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On some curvature functionals over homogeneous Siklos space-times

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Abstract. Some curvature functionals which are defined according to the quadratic curvature invariants were studied on a special class of space-times. We exactly determine metrics that are critical for those considering curvature functionals, through homogeneous classes.

Keywords: Siklos space-times, quadratic curvature functional, critical metric, homogeneous space.

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1. Introduction

The British mathematician, Stephen Siklos, introduced a novel family of space-times in 1985. These spaces were identified as exact solutions of the Einstein field equations and he called them "Lobatchevski plane waves" [23]. Siklos spaces-times are gravitational waves with a negative cosmological constant, propagating in the anti-de-Sitter universe [19]. They fall in the conformal class of pp-waves and be developed in Petrov classification as space-times of type N [24]. One can assign a non-twisting null Killing field to all Siklos space-times. In a vacuum, Siklos space-times belong to a particular category of shear-free solutions of Kundt type which are non-twisting and non-expanding [13].

Siklos metrics are ubiquitous in mathematical physics and geometry. In particular,

- Siklos space-times can be considered as exact gravitational waves propagating in the anti-de Sitter universe [20].
- A classification of plane-fronted waves in space-times was proposed in [18], which is related to the sign of the cosmological constant Λ and a second-order invariant

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(specified by the sign of some constant k) that is related to the compatibility of null rays. Siklos space-times are found as one of the two sub-cases with $\Lambda < 0$ and $k = 0$ in this category and occur simultaneously by the subclass $(IV)_0$ of Kundt space-times.
- Siklos space-times are included in the study of all non-twisting type N solutions of Einstein field vacuum equations [3, 4]. A physical interpretation was obtained as a result of this general study by inspecting the geodesic equation. deviation.
- Siklos metrics are studied as impulsive gravitational waves propagating in an anti-de-Sitter background in [21].
- The photons vacuum polarization equations which propagate in a common Siklos space-time were studied in [16] to explore the impact of one-loop vacuum polarization in the limit of geometric optics.

In addition, lots of special subclasses and representatives of Siklos space-times are significant and their physical interpretation and geometrical features have been examined in the literature; notable subclasses contain Defrise space-times [11], Kaigodorov space-times [15] and generalized Defrise space-times [19]. Moreover, Ricci solitons [6, 7, 26, 27], conformal geometry and symmetries [8] were investigated in the category of Siklos metrics.

One of the interesting topics in the field of differential geometry with evident applications in mathematical physics is the study of critical metrics over a family of (pseudo-)Riemannian manifolds. Let M^n be an oriented closed manifold. Also, we equip M with a family $\mathcal{M}_1$ of volume one (pseudo-)Riemannian metrics. The problem of critical metrics refers to the determination of such metrics from $\mathcal{M}_1$, which are extremum for some specific curvature functional.

Einstein metrics are well-known families of critical metrics. In fact, the Euler-Lagrange system related to the Einstein-Hilbert functional $g \mapsto \int_M \tau \mathrm{d}vol_g$ is the same condition for Einstein metrics, i.e., $\varrho = \lambda g$ for a real constant λ, where τ and ϱ denote the scalar curvature and Ricci tensor, respectively.

Some curvature functionals which are based on quadratic scalar curvature invariants, have received deep attention in the literature. This topic, started in the Riemannian settings in [1], has been studied by many scholars. For a nice survey on quadratic curvature functionals and critical metrics, we refer to [10, 25] and the references therein. Although quadratic curvature functionals may be studied on different dimensions of the base manifold, dimension 4, which is the framework of space-times receives more attention. On the other hand, some quadratic curvature functionals occur physically in the theory of gravitational field [12]. To study quadratic curvature invariants, we mention that $\{\Delta\tau, \tau^2, ||\varrho||^2, ||R||^2\}$ is a basis for them and so a generic functional on quadratic curvature invariants is as follows:

$$g \mapsto \int_M \left(c\tau^2 + b||\varrho||^2 + a||R||^2\right) \mathrm{d}vol_g, \tag{1}$$

where a, b, c are arbitrary real scalars and R is the curvature tensor. The Gauss-Bonnet Theorem in dimension 4 yields

$$32\pi^2\chi(M) = \int_M \left(||R||^2 - 4||\varrho||^2 + \tau^2\right) \mathrm{d}vol_g.$$

This indicates that the functional critical points related to the curvature tensor coincide with the critical points for the functional $4||\varrho||^2 + \tau^2$. Thus, the functional of equation

(1) is identical to

$$g \mapsto \int_M \left((b+4a)||\varrho||^2 - (a-c)\tau^2\right) \mathrm{d}vol_g.$$

