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When an operator gives a unique generalized topology

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Abstract. In topology, we found enough literature on topological operators but in generalized topology, there is only μ-interior, μ-closure, and μ-boundary operator. In this article, we explore different types of operators like μ-derived set operator, μ-exterior operator, μ-preboundary operator in generalized topology. We have shown that any operator can be developed as the above operators impose certain conditions, giving a unique generalized topology in each case.

Keywords: μ-interior, μ-derived set, μ-exterior, μ-boundary, generalized topology.

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1. Introduction and preliminaries

Kuratowski first showed that the topological spaces can be axiomatically defined by its closure operator. There are other important set operators like interior operator, exterior operator, derived set operator, and boundary operator give a different characterization of a topological space. Topological set operators are used in many modern fields, including formal concept analysis, category theory, domain theory, and geographic information systems. Due to the huge application of topological set operators we are interested in studying this operator in general set-theoretic structure which is generalized topology.

In [1], Császár introduced the concept of generalized topology. Let X be a non-empty set and $expX$ denote the collection of all subsets of X. Then μ is called a generalized topology on X (in short, GT) if $\emptyset \in \mu$ and arbitrary union of elements of μ belongs to μ. If μ is GT on X and $S \subseteq X$, then S is μ-open iff $S \in \mu$. S is μ-closed iff $X - S \in \mu$.

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If μ is a GT on X and $X \in \mu$, then μ is said to be strong generalized topology [3]. We denote M_μ is the union of all μ-open sets in μ and $\mu_x = \{ G : x \in G \in \mu\}$.

Suppose μ is a GT on X, then one can define the above-mentioned operators on X. Now the question arises that whether the converse is true or not, i.e., whether an operator $j : expX \rightarrow expX$ may give a unique GT on X or not. We are looking for an affirmative answer even if there are some conditions imposed on the operators. Császár in [2] answered this question for the μ-interior and μ-closure operators. In this article, we have shown that the above question has an affirmative answer for some more operators on X with certain conditions. We have investigated the basic properties of these operators and the relationship between them. We have also introduced the μ-preboundary operator to understand the concept in a more formal way. Straight forward proofs are omitted.

μ-Interior Operator: If μ is a GT on X and $S \subseteq X$, then the μ-interior of S is the largest μ-open subset of S and denoted by $i_\mu S$. Equivalently, $i_\mu S$ is the union of all μ-open subsets of S i.e., $i_\mu S = \cup\{G \in \mu : G \subseteq S\}$.

Theorem 1.1 [2, 5] If μ is a GT on X and $S, T \subseteq X$, then

(a) $i_\mu \emptyset = \emptyset$,
(b) $i_\mu X = M_\mu$,
(c) $i_\mu S \subseteq S$,
(d) if $S \subseteq T$, then $i_\mu S \subseteq i_\mu T$,
(e) $i_\mu i_\mu S = i_\mu S$,
(f) $i_\mu S \cup i_\mu T \subseteq i_\mu(S \cup T)$,
(g) $i_\mu S \cap i_\mu T \supseteq i_\mu(S \cap T)$,
(h) S is μ-open if and only if $i_\mu S = S$.

All the above results are same as in topology except for (b) and (g). In a topological space X is open and intersection of two open sets are open set, which are not true for GT.

Theorem 1.2 [2] Let X be a nonempty set. If $j : expX \rightarrow expX$ satisfies

(I_1) $jS \subseteq S$;
(I_2) if $S \subseteq T$, then $jS \subseteq jT$;
(I_3) $jjS = jS$;

for any $S, T \subseteq X$, then there exists a unique GT $\mu \subseteq expX$ such that $j = i_\mu$.

μ-Closure Operator: If μ is a GT on X and $S \subseteq X$, then the μ-closure of S in X is the smallest μ-closed subset which contains S and denoted by $c_\mu S$. Equivalently, $c_\mu S$ is the intersection of all μ-closed super sets of S, i.e., $c_\mu S = \cap\{M : X - M \in \mu, S \subseteq M\}$.

Theorem 1.3 [2, 5] If μ is a GT on X and $S, T \subseteq X$, then

(a) $c_\mu \emptyset = X - M_\mu$,
(b) $c_\mu X = X$,
(c) $S \subseteq c_\mu S$,
(d) if $S \subseteq T$, then $c_\mu S \subseteq c_\mu T$,
(e) $c_\mu c_\mu S = c_\mu S$,
(f) $c_\mu S \cup c_\mu T \subseteq c_\mu(S \cup T)$,
(g) $c_\mu S \cap c_\mu T \supseteq c_\mu(S \cap T)$,
(h) S is μ-closed if and only if $c_\mu S = S$.

