{p}_{2n-1}\}$ is a decreasing sequence. Consider $\mathcal{D}(\mathfrak{p}_{2n-2}, \mathfrak{p}_{2n-1}) \to \mathfrak{u}$. Since for all $n = 1, 2, \cdots$, $\mathcal{D}(\mathcal{H}, \mathcal{F}) \leqslant \mathcal{D}(\mathfrak{p}_{2n-2}, \mathfrak{p}_{2n-1})$, we have $\mathcal{D}(\mathfrak{p}_{2n-2}, \mathfrak{p}_{2n-1}) \to \mathcal{D}(\mathcal{H}, \mathcal{F})$.

Now, suppose $\{\mathfrak{p}_{2n_j}\}$ is a subsequence of $\{\mathfrak{p}_{2n}\}$ converging to $\mathfrak{p}^* \in \mathcal{H}$. Then

$$\begin{aligned} \mathcal{D}(\mathcal{H}, \mathcal{F}) &\leqslant \mathcal{D}(\mathfrak{p}^*, \mathfrak{p}_{2n_j-1}) \\ &\leqslant \mathcal{D}(\mathfrak{p}^*, \mathfrak{p}_{2n_j}) + \mathcal{D}(\mathfrak{p}_{2n_j}, \mathfrak{p}_{2n_j-1}). \end{aligned}$$

Next, taking limit, we get $\lim_{j\to\infty} \mathcal{D}(\mathfrak{p}^*, \mathfrak{p}_{2n_j-1}) = \mathcal{D}(\mathcal{H}, \mathcal{F})$. As $\mathcal{K}^2$ keeps the edges of $\mathcal{J}$ and $\mathcal{J}$ is a C-graph, $(\mathfrak{p}_{2n_j}, \mathfrak{p}^*) \in \mathcal{E}(\mathcal{J})$ for all $j \in \mathbb{N}$. Using (1), we obtain

$$\begin{aligned} \int_0^{\mathcal{D}(\mathfrak{p}_{2n_j+1}, \mathcal{K}\mathfrak{p}^*)} \chi(t)dt &= \int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}^*, \mathcal{K}^2\mathfrak{p}_{2n_j-1})} \chi(t)dt \\ &\leqslant \eta \int_0^{\mathcal{D}(\mathfrak{p}^*, \mathcal{K}\mathfrak{p}_{2n_j-1})} \chi(t)dt \\ &= \eta \int_0^{\mathcal{D}(\mathfrak{p}^*, \mathfrak{p}_{2n_j})} \chi(t)dt. \end{aligned}$$

So $\{\mathfrak{p}_{2n_j+1}\}$ possesses a subsequence converging to $\mathcal{K}\mathfrak{p}^*$, which concludes

$$\mathcal{D}(\mathfrak{p}^*, \mathcal{K}\mathfrak{p}^*) = \lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2n_j}, \mathfrak{p}_{2n_j+1}) = \mathcal{D}(\mathcal{H}, \mathcal{F}).$$

■

Theorem 2.3 Assume $(\mathcal{W}, \mathcal{D})$ is a GMS, $\mathcal{H}$ is complete and $(\mathcal{H}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{H})$ have the UC propert. In addition, assume $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ is a cyclic $\mathcal{J}$-integral Banach type contractions on $\mathcal{H}$ (and $\mathcal{F}$) in which $\mathcal{K}$ and $\mathcal{K}^2$ preserve the edges of $\mathcal{J}$ on $\mathcal{H}$. If $\mathcal{K}$ is orbitally $\mathcal{J}$-continuous on $\mathcal{H}$ or $\mathcal{J}$ is a C-graph on $\mathcal{H}$, $\mathcal{K}$ has a bpp $\mathfrak{p}^* \in \mathcal{H}$ whenever there is $\mathfrak{p}_0 \in \mathcal{H}$ with $\mathfrak{p}_0 \in C_{\mathcal{K}}$.

