Journal of Linear and Topological Algebra
Vol. 11*, No.* 03*,* 2022*,* 159- 168
DOR: 20.1001.1.22520201.2022.11.03.1.6
DOI: 10.30495/JLTA.2022.693226

$\mathcal{E}$-metric spaces and common fixed point theorems

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Received 3 April 2022; Revised 9 June 2022; Accepted 29 July 2022.
Communicated by Vishnu Narayan Mishra

Abstract. In this work we review some common fixed point theorems for four non-continuous mappings in $\mathcal{E}$-metric spaces, where the metric is Riesz space valued. These results cover famous comparable results in the existing literature by considering fewer conditions.

Keywords: $\mathcal{E}$-metric space, Riesz space, coincidence point, weakly compatible pairs, common fixed point.

2010 AMS Subject Classification: 47B60, 54H25.

1. Introduction and preliminaries

We apply the following definitions introduced by Aliprantis and Border [3] and Altun and Cevik [4, 5].
Let $(\mathcal{E}, \leqslant)$ be a partially ordered set. The notation $a < b$ stands for $a \leqslant b$ but $a \neq b$. An order interval $[a, b]$ in $\mathcal{E}$ is the set $\{c \in \mathcal{E} : a \leqslant c \leqslant b\}$. A real linear space $\mathcal{E}$ along with an order relation $\leqslant$ on $\mathcal{E}$ which is compatible with the algebraic structure of $\mathcal{E}$ is named an ordered vector space. Then $(\mathcal{E}, \leqslant)$ is named a Riesz space if for every $a, b \in \mathcal{E}$, there exist $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$. If $a^+ = 0 \vee a$, $a^- = 0 \vee (-a)$ and $|a| = a \vee (-a)$, then $a = a^+ - a^-$ and $|a| = a^+ + a^-$. The cone $\{a \in \mathcal{E} : a \geqslant 0\}$ in a Riesz space $\mathcal{E}$ is marked by $\mathcal{E}_+$. A sequence of vectors $\{a_n\}$ in $\mathcal{E}$ is said to decrease (increase) to an element $a \in \mathcal{E}$ if $a_{n+1} \leqslant a_n$ ($a_n \leqslant a_{n+1}$) for each $n \in \mathbb{N}$ and $a = \inf\{a_n : n \in \mathbb{N}\} = \wedge_{n\in\mathbb{N}} a_n$ ($a = \sup\{a_n : n \in \mathbb{N}\} = \vee_{n\in\mathbb{N}} a_n$), and we denote it by $a_n \downarrow a$ ($a_n \uparrow a$). Also, $\mathcal{E}$ is named Archimedean if $\frac{1}{n}a \downarrow 0$ for each $a \in \mathcal{E}_+$. Moreover, a sequence (b_n) is named order convergent or o-convergent to b if there is a sequence (a_n) in $\mathcal{E}$ provided that $a_n \downarrow 0$ and

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

$|b_n - b| \leqslant a_n$ for all n. We denote this by $b_n \xrightarrow{o} b$. Further, (b_n) is o-Cauchy if there exists a sequence (a_n) in $\mathcal{E}$ so that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leqslant a_n$ for all n and p. $\mathcal{E}$ is o-complete if every o-Cauchy sequence is o-convergent. For other things about Riesz spaces, see [3].

Definition 1.1 [4, 5] Assume that $\mathcal{X} \neq \emptyset$ and $\mathcal{E}$ is a Riesz space. The function $d : \mathcal{X} \times \mathcal{X} \to \mathcal{E}$ is named a vector metric or $\mathcal{E}$-metric if

($\mathcal{E}_1$) $d(x, y) = 0$ iff $x = y$;
($\mathcal{E}_2$) $d(x, y) \leqslant d(x, z) + d(y, z)$.

for all $x, y, z \in \mathcal{X}$

In this case, the triple $(\mathcal{X}, d, \mathcal{E})$ is named a $\mathcal{E}$-metric space.

***Example* 1.2** [4, 5] A Riesz space $\mathcal{E}$ with $d : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ given by $d(x, y) = |x - y|$ is a $\mathcal{E}$-metric space. This metric is called the absolute valued metric.

