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Some new classes of distance integral graphs constructed from integral graphs

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Abstract. The distance eigenvalues of a connected graph G are the eigenvalues of its distance matrix $D(G)$. A graph is called distance integral if all of its distance eigenvalues are integers. In this paper, we introduce some new classes of distance integral graphs. In particular, we show that if n, k are integers such that $n \geqslant 3k > 0$, then the bipartite Kneser graph $H(n, k)$ is distance integral. Moreover, we determine the distance spectrum of $H(n, k)$. Also, we show that every distance regular integral graph is distance integral.

Keywords: Distance integral, vertex-transitive, distance regular, bipartite Kneser graph.

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1. Introduction and preliminaries

In this paper, a graph $G = (V, E)$ is considered as an undirected simple graph where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For all the terminology and notation not defined here, we follow [3–5].

Let $G = (V, E)$ be a graph and $A = A(G)$ be an adjacency matrix of G. The *characteristic polynomial* of G is defined as $P(G; x) = P(x) = |xI - A|$. A zero of $P(x)$ is called an *eigenvalue* of the graph G. The *spectrum* of G is the (multi)set of all eigenvalues of A and is denoted by $Spec(G) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and indexed such that $\lambda_1 \geqslant \lambda_2 \ldots \geqslant \lambda_n$. If the eigenvalues of G are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we write

$$Spec(G) = \binom{\lambda_1,\lambda_2,\ldots,\lambda_r}{m_1,m_2,\ldots,m_r} \text{ or } Spec(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, ..., \lambda_r^{m_r}\}.$$

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A graph is called *integral* if all its eigenvalues are integers. The study of integral graphs was initiated by Harary and Schwenk in 1974 (see [6]). A survey of papers up to 2002 has been appeared in [2], but more than a hundred new studies on integral graphs have been published in the last 20 years (for instance, see [7, 11, 14, 18]).

Let n be the number of vertices of the graph G. The *distance matrix* $D = D(G)$ is an $n \times n$ matrix indexed by V, such that $D_{uv} = d_G(u,v) = d(u,v)$, where $d_G(u,v)$ is the distance between the vertices u and v in the graph G. The characteristic polynomial $P(D;x) = |xI - D| = P_D(x)$ is the *distance characteristic polynomial* of G. Since D is a real symmetric matrix, the distance characteristic polynomial $P_D(x)$ has real zeros. Every zero of the polynomial $P_D(x)$ is called a *distance eigenvalue* of the graph G. Surveys on the distance spectra of graphs have been appeared in [1, 8]. A graph G is called *distance integral* (briefly, *D-integral*) if all the distance eigenvalues of G are integers. Although there are many papers that study distance spectrum of graphs and their applications, the D-integral graphs are studied only in a few papers (see [16, 17, 19]).

2. Main results

Let G_1, G_2 be graphs. Then the *direct product* (or tensor product) G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set $\{(v_1, v_2) \mid v_1 \in G_1, v_2 \in G_2\}$, and for which vertices (v_1, v_2) and (w_1, w_2) are adjacent precisely if v_1 is adjacent to w_1 in G_1 and v_2 is adjacent to w_2 in G_2. If K_2 is the complete graph on the set $\{0,1\}$, then the direct product $B(G) = G \times K_2$ is a bipartite graph, and is called the *bipartite double cover* of G (or the *bipartite double* of G). Then, $V(B(G)) = \{(v,i) | v \in V(G), i \in \{0,1\}\}$ and two vertices (x,i) and (y,j) are adjacent in the graph $B(G)$ if and only if $i \neq j$ and x is adjacent to y in the graph G. The notion of the bipartite double cover of G has many applications in algebraic graph theory [4].

Theorem 2.1 Let $G = (V,E)$ be a regular graph of order $n > 2$ such that every pair of its vertices has some common neighbor(s). If G is an integral graph, then $B(G)$, the bipartite double cover of G, is a distance integral graph. Moreover, if the spectrum of G is $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_n$, then the (multi)set of all distance eigenvalues of $B(G)$ is