Finally, if the manifold (M, g) is critical for the following functionals $\mathcal{S}$ and $\mathcal{F}_t$, then it is critical for the functional (1).

$$\mathcal{S} : g \mapsto \int_M \tau^2 \mathrm{d}vol_g, \qquad \mathcal{F}_t : g \mapsto \int_M \left(||\varrho||^2 + t\tau^2\right) \mathrm{d}vol_g, \quad t \in \mathbb{R}.$$

In [5], for all quadratic curvature functionals, the critical metrics were studied and it was shown that critical non-Einstein cones exist for all these functionals. For quadratic curvature functionals, Einstein metrics are always critical which involve the scalar curvature and Ricci curvature. A partial converse is true for Bach-flat Riemannian metrics as proved in [22]. Although most of the investigations on critical metrics are focused on the Riemannian settings for quadratic curvature functionals, some interesting results were expressed for the pseudo-Riemannian cases. For example, all quadratic curvature functionals have been studied on semi-direct extensions for the Heisenberg group [9].

For all quadratic curvature functionals $\mathcal{F}_t$, $t \in \mathbb{R}$, the Einstein metrics are critical (see [2]). Known results in the Riemannian settings show that for functional $\mathcal{F}_{-1/3}$, the Bach-flat metrics are critical and for functionals $\mathcal{F}_{-1/4}$ and $\mathcal{S}$, the Weyl metric with zero scalar curvature is critical. From these known results, it is interesting to consider critical metrics of the quadratic curvature functionals in different signatures.

In the present study, we encounter the problem of critical metrics for the functionals $\mathcal{F}_t$ and $\mathcal{S}$ on the homogeneous classes of Siklos space-times.

2. Preliminaries

Siklos space-times: By applying the coordinates (x_1, x_2, x_3, x_4) in the global case, the following general form is assigned to Siklos metrics:

$$g = -\frac{3}{\Lambda x_3^2}\left(2dx_1dx_2 + Hdx_2^2 + dx_3^2 + dx_4^2\right),$$

where $H = H(x_2, x_3, x_4)$ is an optional smooth function ([20, 23]). An analysis of the possible Killing vector fields was performed on the Siklos space-times in [26]. By this study, a homogeneous class of Siklos space-times is the case for $H(x_2, x_3, x_4) = \varepsilon x_3^{\alpha}$, where $\varepsilon = \pm 1$ and $\alpha \in \mathbb{R}$ is an arbitrary constant. Ricci solitons have been studied on the homogenous classes of Siklos space-times in [6]. It is worth knowing by assuming $\alpha = 2k$, this case has received more attention in the literature.

- For $k = -1$, one obtains the solution of Petrov type N of a pure radiation problem. The isometry group is G_6 and this case was first introduced by Defrise [11].
- The case $k = \frac{3}{2}$ is related to Kaigorodov space-time, that is, the solely homogeneous solution N-type for the Einstein vacuum field equations with $\Lambda \neq 0$ ([15, 20], [24, Theorem 12.5]).
- For $k = 2$, it gives the solutions of Einstein-Maxwell equations which are homogeneous and were first introduced by Ozsváth [17].

Quadratic curvature functionals: For a quadratic curvature functional, the Euler-

Lagrange equations are well known and are calculated in the Riemannian settings ([1, 14]). The results of the Riemannian case can be generalized to the pseudo-Riemannian settings since the signature of the base metric does not involve in arguments.

For the functionals $\mathcal{F}_t : g \mapsto \int_M \left(||\varrho||^2 + t\tau^2\right) \mathrm{d}vol_g$ and $\mathcal{S} : g \mapsto \int_M \tau^2 \mathrm{d}vol_g$, one can give the gradient as follows:

$$\begin{aligned}(\nabla\mathcal{S})_{ij} &= 2\nabla^2_{ij}\tau - 2(\Delta\tau)g_{ij} - 2\tau\varrho_{ij} + \tfrac{1}{2}\tau^2 g_{ij},\\ (\nabla\mathcal{F}_t)_{ij} &= -\Delta\varrho_{ij} - 2\varrho_{kl}R_{ikjl} - 2t\tau\varrho_{ij} - \tfrac{1+4t}{2}(\Delta\tau)g_{ij} + (1+2t)\nabla^2_{ij}\tau\\ &\quad + \tfrac{1}{2}\left(t\tau^2 + ||\varrho||^2\right) g_{ij}.\end{aligned}$$

