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Densities and fluxes of the conservation laws for the Kuramoto-Sivashinsky equation

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Abstract. In this paper, the main purpose is to calculate the conservation laws of Kuramoto-Sivashinsky equation using the scaling method. Linear algebra and calculus of variations are used in this algorithmic method. Also the density of the conservation law is obtained by scaling symmetries of the equation and the flux corresponding to the density is calculated using the homotopy operator.

Keywords: KS equation, conversation laws, scaling symmetry, homotopy operator.

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1. Introduction

In applied sciences such as physical chemistry, quantum physics, particle physics, fluid mechanics, etc., there are nonlinear partial differential equations that admit the conservation laws. The basic laws in physics which state that a quantity of an isolated system remains unchanged over time are known as the conservation laws. There are several methods for calculating the conservation laws, some of them can be found in references [2, 4, 5, 7]. Noether's theorem which is used in common methods, relates the conversation laws and variational symmetry of the considered PDE [11, 12]. In contrast, the scaling method uses the calculus of variations and linear algebra, which is described below [14]. First, as a default density, we consider a linear combination of polynomials with arbitrary coefficients that are invariant under the scaling symmetry of the PDE. The concept of symmetries is one of the most important topics in the theory of Lie groups, which can

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be referred to [9, 10] for further study. Then we get the derivative with respect to time from candidate density and combine it with the PDE. Next, using the Euler operator, a system of linear equations is obtained, which solves the desired coefficients, and as a result, the real density is obtained by substituting the specified coefficients instead of the unknown coefficients. After calculating the actual density, the corresponding flux must be obtained using inversion of divergence operator. To do this, the homotopy operator is used. With the help of the homotopy operator, the inversion from divergence is reduced to a one-dimensional integration.
In this study, we construct the new conservation laws of the Kuramoto-Sivashinsky (KS) equation by the scaling method. This equation is a nonlinear PDE that has been used as a model for complex spatiotemporal dynamics in extended systems driven far from equilibrium by intrinsic instabilities. As can be expected, the solutions of the equation exhibit chaotic behavior. Also, this equation plays a dominant role in stability of flame fronts, reaction diffusion, and other physical phenomena [1, 13]. The KS equation is

$$u_t + \alpha u u_x + \beta u_{2x} + \kappa u_{4x} = 0, \tag{1}$$

where α, β and κ are arbitrary constants. The applications of the Kuramoto-Sivashinsky equation go beyond the main field of flame propagation and reaction-diffusion systems. These additional applications include flows in pipes and at interfaces, plasmas, chemical reaction dynamics, and models of ion-sputtered surfaces [8]. Recently, extensive studies have been conducted to find exact solutions and conservation laws for different types of this equation [16, 18]. The novelty of this study is that we have obtained new conservation laws using the scaling method. This paper is organized as follows: In section 2, we have referred to some definitions and previous results that are used in the later sections. We will show KS equation is uniform in rank and admits a scaling symmetry in section 3. In section 4, the primitive density of rank 5 is constructed and actual density is obtained by removing the divergence and divergence-equivalent terms. Finally, we calculate the corresponding fluxes using the homotopy operator. In addition, the conservation laws of orders 1, 3 and 7 are obtained.

2. Preliminaries

In this section, some of the main definitions and concepts that we need are stated. Let $\Delta(x, u^{(n)}) = 0$ be a differential equation, where $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ are independent and dependent variables respectively and $u^{(n)}$ is all the derivatives of u up to nth-order. A conversation law is the following divergence expression

$$\mathrm{Div} Q = 0,$$

which is vanished for all solutions $u = f(x)$ of the given system. The time variable t and the spatial variables $x = (x^1, \ldots, x^p)$ are identified distinguishably in dynamical problems, therefore the conservation law takes the form

$$D_t \rho + \mathrm{Div} J = 0, \tag{2}$$

where ρ is conserved density, J is the corresponding flux, D_t is the total time derivative and Div is the total divergence of $J = (J_1, \ldots, J_p)$ with respect to the spatial variables

[12]. Conserved densities are polynomial in $u^{(n)}$, including polynomial functions of independent variables multiplied to terms. Fluxes have fewer restrictions and are determined by (2) once $D_t\rho$ is computed. If the PDE has rational terms, the flux may also contain rational terms. Transcendental terms are not allowed at this point, but may be included at a later date. The total derivative is an algorithmic tool to compute derivatives with respect to a single independent variable on differential expressions, defined on the jet space. The total time derivative is defined below.