Proof. Assume $\mathfrak{p}_0 \in C_{\mathcal{K}}$ with $\mathfrak{p}_0 \in \mathcal{H}$. Since $\mathcal{K}$ and $\mathcal{K}^2$ preserve the edges of $\mathcal{J}$ on $\mathcal{H}$ and $(\mathfrak{p}_0, \mathcal{K}^2\mathfrak{p}_0) \in \mathcal{E}(\mathcal{J})$ on $\mathcal{H}$, we have $(\mathfrak{p}_{2n}, \mathfrak{p}_{2n+2}) \in \mathcal{E}(\mathcal{J})$ and $(\mathfrak{p}_{2n+1}, \mathfrak{p}_{2n+3}) \in \mathcal{E}(\mathcal{J})$ for $n = 0, 1, \cdots$. As the similar proof is done in Theorem 2.2, we will have $\mathcal{D}(\mathfrak{p}_{2n}, \mathfrak{p}_{2n+1}) \to \mathcal{D}(\mathcal{H}, \mathcal{F})$ and $\mathcal{D}(\mathfrak{p}_{2n+2}, \mathfrak{p}_{2n+1}) \to \mathcal{D}(\mathcal{H}, \mathcal{F})$. From the property UC for $(\mathcal{H}, \mathcal{F})$, we obtain $\mathcal{D}(\mathfrak{p}_{2n}, \mathfrak{p}_{2n+2}) \to 0$. Also, since $(\mathcal{F}, \mathcal{H})$ has the property UC we conclude that $\mathcal{D}(\mathfrak{p}_{2n+1}, \mathfrak{p}_{2n+3}) \to 0$. Now, we show that for all $\mu > 0$, there is a $N \in \mathbb{N}$ so that for every $m > n \geqslant N$

$$\mathcal{D}^*(\mathfrak{p}_{2m}, \mathfrak{p}_{2n+1}) < \mu, \tag{2}$$

where $\mathcal{D}^*(\mathfrak{p}, \mathfrak{q}) = \mathcal{D}(\mathfrak{p}, \mathfrak{q}) - \mathcal{D}(\mathcal{H}, \mathcal{F})$ for all $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{H} \times \mathcal{F}$. To contrary, assume there is $\mu_0 > 0$ such that for each $j \geqslant 1$, there is $m_j > n_j \geqslant j$ satisfying $\mathcal{D}^*(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2n_j+1}) \geqslant \mu_0$ and $\mathcal{D}^*(\mathfrak{p}_{2m_j-2}, \mathfrak{p}_{2n_j+1}) < \mu_0$. Then

$$\begin{aligned} \mu_0 &\leqslant \mathcal{D}^*(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2n_j+1}) \\ &\leqslant \mathcal{D}(\mathfrak{p}_{2m_j-2}, \mathfrak{p}_{2m_j}) + \mathcal{D}^*(\mathfrak{p}_{2m_j-2}, \mathfrak{p}_{2n_j+1}) \\ &\leqslant \mathcal{D}(\mathfrak{p}_{2m_j-2}, \mathfrak{p}_{2m_j}) + \mu_0, \end{aligned}$$

so $\lim_{k\to\infty} \mathcal{D}^*(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2n_j+1}) = \mu_0$. Since $\mathcal{K}$ and $\mathcal{K}^2$ preserve the edges of $\mathcal{J}$ on $\mathcal{H}$,

$$\begin{aligned}
\lim_{j\to\infty} \int_0^{\mathcal{D}(\mathfrak{p}_{2m_j+2},\mathfrak{p}_{2n_j+3})} \chi(t)dt &= \lim_{j\to\infty} \int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}_{2m_j+1},\mathcal{K}^2\mathfrak{p}_{2n_j+1})} \chi(t)dt \qquad (3)\\
&\leqslant \eta \lim_{j\to\infty} \int_0^{\mathcal{D}(\mathfrak{p}_{2m_j+1},\mathcal{K}\mathfrak{p}_{2n_j+1})} \chi(t)dt\\
&\leqslant \eta \lim_{j\to\infty} \int_0^{\mathcal{D}(\mathfrak{p}_{2m_j+1},\mathfrak{p}_{2n_j+2})} \chi(t)dt\\
&\leqslant \lim_{j\to\infty} \int_0^{\mathcal{D}(\mathfrak{p}_{2m_j+1},\mathfrak{p}_{2n_j+2})} \chi(t)dt\\
&= \lim_{j\to\infty} \int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}_{2m_j},\mathcal{K}^2\mathfrak{p}_{2n_j})} \chi(t)dt\\
&\leqslant \eta \lim_{j\to\infty} \int_0^{\mathcal{D}(\mathfrak{p}_{2m_j},\mathfrak{p}_{2n_j+1})} \chi(t)dt\\
&\leqslant \lim_{j\to\infty} \int_0^{\mathcal{D}(\mathfrak{p}_{2m_j},\mathfrak{p}_{2n_j+1})} \chi(t)dt.
\end{aligned}$$