All the above results are same as in topology except for (a) and (f). In a topological space, empty set is closed and union of two closed sets are closed set, which are not true in GT.

Theorem 1.4 [2] Let X be a nonempty set. If $j : expX \to expX$ satisfies

(C_1) $S \subseteq jS$;
(C_2) if $S \subseteq T$ then $jS \subseteq jT$;
(C_3) $jjS = jS$;

for $S, T \subset X$, then there exists a unique GT $\mu \subset expX$ such that $j = c_\mu$.

Theorem 1.5 [2] If μ is a GT on X and $S \subseteq X$, then

(a) $c_\mu S = X - i_\mu(X - S)$,
(b) $i_\mu S = X - c_\mu(X - S)$.

Proposition 1.6 [6] If μ is a GT on X and $S \subseteq X$, then $y \in c_\mu S$ if and only if either $\mu_y = \emptyset$ or $S \cap G \neq \emptyset$ for all $G \in \mu_y$.

2. Main results

μ-Derived Set Operator; If μ is a GT on X and $S \subseteq X$, then a point $p \in X$ is called a μ-cluster point of S if either $\mu_p = \emptyset$ or $S \cap (G - \{p\}) \neq \emptyset$ for all $G \in \mu_p$. The set of all μ-cluster points of S is the μ-derived set of S and denoted by $d_\mu S$.

$$d_\mu S = \{\ p \in X :\ either\ \mu_p = \emptyset\ or\ \ S \cap (G - \{p\}) \neq \emptyset\ for\ all\ G \in \mu_p\ \}.$$

Proposition 2.1 If μ is a GT on X and $S, T \subseteq X$, then

(a) $d_\mu \emptyset = X - M_\mu$,
(b) if $S \subseteq T$, then $d_\mu S \subseteq d_\mu T$,
(c) $d_\mu S \cup d_\mu T \subseteq d_\mu(S \cup T)$,
(d) $d_\mu(S \cap T) \subseteq d_\mu S \cap d_\mu T$,
(e) $x \in d_\mu S \Leftrightarrow x \in d_\mu(S - \{x\}) \Leftrightarrow x \in c_\mu(S - \{x\})$,
(f) $d_\mu S \subseteq c_\mu S$,
(g) $c_\mu S = S \cup d_\mu S$,
(h) $d_\mu(S \cup d_\mu S) \subseteq S \cup d_\mu S$.

Proof.

(g) For any set S we have $S \subseteq c_\mu S$ and by (f) $d_\mu S \subseteq c_\mu S$, both implies $S \cup d_\mu S \subseteq c_\mu S$. Let $y \in c_\mu S$ and $y \notin S$ (if $y \in S$ the proof is done). Then, either $\mu_y = \emptyset$ or $S \cap G \neq \emptyset$ for all $G \in \mu_y$. Since $y \notin S$, therefore either $\mu_y = \emptyset$ or $S \cap (G - \{y\}) \neq \emptyset$ for all $G \in \mu_y$. Thus, $y \in d_\mu S$. Hence, $S \cup d_\mu S \supseteq c_\mu S$.
(h) By (f) and (g) $d_\mu(S \cup d_\mu S) \subseteq c_\mu(S \cup d_\mu S)$.

■

All the above results are same as in topology except for (a) and (c). In topology derived set of an empty set is empty and derived set operator splits under union, i.e., option (c) has become equality.

***Example* 2.2** Let $X = \{\alpha, \beta, \gamma\}$ and a GT $\mu = \{\emptyset, \{\alpha, \beta\}, \{\beta, \gamma\}, \{\gamma, \alpha\}, X\}$. Consider $S = \{\alpha, \beta\}$ and $T = \{\beta, \gamma\}$, then $d_\mu S = \{\gamma\}$, $d_\mu T = \{\alpha\}$ and $d_\mu(S \cup T) = X$. This example shows that equality does not hold for (c).

Theorem 2.3 Let X be a nonempty set. If $j : expX \to expX$ satisfies

(D_1) $x \in j(S - \{x\}) \Leftrightarrow x \in jS$;
(D_2) if $S \subseteq T$, then $jS \subseteq jT$;
(D_3) $j(S \cup jS) \subseteq S \cup jS$;

for $S, T \subseteq X$, then there exists a unique GT $\mu \subseteq expX$ such that $j = d_\mu$.