Obviously, $\mathcal{E}$-spaces extend metric spaces.

Definition 1.3 [4, 5] Let (x_n) be a sequence in a $\mathcal{E}$-metric space. Then

(s_1) (x_n) $\mathcal{E}$-converges to $x \in \mathcal{X}$ (we write $x_n \xrightarrow{d,\mathcal{E}} x$) if there is a sequence (a_n) in $\mathcal{E}$ provided that $a_n \downarrow 0$ and $d(x_n, x) \leqslant a_n$ for each n;
(s_2) (x_n) is $\mathcal{E}$-Cauchy if there is a sequence (a_n) in $\mathcal{E}$ so that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leqslant a_n$ for each n and p.

A $\mathcal{E}$-metric space is named $\mathcal{E}$-complete if every $\mathcal{E}$-Cauchy sequence in $\mathcal{X}$ $\mathcal{E}$-converges to a $x \in \mathcal{X}$.

The difference between $\mathcal{E}$-metric and Zabrejko's metric [15] is that the Riesz space has a lattice structure. Also, the difference between $\mathcal{E}$-metric and Huang-Zhang's metric [7] is that there exists a cone due to the existence of ordering on Riesz space. The other difference is that $\mathcal{E}$-metric omits the requirement for the vector space to be a Banach space. Moreover, if $\mathcal{E} = \mathbb{R}$, then the definitions of $\mathcal{E}$-convergence and convergence in metric are the same. Further, if $\mathcal{X} = \mathcal{E}$ and d is a absolute valued metric, then $\mathcal{E}$-convergence ($\mathcal{E}$-Cauchy) and convergence (Cauchy) in order are the same.

Lemma 1.4 [4] If $\mathcal{E}$ is a Riesz space and $a \leqslant ka$ where $a \in \mathcal{E}_+$ and $k \in [0, 1)$, then $a = 0$.

Definition 1.5 [11] Assume that $f, g : \mathcal{X} \to \mathcal{X}$ are two arbitrary mappings. If $fw = gw = z$ for a $w \in \mathcal{X}$, then w is called a coincidence point of f and g, and z is called a point of coincidence of f and g.

Sessa [13] introduced the concept of weakly commuting to obtain common fixed point for a pairs of mappings. Also, Jungck extended the concept of commuting mappings to compatible mappings in [10] and to weakly compatible mappings in [9]. There exist some examples show that each of these extensions of commutativity are a proper generalization of former definitions.

Definition 1.6 [11] Assume $f, g : \mathcal{X} \to \mathcal{X}$ are two mappings. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Lemma 1.7 [1] Assume that f and g are two weakly compatible self-mappings on $\mathcal{X}$ provided that they have a unique point of coincidence $z = fw = gw$. Then z is the unique common fixed point of f and g.

2. Results

The following theorem is the $\mathcal{E}$-metric version of Theorem 2.1 of [8] and Theorem 2.2 of [2].

Theorem 2.1 Assume that $\mathcal{X}$ is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, g, S, T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(i) for each $x, y \in \mathcal{X}$,

$$d(fx, gy) \leqslant k u_{x,y}(f, g, S, T) \quad (1)$$

in which $k \in (0, 1)$ and

$$u_{x,y}(f, g, S, T) \in \Big\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\Big\}; \quad (2)$$

(ii) $f(\mathcal{X}) \subset T(\mathcal{X})$ and $g(\mathcal{X}) \subset S(\mathcal{X})$.

If one of $f(\mathcal{X})$, $g(\mathcal{X})$, $S(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence. Also, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. Let $x_0 \in \mathcal{X}$ be arbitrary. Since $f(\mathcal{X}) \subset T(\mathcal{X})$, there exists $x_1 \in \mathcal{X}$ so that $f(x_0) = T(x_1) = z_1$. Since $g(\mathcal{X}) \subset S(\mathcal{X})$, there exists $x_2 \in \mathcal{X}$ so that $g(x_1) = S(x_2) = z_2$. By repeating this procedure, construct a sequence $\{z_n\}$ defined by

$$fx_{2n-2} = Tx_{2n-1} = z_{2n-1};$$
$$gx_{2n-1} = Sx_{2n} = z_{2n}.$$

We first establish

$$d(z_{2n+1}, z_{2n+2}) \leqslant k d(z_{2n}, z_{2n+1}) \quad (3)$$

for each n. Using (1), we get

$$d(z_{2n+1}, z_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leqslant k u_{x_{2n}, x_{2n+1}}(f, g, S, T)$$

for all $n \in \mathbb{N}$, where

$$u_{x_{2n}, x_{2n+1}}(f, g, S, T) \in \Big\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\Big\}.$$