Since integral is nondecreasing, we get $\lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2m_j+2}, \mathfrak{p}_{2n_j+3}) \leqslant \lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2n_j+1})$, and so by (3), we obtain

$$\begin{aligned}
\underbrace{\lim_{j\to\infty} \mathcal{D}^*(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2n_j+1})}_{=\mu_0} &\leqslant \underbrace{\lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2m_j+2})}_{=0} + \lim_{j\to\infty} \mathcal{D}^*(\mathfrak{p}_{2m_j+2}, \mathfrak{p}_{2n_j+3})\\
&\quad + \underbrace{\lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2n_j+1}, \mathfrak{p}_{2n_j+3})}_{=0}\\
&\leqslant \underbrace{\lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2m_j+2})}_{=0} + \underbrace{\lim_{j\to\infty} \mathcal{D}^*(\mathfrak{p}_{2m_j}, \mathfrak{p}_{2n_j+1})}_{\mu_0}\\
&\quad + \underbrace{\lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2n_j+1}, \mathfrak{p}_{2n_j+3})}_{=0}.
\end{aligned}$$

This implies that

$$\lim_{j\to\infty} \mathcal{D}^*(\mathfrak{p}_{2m_j+2}, \mathfrak{p}_{2n_j+3}) = \mu_0 \text{ and } \lim_{j\to\infty} \mathcal{D}(\mathfrak{p}_{2m_j+2}, \mathfrak{p}_{2n_j+3}) = \mu_0 + \mathcal{D}(\mathcal{H}, \mathcal{F}).$$

Now,

$$\begin{aligned}
\int_0^{\mu_0+\mathcal{D}(\mathcal{H},\mathcal{F})}\chi(t)dt &= \lim_{j\to\infty}\int_0^{\mathcal{D}(\mathfrak{p}_{2m_j+2},\mathfrak{p}_{2n_j+3})}\chi(t)dt\\
&= \lim_{j\to\infty}\int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}_{2m_j+1},\mathcal{K}^2\mathfrak{p}_{2n_j+1})}\chi(t)dt\\
&\leqslant \eta\lim_{j\to\infty}\int_0^{\mathcal{D}(\mathfrak{p}_{2m_j+1},\mathfrak{p}_{2n_j+2})}\chi(t)dt\\
&= \eta\lim_{j\to\infty}\int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}_{2m_j},\mathcal{K}^2\mathfrak{p}_{2n_j})}\chi(t)dt\\
&\leqslant \eta^2\lim_{j\to\infty}\int_0^{\mathcal{D}(\mathfrak{p}_{2m_j},\mathcal{K}\mathfrak{p}_{2n_j})}\chi(t)dt\\
&= \eta^2\lim_{j\to\infty}\int_0^{\mathcal{D}(\mathfrak{p}_{2m_j},\mathfrak{p}_{2n_j+1})}\chi(t)dt\\
&= \eta^2\int_0^{\mu_0+\mathcal{D}(\mathcal{H},\mathcal{F})}\chi(t)dt,
\end{aligned}$$

concluding $\mu_0+\mathcal{D}(\mathcal{H},\mathcal{F})\leqslant\eta^2\big(\mu_0+\mathcal{D}(\mathcal{H},\mathcal{F})\big)$, which is impossible as $\eta\in(0,1)$, so (2) holds and

$$\lim_{m\to\infty}\sup_{n\geqslant m}\mathcal{D}^*(\mathfrak{p}_{2m},\mathfrak{p}_{2n+1})=0.$$

Since $(\mathcal{H},\mathcal{F})$ has the property UC and by Lemma 1.8, $\{\mathfrak{p}_{2n}\}$ is a Cauchy sequence in $\mathcal{H}$. Because $\mathcal{H}$ is complete, $\{\mathfrak{p}_{2n}\}$ converges to some point $\mathfrak{p}^*\in\mathcal{H}$.