Definition 2.1 Let $f = f(x, t, u^{(M)}(x, t))$ be given. The total time derivative D_t is defined as follows:

$$D_t f = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{q} \sum_{J} u^{\alpha}_{J,t} \frac{\partial f}{\partial u^{\alpha}_{J}},$$

where $J = (j_1, ..., j_k)$ is multi-index with $0 \leqslant k \leqslant M$ and

$$u^{\alpha}_{J,t} = \frac{\partial u^{\alpha}_{J}}{\partial t} = \frac{\partial^{k+1} u^{\alpha}}{\partial t \partial x^{j_1} ... \partial x^{j_k}}.$$

Definition 2.2 Let f be a scalar differential function. The zeroth-Euler operator acting on f is definitionned as

$$\mathcal{L}_{u(x)} f = (\mathcal{L}_{u^1(x)} f, \mathcal{L}_{u^2(x)} f, ..., \mathcal{L}_{u^q(x)} f),$$

where

$$\mathcal{L}_{u^{\alpha}(x)} f = \sum_{k=0}^{M_1^{\alpha}} (-D_x)^k \frac{\partial f}{\partial u^{\alpha}_{kx}}, \qquad \alpha = 1, ..., q, \tag{3}$$

where M_1^{α}'s are the orders of f for the component u^{α} with respect to x [15].

Definition 2.3 Let $f = f(x, u^{(M)}(x))$ of order M be given. f is called exact (or divergence) if a differential vector function $F(x, u^{(M-1)}(x))$ exists such that $f = \mathrm{Div} F$.

Using the zeroth-Euler operator, the following theorem (called Exactness theorem) provides a condition for exactness of a differential function. This theorem is essential for computing conservation laws.

Theorem 2.4 Exactness of a differential function $f = f(x, u^{(M)}(x))$ is equivalent to the condition $\mathcal{L}_{u(x)} f = 0$.

Definition 2.5 Two or more terms are divergence-equivalent when a linear combination of the terms is a divergence.

For example, $u_x u_y$ and $u u_{xy}$ are divergence-equivalent since $u_x u_y + u u_{xy} = \mathrm{Div}(u u_y, 0)$. The zeroth-Euler operator is also used to identify divergence-equivalent terms by applying the following theorem.

Theorem 2.6 When the zeroth-Euler operator is applied to a set of divergence-equivalent terms, their images under the zeroth-Euler operator are linearly dependent [14].

In the following, we will definitionne the homotopy operator. This operator provides a reliable method for integrating exact functions in one independent variable and for inverting divergences on multi-variable exact functions. Homotopies play a large role in topological theory. The homotopy operator first appeared in works by Volterra, where he uses the homotopy operator in the inverse problem of the calculus of variations [17].

Definition 2.7 Given a differential function $f = f(x, u^{(M)}(x))$ of one variable x, the homotopy operator is the following vector operator

$$\mathcal{H}_{u(x)} f = \int_0^1 (\sum_{\alpha=1}^{q} \mathcal{I}_{u^\alpha(x)} f)[\lambda u] \frac{d\lambda}{\lambda}. \tag{4}$$

The integrand is definitionned as

$$\mathcal{I}_{u^\alpha(x)} f = \sum_{k=1}^{M_1^\alpha} (\sum_{j=0}^{k-1} u_{jx}^\alpha (-D_x)^{k-(j+1)}) \frac{\partial f}{\partial u_{kx}^\alpha}. \tag{5}$$

Theorem 2.8 Suppose that an exact differential function $f = f(x, u^{(M)}(x))$ is given. That is, there exists a function $F = F(x, u^{(M-1)}(x))$ such that $f = \text{Div}F$. Then,

$$F = \mathcal{H}_{u(x)} f.$$

3. Scaling symmetry of the KS equation

In calculation of the conservation laws by the scaling method, the scaling (dilation) symmetry is applied to obtain a candidate density. Considering the scaling symmetry, which is computed below, the equation (1) stays invariant under the transformation

$$(x, t, u) \to (\lambda^{-1} x, \lambda^{-4} t, \lambda u, \lambda^2 \alpha, \lambda^2 \beta, \lambda^0 \kappa), \tag{6}$$

where λ is an arbitrarily scaling constant. There are some algorithmic methods for computing the scaling symmetries [3, 4, 6], but we use the concept of weight for variables to find (6) [15].