If $u_{x_{2n}, x_{2n+1}}(f, g, S, T) = d(z_{2n}, z_{2n+1})$, then (3) holds. If $u_{x_{2n}, x_{2n+1}}(f, g, S, T) = d(z_{2n+1}, z_{2n+2})$, then $d(z_{2n+1}, z_{2n+2}) = 0$ by Lemma 1.4 and (3) holds. Ultimately, assume that

$$u_{x_{2n}, x_{2n+1}}(f, g, S, T) = \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}.$$

Then

$$d(z_{2n+1}, z_{2n+2}) \leqslant \frac{k}{2} d(z_{2n}, z_{2n+1}) + \frac{1}{2} d(z_{2n+1}, z_{2n+2})$$

holds. Analogously, we get

$$d(z_{2n+2}, z_{2n+3}) \leqslant kd(z_{2n+1}, z_{2n+2}). \tag{4}$$

So, by (3) and (4), we obtain

$$d(z_n, z_{n+1}) \leqslant k^n d(z_0, z_1). \tag{5}$$

Now, using $(\mathcal{E}_2)$ and (5). Then, we get

$$d(z_n, z_{n+p}) \leqslant \frac{k^n}{1-k} d(z_0, z_1)$$

for all n and p. Now, due to being Archimedean $\mathcal{E}$, $\{z_n\}$ is a $\mathcal{E}$-Cauchy sequence. Assume that $S(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$. Then there is $z \in S(\mathcal{X})$ so that $Sx_{2n} = z_{2n} \xrightarrow{d,\mathcal{E}} z$. Hence, there exists a sequence $\{a_n\}$ in $\mathcal{E}$ provided that $a_n \downarrow 0$ and $d(Sx_{2n}, z) \leqslant a_n$. Beside, there is $w \in \mathcal{X}$ so that $Sw = z$. Now, we establish that $fw = z$. Using $(\mathcal{E}_2)$ and (1), we get

$$d(fw, z) \leqslant d(fw, gx_{2n+1}) + d(gx_{2n+1}, z) \leqslant ku_{w,x_{2n+1}}(f, g, S, T) + a_{n+1}$$

in which

$$u_{w,x_{2n+1}}(f, g, S, T) \in \Big\{ d(Sw, Tx_{2n+1}), d(fw, Sw), d(gx_{2n+1}, Tx_{2n+1}), \frac{d(fw, Tx_{2n+1}) + d(gx_{2n+1}, Sw)}{2} \Big\}$$

for every n. There are four cases.

1. $d(fw, z) \leqslant d(Sw, Tx_{2n+1}) + a_{n+1} \leqslant a_{n+1} + a_{n+1} \leqslant 2a_n$.
2. $d(fw, z) \leqslant kd(fw, Sw) + a_{n+1} \leqslant kd(fw, z) + a_n$. Hence,
$$d(fw, z) \leqslant \tfrac{1}{1-k} a_n.$$
3. $d(fw, z) \leqslant d(gx_{2n+1}, Tx_{2n+1}) + a_{n+1} \leqslant 2a_{n+1} + a_{n+1} \leqslant 3a_n$.
4.
$$\begin{aligned} d(fw, z) &\leqslant \frac{d(fw, Tx_{2n+1}) + d(gx_{2n+1}, Sw)}{2} + a_{n+1} \\ &\leqslant \frac{d(fw, z) + d(z, Tx_{2n+1}) + d(gx_{2n+1}, z)}{2} + a_{n+1} \\ &\leqslant \frac{1}{2} d(fw, z) + 2a_n. \end{aligned}$$

Thus, $d(fw, z) \leqslant 4a_n$.
Since the infimum of sequences on the right side of last inequality are zero, then