To continue, note first that from $\mathfrak{p}\in C_{\mathcal{K}}$, we get $(\mathfrak{p}_{2n},\mathfrak{p}_{2n+1})\in\mathcal{E}(\mathcal{J})$ for every $n\in\mathbb{N}$. When $\mathcal{K}$ is orbitally $\mathcal{J}$-continuous on $\mathcal{H}$, $\mathfrak{p}_{2n}\to\mathfrak{p}^*$ implies $\mathcal{K}(\mathfrak{p}_{2n})\to\mathcal{K}\mathfrak{p}^*$. Thus,

$$\mathcal{D}(\mathfrak{p}^*,\mathcal{K}\mathfrak{p}^*)=\lim_{n\to\infty}\mathcal{D}(\mathfrak{p}_{2n},\mathfrak{p}_{2n+1})=\mathcal{D}(\mathcal{H},\mathcal{F}),$$

i.e. $\mathfrak{p}^*$ is a bpp. Second, let $\mathcal{J}$ be a C-graph. Since $\mathfrak{p}_{2n}\to\mathfrak{p}^*$, there is a strictly increasing sequence $\{n_j\}$ of positive integers such that $(\mathfrak{p}_{2n_j},\mathfrak{p}^*)\in\mathcal{E}(\mathcal{J})$ for all $k\in\mathbb{N}$. As $\mathcal{K}$ satisfies (1) for the graph $\mathcal{J}$, we get

$$\begin{aligned}
\lim_{j\to\infty}\int_0^{\mathcal{D}(\mathfrak{p}_{2n_j+1},\mathcal{K}\mathfrak{p}^*)}\chi(t)dt &\leqslant \lim_{j\to\infty}\int_0^{\mathcal{D}(\mathcal{K}\mathfrak{p}^*,\mathcal{K}^2\mathfrak{p}_{2n_j-1})}\chi(t)dt\\
&\leqslant \eta\lim_{j\to\infty}\int_0^{\mathcal{D}(\mathfrak{p}^*,\mathfrak{p}_{2n_j})}\chi(t)dt.
\end{aligned}$$

Thus, $\{\mathfrak{p}_{2n_j+1}\}$ has a subsequence converging to $\mathcal{K}\mathfrak{p}^*$. This implies that

$$\mathcal{D}(\mathfrak{p}^*,\mathcal{K}\mathfrak{p}^*)=\lim_{n\to\infty}\mathcal{D}(\mathfrak{p}_{2n_j},\mathfrak{p}_{2n_j+1})=\mathcal{D}(\mathcal{H},\mathcal{F}).$$

■

***Example* 2.4** Take $\mathcal{W} = \mathbb{R}^2$ and usual metric

$$\mathcal{D}\big((\mathfrak{p}_1, \mathfrak{q}_1), (\mathfrak{p}_2, \mathfrak{q}_2)\big) = \sqrt{(\mathfrak{p}_1 - \mathfrak{p}_2)^2 + (\mathfrak{q}_1 - \mathfrak{q}_2)^2}$$

for $(\mathfrak{p}_1, \mathfrak{q}_1), (\mathfrak{p}_2, \mathfrak{q}_2) \in \mathbb{R}^2$ and set

$$\mathcal{H} = \big\{(\mathfrak{p}, 1) : \mathfrak{p} \in [0,1]\big\}\ ,\ \mathcal{F} = \big\{(\mathfrak{q}, 0) : \mathfrak{q} \in [0,1]\big\}.$$

Define the function $\chi : [0, +\infty) \to [0, +\infty)$ by the rule $\chi(t) = t^\alpha$ for all $t > 0$ where $\alpha \geqslant 0$ is constant. It is clear that χ is Lebesgue integrable on $[0, +\infty)$ and $\int_0^\nu \chi(t)dt = \frac{\nu^{\alpha+1}}{\alpha+1}$ which is positive and finite for all $\nu > 0$, that is, $\chi \in \Upsilon$. Also, define $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ by

$$\mathcal{K}(\mathfrak{p}, 1) = \begin{cases} (0,0),\ 0 \leqslant \mathfrak{p} < 1 \\ (\frac{1}{6}, 0),\ \mathfrak{p} = 1 \end{cases}$$