$$\{5n - 2\lambda_1 - 2, -2\lambda_2 - 2, -2\lambda_3 - 2, \ldots, -2\lambda_n - 2, -n + 2\lambda_1 - 2, 2\lambda_2 - 2, \ldots, 2\lambda_n - 2\}.$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G and $P_i = \{(v_1, i), \ldots, (v_n, i)\}$, $i \in \{0,1\}$. Thus $W = P_0 \cup P_1$ is the vertex set of $B = B(G)$, the bipartite double cover of G. Let $x = (v,i)$ and $y = (w,i)$ be two vertices of B and u be a common neighbor of v and w in G. Hence, (u,j) is a common neighbor of x and y in $B(G)$, where $j \in \{0,1\}, j \neq i$. Therefore every two vertices in P_i are at distance 2 from each other. Now assume that $(v,0) \in P_0$ and $(w,1) \in P_1$ are not adjacent. Let t be a vertex in G adjacent to w. Thus, $(t,0) \in P_0$ is adjacent to $(w,1)$. Since $(t,0)$ is at distance 2 from $(v,0)$ in graph B, $(v,0)$ is at distance 3 from $(w,1)$ in B. It follows that the diameter of B is 3. Let $\{v_1, v_2, \ldots v_n\}$ be an ordered set of vertices of the graph G. Let A be the adjacency matrix of the graph G respect to the ordering. Let $\{(v_1,0), (v_2,0), \ldots, (v_n,0), (v_1,1), (v_2,1), \ldots, (v_n,1)\}$ be the ordered set of vertices of the graph $B = B(G)$. Now it can be checked that the distance matrix of $D = D(B)$ can be of the following form

$$D = \begin{bmatrix} 2J - 2I & 3J - 2A \\ 3J - 2A & 2J - 2I \end{bmatrix},$$

where $I = I_{n\times n}$ is the identity matrix and $J = J_{n\times n}$ is the matrix with all entries equal to one. Since G is a regular graph, $AJ = JA$ ([3]). Note that A, I and J are symmetric matrices. Thus, there is an orthonormal basis $E = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$ such that each element of E is an eigenvector of A and J. If $x = [x_1, x_2, \ldots, x_n]^t \in E$ is an eigenvector of A corresponding to the eigenvalue λ, then $(x * x)^t = [x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n]^t$ is an eigenvector of D corresponding to the eigenvalue $2\gamma - 2 + 3\gamma - 2\lambda = 5\gamma - 2 - 2\lambda$ where γ is the corresponding eigenvalue for J. Also,

$$(x * (-x))^t = [x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n]^t$$

is an eigenvector for D corresponding to the eigenvalue $2\gamma - 2 - 3\gamma + 2\lambda = -\gamma + 2\lambda - 2$ for D. Nothing that the spectrum of the matrix J is $\{n^1, 0^{n-1}\}$ [5], the result follows. ■

From Theorem 2.1, we have the following result.

Corollary 2.2 Let G be an integral regular graph of diameter 2 such that every pair of its vertices has at list a common neighbor. Then the bipartite double cover of G is distance integral.

Let $n, k \in \mathbb{N}$ with $k < \frac{n}{2}$ and $[n] = \{1, ..., n\}$. The *Kneser graph* $K(n, k)$ is the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices v and w are adjacent if and only if $|v \cap w|$=0. It is clear that this graph is a regular graph of regularity $\binom{n-k}{k}$ and order m=$\binom{n}{k}$. A well-known example of Kneser graph is the Petersen graph P. In fact, P is the Kneser graph $K(5, 2)$. Concerning the spectrum of the Kneser graph $K(n, k)$, we have the following fact.

Theorem 2.3 [20] The eigenvalues of the Kneser graph $K(n, k)$ are the integers

$$\lambda_i = (-1)^i \binom{n-k-i}{k-i}, \; i \in \{0, 1, \ldots, k\}.$$

Moreover, the multiplicity of λ_0=$\binom{n-k}{k}$ is 1 and if $i \geqslant 1$, then the multiplicity of λ_i is $\binom{n}{i}$-$\binom{n}{i-1}$.

Let X be the set of all k-subsets and $(n-k)$-subsets of $[n]$. The *bipartite Kneser graph* $H(n, k)$ has X as its vertex-set, and two vertices v, w are adjacent if and only if $v \subset w$ or $w \subset v$ [4, 5]. It is clear that $H(n, k)$ is a bipartite graph. In fact, if $V_1 = \{v \subset [n] \mid |v| = k\}$ and $V_2 = \{v \subset [n] \mid |v| = n - k\}$, then $\{V_1, V_2\}$ is a partition of $V(H(n, k))$ and every edge of $H(n, k)$ has a vertex in V_1 and a vertex in V_2 and $|V_1| = |V_2|$. It is easy to see that if $H(n, k)$ is a bipartite Kneser graph, then $H(n, k) \cong K(n, k) \times K_2$. Recently, some algebraic properties of the graph $H(n, k)$ has been studied in various papers (see [9, 10, 12, 13, 15]).

Note that if $n \geqslant 3k$ in the Kneser graph $K(n, k)$, then every pair of its vertices has some common neighbor(s). Now, from Theorem 2.1 and Theorem 2.3, the following result follows.