$d(fw, z) = 0$; namely, $fw = z$. So, $fw = Sw = z$. Since $z \in f(\mathcal{X}) \subset T(\mathcal{X})$, there is $v \in \mathcal{X}$ provided that $Tv = z$. Now, we establish that $gv = z$. Using $(\mathcal{E}_2)$ and (1), we get

$$d(z, gv) \leqslant d(z, fx_{2n}) + d(fx_{2n}, gv) \leqslant a_n + ku_{x_{2n},v}(f, g, S, T)$$

in which

$$u_{x_{2n},v}(f, g, S, T) \in \Big\{d(Sx_{2n}, Tv), d(fx_{2n}, Sx_{2n}), d(gv, Tv), \frac{d(fx_{2n}, Tv) + d(gv, Sx_{2n})}{2}\Big\}$$

for every n. There are four cases.

1. $d(z, gv) \leqslant a_n + d(Sx_{2n}, Tv) \leqslant a_n + a_{n+1} \leqslant 2a_n$.
2. $d(z, gv) \leqslant a_n + d(fx_{2n}, Sx_{2n}) \leqslant a_n + 2a_n \leqslant 3a_n$.
3. $d(z, gv) \leqslant a_n + kd(gv, Tv) \leqslant a_n + kd(gv, z)$. Hence, $d(v, gz) \leqslant \frac{1}{1-k}a_n$.
4.

$$\begin{aligned} d(z, gv) &\leqslant a_n + \frac{d(fx_{2n}, Tv) + d(gv, Sx_{2n})}{2} \\ &\leqslant a_n + \frac{d(fx_{2n}, z) + d(gv, z) + d(z, Sx_{2n})}{2} \\ &\leqslant 2a_n + \frac{1}{2}d(z, gv). \end{aligned}$$

So, $d(z, gv) \leqslant 4a_n$.
Since the infimum of sequences on the right side of last inequality are zero, then $d(z, gv) = 0$; namely, $gv = z$. So, $gv = Tv = z$. Thus, $\{f, S\}$ and $\{g, T\}$ have a common point of coincidence. Now, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, $fz = fSw = Sfw = Sz = z_1 \in \mathcal{X}$ and $gz = gTv = Tgv = Tz = z_2 \in \mathcal{X}$. Hence,

$$d(z_1, z_2) = d(fz, gz) \leqslant ku_{z,z}(f, g, S, T),$$

where

$$\begin{aligned} u_{z,z}(f, g, S, T) &\in \Big\{d(Sz, Tz), d(fz, Sz), d(gz, Tz), \frac{d(fz, Tz) + d(gz, Sz)}{2}\Big\} \\ &= \Big\{0, d(z_1, z_2)\Big\}. \end{aligned}$$

So $d(z_1, z_2) = 0$; namely, $z_1 = z_2$. If $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then z is a unique common fixed point of f, g, S and T by Lemma 1.7. The proof is analogous above when $f(\mathcal{X})$, $g(\mathcal{X})$ or $T(\mathcal{X})$ is complete. ■

The following corollary generalizes Fisher's theorem [6] and Theorem 2.1 of Abbas and Jungck [1] to $\mathcal{E}$-metric spaces.

Corollary 2.2 Assume that X is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, g, S, T : \mathcal{X} \to \mathcal{X}$ satisfy

$$d(fx, gy) \leqslant kd(Sx, Ty)$$

for every $x, y \in \mathcal{X}$ in which $k \in [0,1)$. If $f(\mathcal{X}) \subset T(\mathcal{X})$ and $g(\mathcal{X}) \subset S(\mathcal{X})$, and one of $f(\mathcal{X})$, $g(\mathcal{X})$, $S(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in $\mathcal{X}$. Also, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. It's sufficient to consider $u_{x,y}(f, g, S, T) = d(Sx, Ty)$ in Theorem 2.1. ■

***Example* 2.3** Let $\mathcal{E} = \mathbb{R}$, $\mathcal{X} = [0, \infty)$ and $d(x, y) = |x - y|$ for $x, y \in \mathcal{X}$. Also, assume that $f, g, T, S : X \to X$ are given as $f(x) = g(x) = \ln(x + 1)$ and $T(x) = S(x) = e^x - 1$. Then $\{f, S\}$ and $\{g, T\}$ are weakly compatible and the range of all of mappings are complete subspace of $\mathcal{X}$. Further, by applying mean value theorem, we get

$$d(fx, gy) = |\ln(x + 1) - \ln(y + 1)| \leqslant k|x - y| \leqslant k|e^x - e^y| = kd(Tx, Sy)$$

for every $x, y \in \mathcal{X}$ in which $k = \frac{1}{1+c} \in [0, 1)$ with $x < c < y$. So f, g, S and T satisfy all conditions in Corollary 2.2. Furthermore, 0 is a unique common fixed point of these mappings.

