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A log-convex approach to Jensen-Mercer inequality

S. M. Davarpanah[a,*], H. R. Moradi[b]

[a]*Department of Mathematics, Ferdows Branch, Islamic Azad University, Ferdows, Iran.*
[b]*Department of Mathematics, Payame Noor University (PNU), P.O. Box, 19395-4697, Tehran, Iran.*

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Abstract. We obtain some new Jensen-Mercer type inequalities for log-convex functions. Indeed, we establish refinement and reverse for the Jensen-Mercer inequality for log-convex functions. Several new Hermite-Hadamard and Fejér types of inequalities are also presented.

Keywords: Inequality, Jensen-Mercer, Fejér inequality, Hermite-Hadamard inequality, log-convex function.

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1. Introduction and Preliminaries

Lately, the theory of convexity has welcomed considerable attention from many researchers. The interconnection between the theory of convex functions and inequalities has tempted many researchers [7–10]. Accordingly, the classical concepts of convex sets and functions have been extended and generalized in several approaches using novel and clever ideas. One of the most broadly studied inequalities for convex functions is Jensen inequality. Suppose that $f : J \to \mathbb{R}$ is a convex function and $J \subseteq \mathbb{R}$ is an interval. Jensen's inequality states that

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i f(x_i), \tag{1}$$

*Corresponding author.
E-mail address: davarpanah@ferdowsiau.ac.ir, ana.kamali.gh@gmail.com (S. M. Davarpanah)
hrmoradi@mshdiau.ac.ir, hrmoradi.68@gmail.com (H. R. Moradi).

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where $x_i \in J$ for $i = 1, 2, \ldots, n$ and $w_1, w_2, \ldots, w_n$ are positive scalars such that $\sum_{i=1}^{n} w_i = 1$. Mercer [5] proved that if $f : [m, M] \to \mathbb{R}$ is a convex function and $x_i \in [m, M]\,(i = 1, 2, \ldots, n)$, then

$$f\left(M + m - \sum_{i=1}^{n} w_i x_i\right) \leq f(M) + f(m) - \sum_{i=1}^{n} w_i f(x_i).$$

The function $f : J \to (0, \infty)$ is called log-convex if

$$f((1-t)a + tb) \leq f^{1-t}(a) f^t(b);\ \ (a, b \in J, t \in [0, 1]).$$

This follows from the fact that $f : J \to (0, \infty)$ is log-convex if $\log f$ is convex. So, Jensen inequality (1) for log-convex functions can be written as

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leq \prod_{i=1}^{n} f^{w_i}(x_i). \tag{2}$$

The Hermite-Hadamard inequality [3] says that if a function $f : J \to \mathbb{R}$ is convex, then the following chain of inequalities holds:

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f((1-t)a + tb)\,dt \leq \frac{f(a) + f(b)}{2},$$

where $a, b \in J$. This inequality has been generalized by Fejér [2] as

$$\int_0^1 p(t)\,dt f\left(\frac{a+b}{2}\right) \leq \int_0^1 p(t) f((1-t)a + tb)\,dt \leq \int_0^1 p(t)\,dt \frac{f(a) + f(b)}{2},$$

where $p : [0, 1] \to [0, \infty)$ is continuous and $p(t) = p(1-t)$.

Using an utterly different approach from the one in the literature, we present a refinement of Jensen-Mercer inequality for log-convex functions. Our technique allows us to obtain complementary inequalities to Mercer's inequality. Hermite-Hadamard type and Fejér type inequalities for this class of functions are also presented.

2. Jensen-Mercer Inequality via Log-convex Functions

If $f : J \to (0, \infty)$ is log-convex, then Jensen-Mercer inequality [6] says that

$$f\left(M + m - \sum_{i=1}^{n} w_i x_i\right) \leq \frac{f(M) f(m)}{\prod_{i=1}^{n} f^{w_i}(x_i)}. \tag{3}$$

Our first attempt in this section is to provide a modification of (3). To this end, we require the following result.