for $(\mathfrak{p}, 1) \in \mathcal{H}$ and

$$\mathcal{K}(\mathfrak{q}, 0) = \begin{cases} (0,1),\ 0 \leqslant \mathfrak{q} < 1 \\ (\frac{1}{6}, 1),\ \mathfrak{q} = 1 \end{cases}.$$

for $(\mathfrak{q}, 0) \in \mathcal{F}$. Note that for $(1,1), (\frac{2}{3}, 1) \in \mathbb{R}^2$, by (1), we have

$$\int_0^{\mathcal{D}\left(\mathcal{K}(\frac{2}{3},1), \mathcal{K}^2(1,1)\right)} \chi(t)dt = \frac{1}{\alpha+1} > \eta \cdot \frac{1}{\alpha+1} = \eta \int_0^{\mathcal{D}\left((\frac{2}{3},1), \mathcal{K}(1,1)\right)} \chi(t)dt$$

for all $\eta \in [0,1)$.

Consequently, (1) is not true for the mapping $\mathcal{K}$ when we take an usual metric (non a GMS) on $\mathcal{H}$.

Now, take a graph $\mathcal{J}$ by $\mathsf{V}(\mathcal{J}) = \mathbb{R}^2$ and

$$\mathcal{E}(\mathcal{J}) = \big\{\big((\mathfrak{p}_1, \mathfrak{p}_2), (\mathfrak{p}_1, \mathfrak{p}_2)\big) : (\mathfrak{p}_1, \mathfrak{p}_2) \in \mathbb{R}^2\big\} \cup \big\{\big((0,1),(1,1)\big), \big((1,1),(0,1)\big), \big((0,0),(1,0)\big), \big((1,0),(0,0)\big)\big\}$$

Then $(\mathbb{R}^2, \mathcal{D})$ is a complete GMS endowed by $\mathcal{J}$. Evidently, $\mathcal{K}$ is orbitally $\mathcal{J}$-continuous. Also, it's clear for $\mathfrak{p}, \mathfrak{q} \in [0,1)$ we have

$$\int_0^{\mathcal{D}\left(\mathcal{K}(\mathfrak{p},1), \mathcal{K}^2(\mathfrak{p},1)\right)} \chi(t)dt = \frac{1}{\alpha+1} \leqslant \frac{1}{\sqrt{2}} \frac{(\sqrt{\mathfrak{p}^2+1})^{\alpha+1}}{\alpha+1} = \frac{1}{\sqrt{2}} \int_0^{\mathcal{D}\left((\mathfrak{p},1), \mathcal{K}(\mathfrak{p},1)\right)} \chi(t)dt$$

and

$$\int_0^{\mathcal{D}\left(\mathcal{K}(\mathfrak{q},0), \mathcal{K}^2(\mathfrak{q},0)\right)} \chi(t)dt = \frac{1}{\alpha+1} \leqslant \frac{1}{\sqrt{\frac{10}{11}}} \frac{(\sqrt{\mathfrak{q}^2+1})^{\alpha+1}}{\alpha+1} = \frac{1}{\sqrt{\frac{10}{11}}} \int_0^{\mathcal{D}\left((\mathfrak{q},0), \mathcal{K}(\mathfrak{q},0)\right)} \chi(t)dt$$

Moreover, for $\mathfrak{p}, \mathfrak{q} = 1$,

$$\int_0^{\mathcal{D}\left(\mathcal{K}(1,1),\mathcal{K}^2(1,1)\right)} \chi(t)dt = \frac{(\frac{\sqrt{37}}{6})^{\alpha+1}}{\alpha+1} \leqslant \frac{1}{\sqrt{\frac{10}{11}}} \frac{(\frac{\sqrt{61}}{6})^{\alpha+1}}{\alpha+1} = \frac{1}{\sqrt{\frac{10}{11}}} \int_0^{\mathcal{D}\left((1,1),\mathcal{K}(1,1)\right)} \chi(t)dt$$

and

$$\int_0^{\mathcal{D}\left(\mathcal{K}(\mathfrak{q},0),\mathcal{K}^2(\mathfrak{q},0)\right)} \chi(t)dt = \frac{1}{\alpha+1} \leqslant \frac{1}{\sqrt{\frac{10}{11}}} \frac{(\sqrt{\mathfrak{q}^2+1})^{\alpha+1}}{\alpha+1} = \frac{1}{\sqrt{\frac{10}{11}}} \int_0^{\mathcal{D}\left((\mathfrak{q},0),\mathcal{K}(\mathfrak{q},0)\right)} \chi(t)dt.$$