The following result is gained from Theorem 2.1 to $\mathcal{E}$-metric spaces.

Corollary 2.4 Assume that $\mathcal{X}$ is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, g, S, T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(i) $d(f^p x, g^q y) \leqslant k u_{x,y}(f^p, g^q, S^p, T^q)$ for all $x, y \in \mathcal{X}$ and some $p, q \in \mathbb{N}$ in which $k \in (0, 1)$ and

$$u_{x,y}(f^p, g^q, S^p, T^q) \in \{d(S^p x, T^q y), d(f^p x, S^p x), d(g^q y, T^q y), \frac{1}{2}[d(f^p x, T^q y) + d(g^q y, S^p x)]\};$$

(ii) $f(\mathcal{X}) \subset T(\mathcal{X})$ and $g(\mathcal{X}) \subset S(\mathcal{X})$.

If one of $f(\mathcal{X})$, $g(\mathcal{X})$, $S(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence. Also, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

The following corollary is same Theorem 2.1 of Soleimani Rad and Altun [14], where generalizes Corollary 2.5 of Abbas et al. [2].

Corollary 2.5 Assume that $\mathcal{X}$ is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, g, T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(i) $d(fx, gy) \leqslant k u_{x,y}(f, g, T)$ for every $x, y \in \mathcal{X}$ in which $k \in (0, 1)$ and

$$u_{x,y}(f, g, T) \in \Big\{d(Tx, Ty), d(fx, Tx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Tx)]\Big\};$$

(ii) $f(\mathcal{X}) \cup g(\mathcal{X}) \subset T(\mathcal{X})$.

If one of $f(\mathcal{X})$, $g(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence. Further, if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g and T have a unique common fixed point.

Proof. It's sufficient to consider $S = T$ in Theorem 2.1. ■

In Corollary 2.5, consider $g = f$. Then we obtain the same Corollary 2.2 in [14] as bellow.

Corollary 2.6 Assume that $\mathcal{X}$ is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(i) $d(fx, fy) \leqslant ku_{x,y}(f, T)$ for every $x, y \in \mathcal{X}$ in which $k \in (0, 1)$ and

$$u_{x,y}(f,T) \in \{d(Tx,Ty), d(fx,Tx), d(fy,Ty), \frac{1}{2}[d(fx,Ty) + d(fy,Tx)]\};$$

(ii) $f(\mathcal{X}) \subset T(\mathcal{X})$.

If one of $f(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, T\}$ have a unique point of coincidence. Further, if $\{f, T\}$ are weakly compatible, then f and T have a unique common fixed point.

Corollary 2.7 Assume that X is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mapping $f : \mathcal{X} \to \mathcal{X}$ satisfies the inequality

$$d(fx, fy) \leqslant ku_{x,y}(f)$$

for all $x, y \in \mathcal{X}$ in which $k \in (0, 1)$ and

$$u_{x,y}(f) \in \{d(x,y), d(fx,x), d(fy,y), \frac{1}{2}[d(fx,y) + d(fy,x)]\}.$$

If $f(X)$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then f has a unique fixed point.

Proof. It's sufficient to set $T = i_{\mathcal{X}}$ in Corollary 2.6 in which $i_{\mathcal{X}}$ is identity mapping on $\mathcal{X}$. ■

The following theorem is the $\mathcal{E}$-metric version of Theorem 2.6 of [2].