We note here that if we have $(\nabla\mathcal{F}_t) = cg$ for a real constant c, then g is a critical metric for $\mathcal{F}_t$. Taking a trace from this equation, we have

$$(4(1-n)t - n)\Delta\tau - (4-n)\left(t\tau^2 + ||\varrho||^2\right) = 2cn$$

and so g is a critical metric of $\mathcal{F}_t$ whenever

$$-\Delta\varrho_{ij} + (1+2t)\nabla^2_{ij}\tau - \frac{2t}{n}(\Delta\tau)g_{ij} - 2\varrho_{kl}R_{ikjl} - 2t\tau\varrho_{ij} + \frac{2}{n}\left(||\varrho||^2 + t\tau^2\right) g_{ij} = 0 \quad (2)$$

and

$$(4-n)\left(t\tau^2 + ||\varrho||^2 - \lambda\right) = (4(1-n)t - n)\Delta\tau, \quad (3)$$

where $\lambda = \mathcal{F}_t(g)$ (refer to [10]). It follows for all t that the Einstein metrics are critical for $\mathcal{F}_t$. In addition, according to $\nabla\mathcal{S}$ representation above, metrics whose scalar curvature is zero and metrics that are Einstein are critical points for $\mathcal{S}$. For spaces that are 4-dimensional and have constant scalar curvature (especially for homogeneous spaces), the above Euler-Lagrange equations could be simplified remarkably.

It is easy to see that in this case, $(\nabla\mathcal{S})_{ij} = -2\tau\left(\varrho_{ij} - \frac{1}{4}\tau g_{ij}\right)$ and equation (3) is satisfied automatically. Also, the equation (2) becomes the following reduced form

$$\Delta\varrho + 2t\tau\varrho + 2R[\varrho] - \frac{1}{2}\left(t\tau^2 + ||\varrho||^2\right) g = 0, \quad (4)$$

where $R[\varrho]$ represents the tensor defined by the $\varrho_{kl}R_{ikjl}$ components. We focus on the equation (4) and its solutions in the forthcoming sections.

3. Geometric computations on homogeneous Siklos space-times

We first derive some geometric features of the homogeneous Siklos space-times. As mentioned in the previous section, one can exhibit the homogeneous metric on the Siklos space-times corresponding to the local coordinates (x_1, x_2, x_3, x_4) as

$$g = -\frac{3}{\Lambda x_3^2}\left(2dx_1dx_2 + \varepsilon x_3^\alpha dx_2^2 + dx_3^2 + dx_4^2\right), \quad (5)$$

where $\Lambda \neq 0$ and α are arbitrary real constants and $\varepsilon = \pm 1$.

Using the explicit coordinate description of the homogeneous Siklos metric (5) and the well-known Koszul formula, one can calculate the non-zero components of the Levi-Civita connection in the local coordinates (x_1, x_2, x_3, x_4) as follows:

$$\begin{array}{ll} \nabla_{\partial_1}\partial_2 = \frac{1}{x_3}\partial_3, & \nabla_{\partial_1}\partial_3 = -\frac{1}{x_3}\partial_1, \\ \nabla_{\partial_2}\partial_2 = -\frac{x_3^{\alpha-1}\varepsilon(\alpha-2)}{2}\partial_3, & \nabla_{\partial_2}\partial_3 = \frac{\varepsilon x_3^{\alpha-1}\alpha}{2}\partial_1 - \frac{1}{x_3}\partial_2, \\ \nabla_{\partial_3}\partial_3 = -\frac{1}{x_3}\partial_3, & \nabla_{\partial_3}\partial_4 = -\frac{1}{x_3}\partial_4, \\ \nabla_{\partial_4}\partial_4 = \frac{1}{x_3}\partial_3, & \end{array} \tag{6}$$

where $\partial_i = \frac{\partial}{\partial x_i}$. Using the relation $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, non-zero components of the $(1,3)$ curvature tensor field R are deduced as