Also, we have

$$\int_0^{\mathcal{D}\left(\mathcal{K}(0,1),\mathcal{K}^2(1,1)\right)} \chi(t)dt = \frac{1}{\alpha+1} \leqslant \frac{1}{\sqrt{\frac{10}{11}}} \frac{(\frac{\sqrt{37}}{6})^{\alpha+1}}{\alpha+1} = \frac{1}{\sqrt{\frac{10}{11}}} \int_0^{\mathcal{D}\left((0,1),\mathcal{K}(1,1)\right)} \chi(t)dt$$

and

$$\int_0^{\mathcal{D}\left(\mathcal{K}(0,0),\mathcal{K}^2(1,0)\right)} \chi(t)dt = \frac{1}{\alpha+1} \leqslant \frac{1}{\sqrt{\frac{10}{11}}} \frac{(\frac{\sqrt{37}}{6})^{\alpha+1}}{\alpha+1} = \frac{1}{\sqrt{\frac{10}{11}}} \int_0^{\mathcal{D}\left((0,0),\mathcal{K}(1,0)\right)} \chi(t)dt,$$

where $\eta = \sqrt{\frac{10}{11}}$. Thus, (1) is valid for the mapping $\mathcal{K}$ on $\mathcal{H}$ (and $\mathcal{F}$). Therefore, all hypotheses of Theorem 2.3 fulfill and $\mathcal{K}$ has a bpp, being $\vartheta = (0,1)$ and $\gamma = (0,0)$.

Taking only the condition orbitally $\mathcal{J}$-continuity version of the mapping $\mathcal{K}$ from Theorem2.3, we can extract some attractive corollaries as follows: First, take $\mathcal{J} = \mathcal{J}_0$ in which $\mathcal{J}_0$ is a complete graph, i.e. $\mathcal{J}_0$ is a graph with $\mathsf{V}(\mathcal{J}_0) = \mathcal{W}$ and $\mathcal{E}(\mathcal{J}_0) = \mathcal{W} \times \mathcal{W}$.

Corollary 2.5 Let $(\mathcal{W}, \mathcal{D})$ be a GMS, $\mathcal{H}$ be complete and $(\mathcal{H}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{H})$ satisfy the property UC. Assume $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ is a cyclic integral Banach type contractions on $\mathcal{H}$ (and $\mathcal{F}$). Then whenever $\mathcal{K}$ is continuous on $\mathcal{H}$, $\mathcal{K}$ has a bpp $\mathfrak{p}^* \in \mathcal{H}$.

Second, presume $(\mathcal{W}, \preceq)$ is a POS and $\mathcal{J}_1$ is a graph on $\mathcal{W}$ in which $\mathsf{V}(\mathcal{J}_1) = \mathcal{W}$ and $\mathcal{E}(\mathcal{J}_1) = \{(\mathfrak{p}, \mathfrak{q}) \in \mathcal{W} \times \mathcal{W} : \mathfrak{p} \preceq \mathfrak{q}\}$. If $\mathcal{J} = \mathcal{J}_1$ in Theorem 2.3, then we gain the second corollary.

Corollary 2.6 Let $(\mathcal{W}, \mathcal{D})$ be a partially ordered MS, $\mathcal{H}$ be complete and $(\mathcal{H}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{H})$ satisfy the property UC. Assume that $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ is a cyclic $\mathcal{J}_1$-integral Banach type contractions on $\mathcal{H}$ (and $\mathcal{F}$) such that $\mathcal{K}$ and $\mathcal{K}^2$ are nondecersing on $\mathcal{H}$.

Then whenever $\mathcal{K}$ is orbitally $\mathcal{J}_1$-continuous on $\mathcal{H}$ or $\mathcal{J}_1$ is a C-graph on $\mathcal{H}$, $\mathcal{K}$ has a bpp $\mathfrak{p}^* \in \mathcal{H}$ if there exists $\mathfrak{p}_0 \in \mathcal{H}$ with $\mathfrak{p}_0 \preceq \mathcal{K}^2\mathfrak{p}_0$.