Theorem 2.8 Assume that $\mathcal{X}$ is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, g, S, T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(i) for all $x, y \in \mathcal{X}$,

$$\begin{aligned} d(fx, gy) \leqslant\ & k_1 d(Sx, Ty) + k_2 d(fx, Sx) + k_3 d(gy, Ty) \\ & + k_4 d(fx, Ty) + k_5 d(gy, Sx), \end{aligned} \tag{6}$$

where k_i for $i = 1, 2, \cdots, 5$ are nonnegative constants and

$$k_1 + k_2 + k_3 + 2\max\{k_4, k_5\} < 1; \tag{7}$$

(ii) $f(\mathcal{X}) \subset T(\mathcal{X})$ and $g(\mathcal{X}) \subset S(\mathcal{X})$.

If one of $f(\mathcal{X})$, $g(\mathcal{X})$, $S(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence. Also, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof. For each arbitrary point $x_0 \in \mathcal{X}$, take the sequences $\{x_n\}$ and $\{z_n\}$ as the proof of Theorem 2.1. Using (6), we obtain

$$d(z_{2n+1}, z_{2n+2}) \leqslant k_1 d(z_{2n}, z_{2n+1}) + k_2 d(z_{2n+1}, z_{2n}) + k_3 d(z_{2n+2}, z_{2n+1}) + k_4 d(z_{2n+1}, z_{2n+1}) + k_5 d(z_{2n+2}, z_{2n}).$$

Thus,

$$d(z_{2n+1}, z_{2n+2}) \leqslant \beta d(z_{2n}, z_{2n+1}), \tag{8}$$

where $\beta = \frac{k_1+k_2+k_5}{1-k_3-k_5} < 1$ by (7). Analogously, we have

$$d(z_{2n+3}, z_{2n+2}) \leqslant k_1 d(z_{2n+2}, z_{2n+1}) + k_2 d(z_{2n+3}, z_{2n+2}) + k_3 d(z_{2n+2}, z_{2n+1}) + k_4 d(z_{2n+3}, z_{2n+1}) + k_5 d(z_{2n+2}, z_{2n+2}).$$

Hence,

$$d(z_{2n+3}, z_{2n+2}) \leqslant \beta d(z_{2n+2}, z_{2n+1}), \tag{9}$$

where $\beta = \frac{k_1+k_3+k_4}{1-k_2-k_4} < 1$ by (7). Using (8) and (9), we get

$$d(z_n, z_{n+1}) \leqslant \beta^n d(z_0, z_1).$$

As the same discussion of Theorem 2.1, $\{z_n\}$ is $\mathcal{E}$-Cauchy. Assume that $S(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$. Then there is $z \in S(\mathcal{X})$ provided that $Sx_{2n} = z_{2n} \xrightarrow{d,\mathcal{E}} z$. Thus, there exists a sequence $\{a_n\}$ in $\mathcal{E}$ so that $a_n \downarrow 0$ and $d(Sx_{2n}, z) \leqslant a_n$. Beside, we can find $w \in \mathcal{X}$ so that $Sw = v$. Now, we establish that $fw = v$. From (6), we obtain

$$\begin{aligned} d(fw, z) &\leqslant d(fw, gx_{2n+1}) + d(gx_{2n+1}, z) \\ &\leqslant (k_1 + k_3 + k_4) d(z, Tx_{2n+1}) + (k_2 + k_4) d(fw, z) \\ &\quad + (k_3 + k_5 + 1) d(gx_{2n+1}, z). \end{aligned}$$

So,

$$d(fw, z) \leqslant \frac{k_1 + 2k_3 + k_4 + k_5 + 1}{1 - k_2 - k_4} a_n$$

for every n. Hence, $d(fw, v) = 0$; namely, $fw = z$. Since $z \in f(X) \subset T(X)$, there is $v \in X$ provided that $Tv = z$. Now, we establish that $gv = z$. From (6), we obtain

$$\begin{aligned} d(z, gv) &\leqslant d(z, fx_{2n}) + d(fx_{2n}, gv) \\ &\leqslant (k_1 + k_2 + k_5) d(z, Sx_{2n}) + (k_3 + k_5) d(gv, z) \\ &\quad + (k_2 + k_4 + 1) d(fx_{2n}, z). \end{aligned}$$