Theorem 2.4 If $n \geqslant 3k$, then the bipartite Kneser graph $H(n, k)$ is distance integral.

Remark 1 *The bipartite Kneser graph $H(n, 1)$ is known as the crown graph [3, 4]. Note* *that the Kneser graph $K(n,1)$ is isomorphic with K_n, the complete graph of order n. Since the spectrum of K_n is $n-1$ with multiplicity 1, and -1 with multiplicity $n-1$ [3], then by Theorem 2.1, the distance spectrum of the crown graph $H(n,1)$ is $5n-2(n-1)-2=3n$ with multiplicity 1, and $-n+2(n-1)-2=n-4$ with multiplicity 1, and $-2(-1)-2=0$ with multiplicity $n-1$, and $2(-1)-2=-4$ with multiplicity $n-1$. The present result has already been determined by a different method [17].*

Let $G=(V,E)$ be a simple connected graph with diameter d. Let $d(x,y)$ denotes the distance between vertices x and y. Let $G_i(x)$ denotes the set of vertices of G at distance i from the vertex x. A *distance-regular* graph $G=(V,E)$, with diameter d, is a regular connected graph of valency k with the following property. There are positive integers

$$b_0=k, b_1, ..., b_{d-1}; c_1=1, c_2, ..., c_d,$$

such that for each pair (u,v), of vertices satisfying $u\in G_i(v)$, we have
(1) the number of vertices in $G_{i-1}(v)$ adjacent to u is c_i, $1\leqslant i\leqslant d$.
(2) the number of vertices in $G_{i+1}(v)$ adjacent to u is b_i, $0\leqslant i\leqslant d-1$.
In this situation, the intersection array of G is $i(G)=\{k, b_1, ..., b_{d-1}; 1, c_2, ..., c_d\}$.

As an example [3, 4], it is easy to show that the hypercube Q_n, $n>2$ is a distance-regular graph with the intersection array,

$$\{n, n-1, n-2, ..., 1; 1, 2, 3, ..., n\}$$

In other words, for hypercube Q_n, we have $b_i=n-i$, $c_i=i$, $1\leqslant i\leqslant n-1$, and $b_0=n$, $c_n=n$.

We know that the hypercube Q_n is an integral graph with the eigenvalues $\lambda_i=n-2i$, $0\leqslant i\leqslant n$ [3, 4]. In the rest of the paper, we show that every distance regular integral graph is a distance integral graph.

Theorem 2.5 Let $G=(V,E)$ be a distance regular integral graph. Then G is a distance integral graph.

Proof. Let d denote the diameter of the graph G. For every integer i, $1\leqslant i\leqslant d$, the distance-i matrix A_i of G is defined as,

$$A_i(u_r,v_s)=\begin{cases}1 & if\ d(u_r,v_s)=i\\ 0 & otherwise.\end{cases}$$

Then $A_0=I$, A_1 is the usual adjacency matrix A of G. Note that $A_0+A_1+...+A_d=J$, where J is the matrix in which each entry is 1. Let the intersection array of G be $i(G)=\{k, b_1, ..., b_{d-1}; 1, c_2, ..., c_d\}$. We let $a_i=k-b_i-c_i$, $1\leqslant i\leqslant d-1$. Then, we have

$$AA_i=b_{i-1}A_{i-1}+a_iA_i+c_{i+1}A_{i+1}\ (1\leqslant i\leqslant d-1).$$

This follows that $A_i=p_i(A)$, where $p_i(x)$ is a polynomial with rational coefficients [3]. Now, it is easy to see that if λ is an eigenvalue of A, then $p_i(\lambda)$ is an eigenvalue of A_i. Since the number of eigenvalues of A and A_i are equal, every eigenvalue of A_i is of this form. We now deduce that if λ is an integer, then $p_i(\lambda)$ is a rational number. On the other hand $p_i(\lambda)$ is an eigenvalue of the matrix A_i which is a matrix with integer entries. Hence, every eigenvalue of A_i is an algebraic integer. We know that every rational algebraic integer is an integer [21]. Thus, $p_i(\lambda)$ is an integer.

Now, let D denote the distance matrix of the graph G. Thus we have $D = I + A + 2A_2 + 3A_3 + \cdots + dA_d$. Hence, if λ is an eigenvalue of A, then $\delta = 1 + \lambda + 2p_2(\lambda) + 3p_3(\lambda) + \cdots + dp_d(\lambda)$ is an eigenvalue of D and every eigenvalue of D is of this form. We now conclude that if λ is an integer, then δ is an integer and hence the graph G is a distance integral graph. ■

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