Theorem 2.1 Let $f : [m, M] \to (0, \infty)$ be a log-convex function and let $x_i \in [m, M]$ for $i = 1, 2, \ldots, n$. Then

$$f\left(M+m-x_i\right) \le \left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f\left(M\right)f\left(m\right)}}\right)^{2\left(1-\left|\frac{M+m-2x_i}{M-m}\right|\right)} \frac{f\left(M\right)f\left(m\right)}{f\left(x_i\right)}.$$

Proof. If $f : J \to (0, \infty)$ is a convex function and $a, b \in J$, then

$$f\left(\left(1-t\right)a+tb\right) \le \left(1-t\right)f\left(a\right)+tf\left(b\right)-2r\left(\frac{f\left(a\right)+f\left(b\right)}{2}-f\left(\frac{a+b}{2}\right)\right), \quad (4)$$

where $r = \min\{t, 1-t\}$ and $0 \le t \le 1$ [1]. It follows from the inequality (4) that

$$f\left(\left(1-t\right)a+tb\right) \le \left(\frac{f\left(\frac{a+b}{2}\right)}{\sqrt{f\left(a\right)f\left(b\right)}}\right)^{2r} f^{1-t}\left(a\right)f^{t}\left(b\right) \quad (5)$$

provided that $f : J \to (0, \infty)$ is log-convex. If $m \le x_i \le M$ for $i = 1, 2, \ldots, n$, then $0 \le \frac{x_i-m}{M-m}, \frac{M-x_i}{M-m} \le 1$. Now, by replacing $1-t$, t, a, and b by $\frac{x_i-m}{M-m}$, $\frac{M-x_i}{M-m}$, M, and m, respectively, in (5), we obtain

$$\begin{aligned} f\left(x_i\right) &= f\left(\frac{x_i-m}{M-m}M+\frac{M-x_i}{M-m}m\right) \\ &\le \left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f\left(M\right)f\left(m\right)}}\right)^{1-\left|\frac{M+m-2x_i}{M-m}\right|} f^{\frac{x_i-m}{M-m}}\left(M\right)f^{\frac{M-x_i}{M-m}}\left(m\right). \end{aligned} \quad (6)$$

Here we used the fact that $\min\{a, b\} = \frac{a+b-|a-b|}{2}$ for $a, b \in \mathbb{R}^+$.

On the other hand, if $m \le x_i \le M$, for $i = 1, 2, \ldots, n$, then $m \le M+m-x_i \le M$. So by replacing x_i by $M+m-x_i$ in (6), we get

$$f\left(M+m-x_i\right) \le \left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f\left(M\right)f\left(m\right)}}\right)^{1-\left|\frac{2x_i-M-m}{M-m}\right|} f^{\frac{M-x_i}{M-m}}\left(M\right)f^{\frac{x_i-m}{M-m}}\left(m\right). \quad (7)$$

Multiplying inequalities (6) and (7) together, we have

$$f\left(x_i\right)f\left(M+m-x_i\right) \le \left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f\left(M\right)f\left(m\right)}}\right)^{2\left(1-\left|\frac{M+m-2x_i}{M-m}\right|\right)} f\left(M\right)f\left(m\right),$$

for $i = 1, 2, \ldots$. So,

$$f\left(M+m-x_i\right) \le \left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f\left(M\right)f\left(m\right)}}\right)^{2\left(1-\left|\frac{M+m-2x_i}{M-m}\right|\right)} \frac{f\left(M\right)f\left(m\right)}{f\left(x_i\right)}.$$

This finishes the proof. ■

Now, we present an improvement of (3).

Corollary 2.2 Let $f:[m,M]\to(0,\infty)$ be a log-convex function and let $x_i\in[m,M]$ for $i=1,2,\ldots,n$. Then

$$f\left(M+m-\sum_{i=1}^{n}w_ix_i\right)\le\prod_{i=1}^{n}\left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f(M)f(m)}}\right)^{2w_i\left(1-\left|\frac{M+m-2x_i}{M-m}\right|\right)}\frac{f(M)f(m)}{f^{w_i}(x_i)}$$

where $w_1,w_2,\ldots,w_n$ are positive scalars such that $\sum_{i=1}^{n}w_i=1$.