So,

$$d(gv, z) \leqslant \frac{k_1 + 2k_2 + k_4 + k_5 + 1}{1 - k_3 - k_5} a_n$$

for every n. Hence $d(gv, z) = 0$; namely, $gv = z$. Now, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, $fz = fSw = Sfw = Sz = z_1 \in \mathcal{X}$ and $gz = gTv = Tgv = Tz = z_2 \in \mathcal{X}$. By using (6) and doing simple calculation, we have

$$d(z_1, z_2) \leqslant (k_1 + k_4 + k_5)d(z_1, z_2),$$

which implies that $d(z_1, z_2) = 0$ by (7) and Lemma 1.4. Hence, $z_1 = z_2$; namely, $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence. If $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then z is a unique common fixed point of f, g, S and T by Lemma 1.7. The proof is analogous above when $f(\mathcal{X})$, $g(\mathcal{X})$ or $T(\mathcal{X})$ is complete. ■

The following corollary is same Theorem 2.5 of Soleimani Rad and Altun [14], where generalizes Corollary 2.10 of Abbas et al. [2] to $\mathcal{E}$-metric spaces.

Corollary 2.9 Assume that $\mathcal{X}$ is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, g, T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(i) for every $x, y \in \mathcal{X}$,

$$d(fx, gy) \leqslant k_1d(Tx, Ty) + k_2d(fx, Tx) + k_3d(gy, Ty) \\ + k_4d(fx, Ty) + k_5d(gy, Tx),$$

where k_i for $i = 1, 2, \cdots, 5$ are nonnegative and

$$k_1 + k_2 + k_3 + 2\max\{k_4, k_5\} < 1;$$

(ii) $f(\mathcal{X}) \cup g(\mathcal{X}) \subset T(\mathcal{X})$.

If one of $f(\mathcal{X})$, $g(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, T\}$ and $\{g, T\}$ have a unique point of coincidence. Also, if $\{f, T\}$ and $\{g, T\}$ are weakly compatible, then f, g and T have a unique common fixed point.

Proof. It's sufficient to consider $S = T$ in Theorem 2.8. ■

Corollary 2.10 Assume that $\mathcal{X}$ is a $\mathcal{E}$-metric space with $\mathcal{E}$ is Archimedean and mappings $f, T : \mathcal{X} \to \mathcal{X}$ satisfy the following conditions:

(i) for every $x, y \in \mathcal{X}$,

$$d(fx, fy) \leqslant k_1d(Tx, Ty) + k_2d(fx, Tx) + k_3d(fy, Ty) \\ + k_4d(fx, Ty) + k_5d(fy, Tx)$$

in which k_i for $i = 1, 2, \cdots, 5$ are nonnegative and

$$k_1 + k_2 + k_3 + 2\max\{k_4, k_5\} < 1;$$

(ii) $f(\mathcal{X}) \subset T(\mathcal{X})$.

If one of $f(\mathcal{X})$ or $T(\mathcal{X})$ is a $\mathcal{E}$-complete subspace of $\mathcal{X}$, then $\{f, T\}$ have a unique point of coincidence in $\mathcal{X}$. Also, if $\{f, T\}$ are weakly compatible, then f and T have a unique common fixed point.

Proof. It's sufficient to consider $g = f$ in Corollary 2.9. ■

Remark 1 *Note that although the proof of our results is completely similar to Rahimi et al.'s paper [12] but our results is different of [12], since we consider vector metric spaces instead of ordered vector metric spaces. Actually, it is hard to find mappings satisfying all of condition in the results of Rahimi et al. Also, the assumptions of our theorems and corollaries is much less than of assumptions of theorems and corollaries in mentioned paper.*

3. Conclusion

In this paper, we introduced $\mathcal{E}$-metric version of some famous fixed point theorems without appealing to continuity of mappings. Note that the $\mathcal{E}$-metric is generalization of usually metric, Zabrejko's metric [15], and Huang and Zhang's metric [7]. Thus, our theorems and corollaries unify, extend and generalize well-known comparable results of fixed point theory in metric spaces and cone metric spaces like Jungck and Rhoades [8–11], Abbas and Jungck [1]. Moreover, Our work was continuing and reviewing work of Altun and Cevik [4, 5] and Soleimani Rad and Altun [14].

Acknowledgments

The author wishes to thank editor and referees for their helpful suggestions to improve this manuscript.

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