Proof. Define $cS := S \cup jS$. If $S \subseteq T$, then $S \cup jS \subseteq T \cup jT$ by (D_2), i.e., $cS \subseteq cT$. Now,

$$\begin{aligned} ccS &= c(S \cup jS) \\ &= (S \cup jS) \cup j(S \cup jS) \\ &\subseteq (S \cup jS) \cup (S \cup jS) && \text{by } (D_3) \\ &= S \cup jS \\ &= cS. \end{aligned}$$

Thus, c satisfies all conditions of Theorem 1.4. Hence, there exists a unique GT $\mu \subseteq \exp X$ such that $c = c_\mu$. Next, we will show that $j = d_\mu$ for the induce GT μ. Suppose that $S \in expX$, then

$$\begin{aligned} x \in d_\mu(S) &\Leftrightarrow x \in c_\mu(S - \{x\}) && \text{by Proposition 2.1 } (e) \\ &\Leftrightarrow x \in c(S - \{x\}) && \text{as } c = c_\mu \\ &\Leftrightarrow x \in (S - \{x\}) \cup j(S - \{x\}) \\ &\Leftrightarrow x \in (S - \{x\}) \cup jS && \text{by } (D_1) \\ &\Leftrightarrow x \in jS. \end{aligned}$$

Hence, the proof is done. ■

μ-Boundary Operator: If μ is a GT on X and $S \subseteq X$, then the μ-boundary of S in X is the intersection of the closure of S with the closure of its complement and denoted by ∂_μ, i.e., $\partial_\mu S = c_\mu S \cap c_\mu(X - S) = c_\mu S - i_\mu S$.

Theorem 2.4 [5] If μ is a GT on X and $S, T \subseteq X$, then

(a) $c_\mu S = S \cup \partial_\mu S$,
(b) $c_\mu S = i_\mu S \cup \partial_\mu S$,
(c) $i_\mu S = S - \partial_\mu S$,
(d) $i_\mu S = c_\mu S - \partial_\mu S$,
(e) $X = i_\mu S \cup \partial_\mu S \cup i_\mu(X - S)$.

Proposition 2.5 If μ is a GT on X and $S, T \subseteq X$, then

(a) $\partial_\mu \emptyset = X - M_\mu$,
(b) $\partial_\mu X = X - M_\mu$,
(c) $\partial_\mu S = \partial_\mu(X - S)$,
(d) $\partial_\mu S \subseteq c_\mu S$,
(e) if $S \subseteq T$, then $\partial_\mu S \subseteq T \cup \partial_\mu T$,
(f) $\partial_\mu(S \cup \partial_\mu S) \subseteq \partial_\mu S$,
(g) $\partial_\mu S$ is μ-closed set,
(h) S is μ-open $\Leftrightarrow S \cap \partial_\mu S = \emptyset$,
(i) S is μ-closed $\Leftrightarrow \partial_\mu S \subseteq S$,

(j) S is μ-clopen $\Leftrightarrow \partial_\mu S = \emptyset$.

All the above results are same as in topology except for (a) and (b).

Theorem 2.6 Let X be a nonempty set. If $j : expX \to expX$ satisfies

(F_1) $j(X - S) = jS$;
(F_2) if $S \subseteq T$, then $jS \subseteq T \cup jT$;
(F_3) $j(S \cup jS) \subseteq jS$;

for any $S, T \subseteq X$, then there exists a unique GT $\mu \subseteq expX$ such that $j = \partial_\mu$.

Proof. Define $cS := S \cup jS$. If $S \subseteq T$, then $S \cup jS \subseteq T \cup jT$ by (F_2), i.e., $cS \subseteq cT$. Therefore,

$$\begin{aligned} ccS &= c(S \cup jS) \\ &= (S \cup jS) \cup j(S \cup jS) \\ &\subseteq (S \cup jS) \cup jS && \textit{by } (F_3) \\ &= S \cup jS \\ &= cS. \end{aligned}$$

Thus, c satisfies all conditions of Theorem 1.4. Hence, there exists a unique GT $\mu \subseteq expX$ such that $c = c_\mu$. Now, we will show that $j = \partial_\mu$ for the induce GT μ. Suppose that $S \in expX$, then

$$\begin{aligned} \partial_\mu S &= c_\mu S \cap c_\mu(X - S) \\ &= cS \cap c(X - S) && \textit{as} \quad c = c_\mu \\ &= (S \cup jS) \cap ((X - S) \cup j(X - S)) \\ &= (S \cup jS) \cap ((X - S) \cup jS) && \textit{by } (F_1) \\ &= (S \cap (X - S)) \cup jS \\ &= jS. \end{aligned}$$

Therefore, the proof is complete. ■

μ-Exterior Operator: If μ is a GT on X and $S \subseteq X$, then the μ-exterior of S is the μ-interior of compliment of S and is denoted by $e_\mu S$. $e_\mu S = i_\mu(X - S)$.