$$\begin{array}{ll} R(\partial_1,\partial_2)\partial_1 = -\frac{1}{x_3^2}\partial_1, & R(\partial_1,\partial_2)\partial_2 = -\varepsilon x_3^{\alpha-2}\partial_1 + \frac{1}{x_3^2}\partial_2, \\ R(\partial_1,\partial_3)\partial_2 = \frac{1}{x_3^2}\partial_3, & R(\partial_1,\partial_3)\partial_3 = -\frac{1}{x_3^2}\partial_1, \\ R(\partial_1,\partial_4)\partial_2 = \frac{1}{x_3^2}\partial_4, & R(\partial_1,\partial_4)\partial_4 = -\frac{1}{x_3^2}\partial_1, \\ R(\partial_2,\partial_3)\partial_1 = \frac{1}{x_3^2}\partial_3, & R(\partial_2,\partial_3)\partial_2 = \frac{\varepsilon x_3^{\alpha-2}(\alpha^2-2\alpha+2)}{2}\partial_3, \\ R(\partial_2,\partial_3)\partial_3 = -\frac{\varepsilon x_3^{\alpha-2}\alpha(\alpha-2)}{2}\partial_1 - \frac{1}{x_3^2}\partial_2, & R(\partial_2,\partial_4)\partial_1 = \frac{1}{x_3^2}\partial_4, \\ R(\partial_2,\partial_4)\partial_2 = -\frac{\varepsilon x_3^{\alpha-2}(\alpha-2)}{2}\partial_4, & R(\partial_2,\partial_4)\partial_4 = \frac{\varepsilon x_3^{\alpha-2}\alpha}{2}\partial_1 - \frac{1}{x_3^2}\partial_2, \\ R(\partial_3,\partial_4)\partial_3 = \frac{1}{x_3^2}\partial_4, & R(\partial_3,\partial_4)\partial_4 = -\frac{1}{x_3^2}\partial_3. \end{array} \tag{7}$$

Then, we calculate the scalar curvature and the Ricci tensor using the identity $\varrho(X,Y) = \mathrm{tr}\{Z \to R(Z,X)Y\}$ and $\tau = \mathrm{tr}_g(\varrho)$. In this case, direct calculations yield

$$\varrho = -\frac{6}{x_3^2}dx_1dx_2 - \frac{\varepsilon x_3^{\alpha-2}(\alpha^2-3\alpha+6)}{2}dx_2^2 - \frac{3}{x_3^2}(dx_3^2 + dx_4^2) \tag{8}$$

and

$$\tau = 4\Lambda. \tag{9}$$

Now, we specify the Weyl conformal tensor field W by using the identity

$$W(X,Y)Z = R(X,Y)Z + \frac{\tau}{(n-1)(n-2)}(X \wedge Y)Z - \frac{1}{n-2}(QX \wedge Y + X \wedge QY)Z,$$

where $(X \wedge Y)(Z) = g(Y,Z)X - g(X,Z)Y$ and Q denotes the Ricci operator. Due to the local coordinates (x_1, x_2, x_3, x_4) and the above description of R, ϱ, and τ, non-zero components of the Weyl conformal curvature tensor field W are thoroughly specified as

$$W_{2323} = -W_{2424} = \frac{3\varepsilon x_3^{\alpha-4}\alpha(\alpha-1)}{4\Lambda}. \tag{10}$$

Einstein metrics are an important family of metric spaces in geometry and physics. By an Einstein (pseudo-)Riemannian manifold (M, g), we mean the space in which its Ricci tensor is a scalar multiple of the metric tensor at each point, i.e., there exists some real constant λ such that $\varrho = \lambda g$.

Some geometric features of the homogeneous Siklos space-times are presented in the following theorem.

Theorem 3.1 Let (M, g) be a homogeneous Siklos space-time, where in the local coordinates (x_1, x_2, x_3, x_4), g is described in the equation (5). In this case, the following properties hold:

1) (M, g) is never flat nor Ricci-flat.
2) (M, g) is Einstein if and only if $\alpha = 0$ or $\alpha = 3$.
3) (M, g) is locally conformally flat if and only if $\alpha = 0$ or $\alpha = 1$.
4) (M, g) is locally symmetric if and only if $\alpha = 0$.

Proof. 1) Equations (7) and (8) clearly show the Siklos space-time (M, g) is not Ricci flat nor flat.

2) By using the equations (5) and (8), we set $\varrho = \lambda g$ for an arbitrary real constant λ. Now the Siklos space-time (M, g) is Einstein if and only if the following equations establish:

$$\begin{cases} \Lambda - \lambda = 0, \\ \left(\alpha^2 - 3\alpha + 6\right)\Lambda - 6\lambda = 0 \end{cases}$$

From the above system, we immediately get

$$\lambda = \Lambda, \quad \alpha(\alpha - 3) = 0.$$

This shows the validity of the second statement.

3) Clearly equation (10) shows that the Weyl conformal tensor W vanishes identically (so (M, g) is locally conformally flat) if and only if $\alpha(\alpha - 1) = 0$.