Proof. We have

$$\begin{aligned}
&f\left(M+m-\sum_{i=1}^{n}w_ix_i\right)\\
&=f\left(\sum_{i=1}^{n}w_i(M+m-x_i)\right)\\
&\le\prod_{i=1}^{n}f^{w_i}(M+m-x_i)\\
&\le\prod_{i=1}^{n}\left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f(M)f(m)}}\right)^{2w_i\left(1-\left|\frac{M+m-2x_i}{M-m}\right|\right)}\frac{f(M)f(m)}{f^{w_i}(x_i)},
\end{aligned}$$

where the first inequality follows from (2) and the second inequality is obtained from Theorem 2.1. ■

Remark 1 *Since for any log-convex function f,*

$$f\left(\frac{M+m}{2}\right)\le\sqrt{f(M)f(m)}$$

and

$$1-\left|\frac{M+m-2x_i}{M-m}\right|\ge0,$$

when $x_i\in[m,M]\,(i=1,2,\ldots,n)$, we get

$$\prod_{i=1}^{n}\left(\frac{f\left(\frac{M+m}{2}\right)}{\sqrt{f(M)f(m)}}\right)^{2w_i\left(1-\left|\frac{M+m-2x_i}{M-m}\right|\right)}\le1.$$

For the counterpart of Theorem 2.1, we have

Proposition 2.3 Let $f:[m,M]\to(0,\infty)$ be a log-convex function and let $x_i\in[m,M]$ for $i=1,2,\ldots,n$. Then

$$\frac{f(M)f(m)}{f(x_i)}\le\left(\frac{\sqrt{f(M)f(m)}}{f\left(\frac{M+m}{2}\right)}\right)^{2\left(1+\left|\frac{M+m-2x_i}{M-m}\right|\right)}f(M+m-x_i).$$

Proof. If $f : J \to (0,\infty)$ is a convex function and $a, b \in J$, then

$$(1-t) f(a) + t f(b) \le f((1-t)a + tb) + 2R\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right)$$

where $R = \max\{t, 1-t\}$ and $0 \le t \le 1$ [1]. So, if $f : J \to (0,\infty)$ is log-convex, we conclude that

$$f^{1-t}(a) f^{t}(b) \le \left(\frac{\sqrt{f(a) f(b)}}{f\left(\frac{a+b}{2}\right)}\right)^{1+|2t-1|} f((1-t)a + tb), \tag{8}$$

due to $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ for $a, b \in \mathbb{R}^+$. By replacing $1-t$, t, a, and b by $\frac{x_i - m}{M-m}$, $\frac{M - x_i}{M-m}$, M, and m, respectively, in (8), we get

$$f^{\frac{x_i-m}{M-m}}(M) f^{\frac{M-x_i}{M-m}}(m) \le \left(\frac{\sqrt{f(M) f(m)}}{f\left(\frac{M+m}{2}\right)}\right)^{1+\left|\frac{M+m-2x_i}{M-m}\right|} f(x_i). \tag{9}$$

By substituting x_i by $M + m - x_i$ in (9), we get

$$f^{\frac{M-x_i}{M-m}}(M) f^{\frac{x_i-m}{M-m}}(m) \le \left(\frac{\sqrt{f(M) f(m)}}{f\left(\frac{M+m}{2}\right)}\right)^{1+\left|\frac{2x_i-M-m}{M-m}\right|} f(M+m-x_i). \tag{10}$$

Multiplying inequalities (9) and (10), together, we have

$$f(M) f(m) \le \left(\frac{\sqrt{f(M) f(m)}}{f\left(\frac{M+m}{2}\right)}\right)^{2\left(1+\left|\frac{M+m-2x_i}{M-m}\right|\right)} f(M+m-x_i) f(x_i),$$

which indicates

$$\frac{f(M) f(m)}{f(x_i)} \le \left(\frac{\sqrt{f(M) f(m)}}{f\left(\frac{M+m}{2}\right)}\right)^{2\left(1+\left|\frac{M+m-2x_i}{M-m}\right|\right)} f(M+m-x_i).$$

Completes the proof. ■

In the following, we concentrate on the Hermite-Hadamard-Mercer inequality. Notice that it has been shown in [4] that for convex function f on the interval $[m, M]$,

$$f\left(M+m-\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(M+m-t)\,dt \le f(M) + f(m) - \frac{f(a)+f(b)}{2}.$$

Clearly, if $f : [m, M] \to (0,\infty)$ is log-convex, we have

$$\log f\left(M+m-\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \log f(M+m-t)\,dt \le \log\frac{f(M) f(m)}{\sqrt{f(a) f(b)}}. \tag{11}$$

Directly, if we apply exp to both sides of (11), we infer that

$$f\left(M+m-\frac{a+b}{2}\right)\le \exp\left(\frac{1}{b-a}\int_a^b \log f\left(M+m-t\right)dt\right)\le \frac{f\left(M\right)f\left(m\right)}{\sqrt{f\left(a\right)f\left(b\right)}}.$$

The following theorem furnishes another variant of Hermite-Hadamard-Mercer inequality.