Proposition 2.7 If μ is a GT on X and $S, T \subseteq X$, then

(a) $e_\mu \emptyset = M_\mu$,
(b) $e_\mu X = \emptyset$,
(c) $e_\mu S \subseteq X - S$,
(d) $e_\mu e_\mu S = e_\mu S$,
(e) if $S \subseteq T$, then $e_\mu T \subseteq e_\mu S$,
(f) $e_\mu S \cup e_\mu T \subseteq e_\mu(S \cap T)$,
(g) $e_\mu S \cap e_\mu T \supseteq e_\mu(S \cup T)$,
(h) $e_\mu S = e_\mu(X - e_\mu S)$,
(i) $c_\mu S = X - e_\mu S$.

All the above results are same as in topology except for (a) and (g). In topology equality hold in (g) but not in GT.

***Example* 2.8** Let $X = \{\alpha, \beta, \gamma\}$ and a GT $\mu = \{\emptyset, \{\alpha, \beta\}, \{\beta, \gamma\}, \{\gamma, \alpha\}, X\}$. Consider $S = \{\alpha\}$ and $T = \{\beta\}$, then $e_\mu S = \{\beta, \gamma\}$, $e_\mu T = \{\alpha, \gamma\}$ and $e_\mu(S \cup T) = \emptyset$. This example shows that equality does not hold for (g).

Theorem 2.9 Let X be a nonempty set. If $j : expX \to expX$ satisfies

(E_1) $jS \subseteq X - S$;
(E_2) if $S \subseteq T$, then $jT \subseteq jS$;
(E_3) $jS = j(X - jS)$;

for any $S,\ T \subseteq X$, then there exists a unique GT $\mu \subseteq expX$ such that $j = e_\mu$.

Proof. Define $cS := X - jS$. If $S \subseteq T$, then $S \cup jS \subseteq T \cup jT$ by (E_2), i.e., $cS \subseteq cT$. Also, (E_3) implies that $S \subseteq cS$. Now,

$$\begin{aligned} ccS &= c(X - jS) \\ &= X - j(X - jS) \\ &= X - jS \qquad\qquad \textit{by } (E_3) \\ &= cS. \end{aligned}$$

Thus, c satisfies all conditions of Theorem 1.4. Hence, there exists a unique GT $\mu \subseteq expX$ such that $c = c_\mu$. Now, we will show that $j = e_\mu$ for the induce GT μ. Suppose that $S \in expX$. Then

$$\begin{aligned} e_\mu S &= X - c_\mu S \\ &= X - cS \qquad\qquad \textit{by } c = c_\mu \\ &= X - (X - jS) \\ &= jS. \end{aligned}$$

Hence, the proof is done. ■

μ-Preboundary Operator: In [4], the author defines two new topological set operators. With the same idea we define an operator called μ-preboundary operator. Note that for a μ-open set μ-preboundary operator becomes μ-boundary operator.

Definition 2.10 If μ is a GT on X and $S \subseteq X$, then the μ-preboundary of S in X is the intersection of the closure of S with its complement and is denoted by p_μ. $p_\mu S = c_\mu S \cap (X - S) = c_\mu S - S$.

Proposition 2.11 If μ is a GT on X and $S, T \subseteq X$, then

(a) $p_\mu \emptyset = X - M_\mu$,
(b) $p_\mu X = \emptyset$,
(c) $S \cap p_\mu S = \emptyset$,
(d) $S \cup p_\mu S = c_\mu S$,
(e) if $S \subset T$ then $p_\mu S \subset T \cup p_\mu T$,
(f) $p_\mu(S \cup p_\mu S) = \emptyset$,
(g) $p_\mu S = \partial_\mu S - S$,
(h) $\partial_\mu S = p_\mu S \cup p_\mu(X - S)$,
(i) $(p_\mu S - T) \cup (p_\mu T - S) \subseteq p_\mu(S \cup T)$,

(j) S is μ-closed iff $p_\mu S = \emptyset$,
(k) S is μ-open iff $p_\mu S = \partial_\mu S$.