For our next consequence, presume $(\mathcal{W}, \preceq)$ is a POS and $\mathcal{J}_2$ is a graph on $\mathcal{W}$ in which $\mathsf{V}(\mathcal{J}_2) = \mathcal{W}$ and $\mathcal{E}(\mathcal{J}_2) = \{(\mathfrak{p}, \mathfrak{q}) \in \mathcal{W} \times \mathcal{W} : \mathfrak{p} \preceq \mathfrak{q} \vee \mathfrak{q} \preceq \mathfrak{p}\}$. If we set $\mathcal{J} = \mathcal{J}_2$ in Theorem 2.3, then the following version of our bpp theorem in metric spaces endowed with endowed with graph $\mathcal{J}_2$.

Corollary 2.7 Presume $(\mathcal{W}, \preceq)$ is POS, $(\mathcal{W}, \mathcal{D})$ is a MS such that $\mathcal{H}$ is complete and $(\mathcal{H}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{H})$ satisfy the property UC. Let $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ be a cyclic $\mathcal{J}_2$-integral Banach type contractions on $\mathcal{H}$ (and $\mathcal{F}$) and for $\mathfrak{p}, \mathfrak{q} \in \mathcal{H}$, if $\mathfrak{p}$ and $\mathfrak{q}$ are comparable we have $\mathcal{K}^2\mathfrak{p}$ and $\mathcal{K}^2\mathfrak{q}$ (also $\mathcal{K}\mathfrak{p}$ and $\mathcal{K}\mathfrak{q}$) are comparable. Then whenever $\mathcal{K}$ is orbitally $\mathcal{J}_2$-continuous on $\mathcal{H}$ or $\mathcal{J}_2$ is a C-graph on $\mathcal{H}$, $\mathcal{K}$ has a bpp $\mathfrak{p}^* \in \mathcal{H}$ if there exists $\mathfrak{p}_0 \in \mathcal{H}$ where $\mathfrak{p}_0$ and $\mathcal{K}^2\mathfrak{p}_0$ are comparable.

At last, consider a fixed value $\vartheta > 0$. Recall that $\mathfrak{p}, \mathfrak{q} \in \mathcal{W}$ are said to be ϑ-close if $\mathcal{D}(\mathfrak{p}, \mathfrak{q}) \preceq \vartheta$. Taking $\mathcal{J}_\vartheta$ by $\mathsf{V}(\mathcal{J}_\vartheta) = \mathcal{W}$ and $\mathcal{E}(\mathcal{J}_\vartheta) = \{(\mathfrak{p}, \mathfrak{q}) \in \mathcal{W} \times \mathcal{W} : \mathcal{D}(\mathfrak{p}, \mathfrak{q}) \preceq \vartheta\}$, we get the latest corollary of this section regarding $\mathcal{J} = \mathcal{J}_\vartheta$ in Theorem 2.2.

Corollary 2.8 Let $(\mathcal{W}, \mathcal{D})$ be a GMS endowed with graph $\mathcal{J}_\vartheta$ and $\mathcal{H}$ be complete. Assume $\mathcal{K} : \mathcal{H} \cup \mathcal{F} \to \mathcal{H} \cup \mathcal{F}$ is a cyclic $\mathcal{J}_\vartheta$-integral Banach type contractions on $\mathcal{H}$ (and $\mathcal{F}$) and if $\mathfrak{p}$ and $\mathfrak{q}$ are ϑ-close for $\mathfrak{p}, \mathfrak{q} \in \mathcal{H}$, we have $\mathcal{K}^2\mathfrak{p}$ and $\mathcal{K}^2\mathfrak{q}$ (also $\mathcal{K}\mathfrak{p}$ and $\mathcal{K}\mathfrak{q}$) are ϑ-close. Then whenever $\mathcal{K}$ is orbitally $\mathcal{J}_\vartheta$-continuous on $\mathcal{H}$ or $\mathcal{J}_\vartheta$ is a C-graph on $\mathcal{H}$, $\mathcal{K}$ has a bpp $\mathfrak{p}^* \in \mathcal{H}$ if there exists $\mathfrak{p}_0 \in \mathcal{H}$ where $\mathfrak{p}_0$ and $\mathcal{K}^2\mathfrak{p}_0$ are ϑ-close.

Similarly, all corollaries stated above hold for Theorem 2.2.

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