4) To check the locally symmetric condition, by using the equations (6) and (7), one can calculate the covariant derivative of the curvature tensor. In this case,

$$(\nabla_{\partial_2} R)(\partial_3, \partial_4)\partial_4 = \frac{\varepsilon \alpha x_3^{\alpha-3}}{2}\partial_1,$$

which shows the necessary condition for being locally symmetric is $\alpha = 0$. But this condition yields that $\nabla R = 0$ at once and the fourth statement establishes. This ends the proof. ■

4. Critical metrics for Quadratic curvature functionals

In this section, we study critical metrics for the functionals $\mathcal{F}_t$ and $\mathcal{S}$, which are calculated on the homogeneous Siklos space-times. We remind from Section two that metrics whose scalar curvature vanishes identically or Einstein metrics are critical metrics for the functional $\mathcal{S}$. We note here, the scalar curvature $\tau = 4\Lambda$ will never vanish and (M, g) is Einstein whenever $\alpha(\alpha - 3) = 0$. Thus, the following remark establishes.

Remark 1 *A homogeneous Siklos space-time (M, g) is critical for the functional $\mathcal{S}$ if and only if be Einstein. That means $\alpha(\alpha - 3) = 0$.*

Now, we calculate critical homogeneous Siklos metrics for the functional $\mathcal{F}_t$. According to the equation (4), it is necessary to first compute tensors that generate this relation.

The results are resumed in the following theorem.

Theorem 4.1 A homogeneous Siklos space-time (M, g) is critical for the functional $\mathcal{F}_t$ if and only if one of the following cases occurs:

I) $\alpha(\alpha - 3) = 0$ for any real value of t.
II) $t = \frac{1}{24}(\alpha^2 - 3\alpha - 6)$ for any real value of α.

Proof. According to the equation (8), we can deduce the Laplacian of the Ricci tensor by direct calculation as

$$\Delta\varrho = \frac{1}{6}\varepsilon\Lambda x_3^{\alpha-2}\alpha(\alpha-3)(\alpha^2-3\alpha-2)dx_2^2.$$

On the other hand, using the equations (7), (8) and by operating a metric contraction on the indices, $\varrho_{kl}R_{ikjl}$, we calculate the tensor field $R[\varrho]$ as

$$R[\varrho] = -\frac{6\Lambda}{x_3^2}dx_1dx_2 - \frac{1}{3}\varepsilon\Lambda x_3^{\alpha-2}(\alpha^2-3\alpha+9)dx_2^2 - \frac{3\Lambda}{x_3^2}(dx_3^2+dx_4^2).$$

Since $\|\varrho\|^2 = 4\Lambda^2$, by applying the above expressions in the equation (4), the homogeneous Siklos metric g is critical for the functional $\mathcal{F}_t$, whenever the following equation establishes.

$$\Lambda x_3^{\alpha-2}\alpha(\alpha-3)(\alpha^2-3\alpha-24t-6) = 0. \tag{11}$$

As we expect, Einstein metrics are critical for any value of t. Except for the Einstein cases, the homogeneous Siklos metric g is critical for the functional $\mathcal{F}_t$ if and only if $\alpha^2 - 3\alpha - 24t - 6 = 0$ and this ends the proof. ■

We note here Bach-flat classes of Siklos metrics have been studied recently. For the homogeneous examples $H = \varepsilon x_3^\alpha$, the Siklos space-time (M, g) is Bach-flat whenever $\alpha \in \{0, 1, 2, 3\}$. This confirms the results of the Riemannian settings are expandable to the pseudo-Rimannian case. In fact, for $\alpha = 0, 3$, the metric is Einstein and so critical for any value of t (especially $t = -\frac{1}{3}$) and the cases $\alpha = 0, 2$ are exactly the critical metrics for $t = -\frac{1}{3}$.

Remark 2 *As mentioned in the Preliminaries section, homogeneous Siklos space-times have come into consideration for special values of α by many scholars. We resume the above outcomes of Theorem4.1 in combination with the known results as follows.*

- *The Defrise space-time is critical for the functional F_t for $t = \frac{1}{6}$ (see [11]).*
- *The Kaigorodov space-time is critical for the functional F_t for any value of t (see [15]).*
- *The Ozsváth space-time is critical for the functional F_t for $t = -\frac{1}{12}$ (see [17]).*

5. Conclusion

In this paper, we examined the class of homogeneous Siklos space-times and we completely specified those types which are critical for the quadratic curvature functional $\mathcal{S}$ and $\mathcal{F}_t$. Our computations show that only examples which are critical for the functional $\mathcal{S}$ are the Einstein metrics while there exist some functionals $\mathcal{F}_t$ which are critical for any homogeneous Siklos space-time. Our results relating to the Bach-flat examples are compatible with the known results of the Riemannian settings.

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