Proof.

(e) Let $S \subseteq T$ then $c_\mu S \subseteq c_\mu T$. Therefore, (d) implies that

$$p_\mu S \subseteq c_\mu S \subseteq c_\mu T = T \cup c_\mu T.$$

(f) By (d), $p_\mu(S \cup p_\mu S) = p_\mu(c_\mu S) = c_\mu c_\mu S \cap (X - c_\mu S) = \emptyset$.

(g) Let $S \subseteq X$.

$$\begin{aligned}\partial_\mu S - S &= (c_\mu S \cap c_\mu(X - S)) - S\\ &= (c_\mu S \cap c_\mu(X - S)) \cap (X - S)\\ &= c_\mu S \cap (c_\mu(X - S) \cap (X - S))\\ &= c_\mu S \cap (X - S)\\ &= p_\mu S.\end{aligned}$$

(h) Let $S \subseteq X$.

$$\begin{aligned}&p_\mu S \cup p_\mu(X - S)\\ &= (c_\mu S \cap (X - S)) \cup (c_\mu(X - S) \cap S)\\ &= \{(c_\mu S \cap (X - S)) \cup c_\mu(X - S)\} \cap \{(c_\mu S \cap (X - S)) \cup S\}\\ &= \{(c_\mu S \cup c_\mu(X - S)) \cap ((X - S) \cup c_\mu(X - S))\} \cap \{(c_\mu S \cup S) \cap ((X - S) \cup S)\}\\ &= \{X \cap c_\mu(X - S)\} \cap \{c_\mu S \cap X\}\\ &= c_\mu(X - S) \cap c_\mu S\\ &= \partial_\mu S.\end{aligned}$$

(i) Let $S, T \subseteq X$.

$$\begin{aligned}&(p_\mu S - T) \cup (p_\mu T - S)\\ &= (\ p_\mu S \cap (X - T)\ ) \cup (\ p_\mu T \cap (X - S)\ )\\ &= \{\ c_\mu S \cap (X - S) \cap (X - T)\ \} \cup \{\ c_\mu T \cap (X - T) \cap (X - S)\ \}\\ &= \{\ c_\mu S \cap (X - (S \cup T))\ \} \cup \{\ c_\mu T \cap (X - (S \cup T))\ \}\\ &= (c_\mu S \cup c_\mu T) \cap (X - (S \cup T))\\ &\subseteq c_\mu(S \cup T) \cap (X - (S \cup T)) \qquad \textit{by Theorem } 1.3\ (f)\\ &= p_\mu(S \cup T).\end{aligned}$$

(j) S is μ-closed $\Leftrightarrow c_\mu S \subseteq S \Leftrightarrow c_\mu S - S = \emptyset \Leftrightarrow p_\mu S = \emptyset$.

■

Theorem 2.12 Let X be a nonempty set. If $j : expX \to expX$ satisfies

(P_1) $jS \subseteq (X - S)$;
(P_2) if $S \subseteq T$, then $jS \subseteq T \cup jT$;
(P_3) $j(S \cup jS) \subseteq S \cup jS$;

for any $S, T \subseteq X$, then there exists a unique GT $\mu \subseteq expX$ such that $j = p_\mu$.

Proof. Define $cA := A \cup jA$. If $S \subseteq T$, then $S \cup jS \subseteq T \cup jT$ by (P_2), i.e., $cS \subseteq cT$. Now,

$$\begin{aligned} ccS &= c(S \cup jS) \\ &= (S \cup jS) \cup j(S \cup jS) \\ &= (S \cup jS) && by\ (P_3) \\ &= cS. \end{aligned}$$

Thus, c satisfies all conditions of Theorem 1.4. Hence, there exists GT $\mu \subseteq expX$ such that $c = c_\mu$. Now, we will show that $j = p_\mu$ for the induce GT μ. Suppose that $S \in expX$. Then

$$\begin{aligned} p_\mu S &= c_\mu S \cap (X - S) \\ &= cS \cap (X - S) \\ &= (S \cup jS) \cap (X - S) \\ &= (S \cap (X - S)) \cup (jS \cap (X - S)) \\ &= jS \cap (X - S) \\ &= jS, && by\ (P_1). \end{aligned}$$

Hence, the proof is done. ■

3. Conclusion

With careful study of these operators, we identify that most of the general topological results on set operators remain the same for generalized topology. Also, point out some results that are true in general topology but not in generalized topology.

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