Theorem 2.4 Let $f:[m,M]\to(0,\infty)$ be a log-convex function and let $a,b\in[m,M]$. Then

$$\begin{aligned}&f\left(M+m-\frac{a+b}{2}\right)\\&\le\int_0^1\sqrt{f\left(\left(M+m-\left(\left(1-t\right)a+tb\right)\right)\right)f\left(\left(M+m-\left(\left(1-t\right)b+ta\right)\right)\right)}dt\\&\le\frac{f\left(M\right)f\left(m\right)}{\sqrt{f\left(a\right)f\left(b\right)}}.\end{aligned}$$

Proof. It observes from the supposition that

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&=f\left(\frac{\left(\left(1-t\right)a+tb\right)+\left(\left(1-t\right)b+ta\right)}{2}\right)\\&\le\sqrt{f\left(\left(1-t\right)a+tb\right)f\left(\left(1-t\right)b+ta\right)}\\&\le\sqrt{f\left(a\right)f\left(b\right)}.\end{aligned}\tag{12}$$

Substituting a and b by $M+m-a$ and $M+m-b$ in (12), we reach

$$\begin{aligned}&f\left(M+m-\frac{a+b}{2}\right)\\&\le\sqrt{f\left(\left(M+m-\left(\left(1-t\right)a+tb\right)\right)\right)f\left(\left(M+m-\left(\left(1-t\right)b+ta\right)\right)\right)}\\&\le\sqrt{f\left(M+m-a\right)f\left(M+m-b\right)}.\end{aligned}\tag{13}$$

Integrating over $t\in[0,1]$, in (13), we obtain

$$\begin{aligned}&f\left(M+m-\frac{a+b}{2}\right)\\&\le\int_0^1\sqrt{f\left(\left(M+m-\left(\left(1-t\right)a+tb\right)\right)\right)f\left(\left(M+m-\left(\left(1-t\right)b+ta\right)\right)\right)}dt\\&\le\sqrt{f\left(M+m-a\right)f\left(M+m-b\right)}\\&\le\frac{f\left(M\right)f\left(m\right)}{\sqrt{f\left(a\right)f\left(b\right)}},\end{aligned}$$

where the last inequality follows from (3). This ends the proof. ■

We present Fejér-Mercer inequality in the coming result.

Theorem 2.5 Let $f : [m, M] \to (0, \infty)$ be a log-convex function, let $a, b \in [m, M]$, and let $p : [0, 1] \to (0, \infty)$ be continuous. Then

$$\begin{aligned}
&\int_0^1 p(t)\,dt f\left(M + m - \frac{a+b}{2}\right) \\
&\le \int_0^1 p(t)\sqrt{f\left((M + m - ((1-t)a + tb)\right) f\left((M + m - ((1-t)b + ta)\right)}dt \\
&\le \int_0^1 p(t)\,dt \frac{f(M) f(m)}{\sqrt{f(a) f(b)}}.
\end{aligned}$$

Proof. If we multiply the inequality (13) by $p(t)$, we obtain

$$\begin{aligned}
&p(t) f\left(M + m - \frac{a+b}{2}\right) \\
&\le p(t)\sqrt{f\left((M + m - ((1-t)a + tb)\right) f\left((M + m - ((1-t)b + ta)\right)} \\
&\le p(t)\sqrt{f(M + m - a) f(M + m - b)}.
\end{aligned}$$

Integrating over $t \in [0, 1]$, we reach to

$$\begin{aligned}
&\int_0^1 p(t)\,dt f\left(M + m - \frac{a+b}{2}\right) \\
&\le \int_0^1 p(t)\sqrt{f\left((M + m - ((1-t)a + tb)\right) f\left((M + m - ((1-t)b + ta)\right)}dt \\
&\le \int_0^1 p(t)\,dt \sqrt{f(M + m - a) f(M + m - b)} \\
&\le \int_0^1 p(t)\,dt \frac{f(M) f(m)}{\sqrt{f(a) f(b)}},
\end{aligned}$$

where the last inequality obeys from (3). ■

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