Definition 3.1 Considering the scaling symmetry $x \to \lambda^{-p} x$, the weight of the variable x, denoted by $W(x)$, is $-p$. If D_x denotes the total derivative with respect to x and $W(x) = -p$, then $W(D_x) = p$ [15].

Definition 3.2 If a monomial has more than one variable and each one has a weight, then the sum of the weights is called the rank of the monomial. If each monomial in a differential function has the same rank, then it is called uniform in rank.

A PDE that admits a scaling symmetry is uniform in rank, so we can compute the scaling symmetry of the KS equation by letting (1) to be uniform in rank. Under this assumption, a system of weight-balance equations, corresponding to the terms in the KS equation can be arranged. By solving this system, we can determine the scaling symmetry. For (1), the weight-balance equations are

$$\begin{aligned} W(u)+W(D_t) &= W(\alpha)+2W(u)+W(D_x) \\ &= W(\beta)+W(u)+2W(D_x) \\ &= W(\kappa)+W(u)+4W(D_x). \end{aligned} \tag{7}$$

Solving the linear system (11) gives $W(\kappa)=0$, $W(u)=1$ and $W(D_x)=1$ then $W(\beta)=W(\alpha)=2$ and $W(D_t)=4$. Since (2) must vanishes on all the solutions of the PDE, the conserved density and its corresponding flux must agree with the scaling symmetry of the PDE. So, the conservation law must be uniform in rank evidently. Thus, according to the scaling symmetry of the KS equation, we can make a primitive density that is a linear combination of monomials of preselected ranks (see [15] for more details).

4. Conservation laws of the KS equation

In this section, we find the conservation laws of the KS equation using the scaling method. For computing the conservation laws, we first need to construct the density ρ. Then, the corresponding flux J is computed. To find the density, we construct a candidate density by taking a linear combination of differential monomials with arbitrary coefficients. Under the scaling symmetry of the KS equation, these terms must be invariant. For the next step, the total derivative of the candidate density with respect to time is calculated and then all the time derivative factors of the terms are substituted with their equivalent expressions using (1). By (2), the obtained expression must be exact. Therefore, arbitrary coefficients are computed by solving the linear system that is found by applying the exactness Theorem 2.4; i.e.,

$$\mathcal{L}_{u(x)}(D_t\rho)=0.$$

Substituting the solution for the coefficients in ρ, we get the actual density. Finally, the corresponding flux is calculated using the homotopy operator

$$J=-\mathrm{Div}^{-1}(D_t\rho).$$

4.1 *Constructing the Candidate Density*

As previously explained, the first step of finding the conservation laws of the KS equation is to compute the candidate density ρ. At first, we choose an arbitrary rank for the primitive density. Then, to construct the terms of ρ, the dependent variables are combined with their partial derivatives to make monomials of the previously specified rank. After that, the monomials are linearly combined with unspecified coefficients to form the candidate density ρ of the same rank.
In the following, we construct the candidate density ρ of rank 5 for the KS equation (1). First, we consider a list $\mathcal{P}$ including all powers of the dependent variables up to rank 5. According to (7), $\mathcal{P}=\{u^5,\alpha u^3,\beta u^3,\alpha^2 u,\beta^2 u,\alpha\beta u,u^4,\alpha u^2,\beta u^2,u^3,\alpha u,\beta u,u^2,u\}$. Then, we extend $\mathcal{P}$ by applying the total derivative operator with respect to the space variables in order to increase the rank of the terms in $\mathcal{P}$ up to 5 and make a new list $\mathcal{Q}$ such as

$$\begin{aligned} \mathcal{Q}=\{&u^5,u^3u_x,u^2u_{2x},u_x^2u,u_{2x}u_x,u_{3x}u,u_{4x},\alpha u_{2x},\alpha uu_x,\alpha u^3,\alpha^2u,\alpha\beta u,\beta u_{2x},\beta uu_x,\\ &\beta^2u,\beta u^3\}. \end{aligned} \tag{8}$$

In order to have a nontrivial density, it should not include divergence terms and one of the terms must be excluded from each pair of the divergence-equivalent terms. By applying (3) over (8), we have

$$L_{u(x)}\mathcal{Q} = \{5u^4, 0, 4u_{2x}u + 2u_x^2, -u_x^2 - 2u_{2x}u, 0, 0, 0, 0, 0, 3\alpha u^2, \alpha^2, \alpha\beta, 0, 0, \beta^2, 3\beta u^2\}. \quad (9)$$

According to the Theorem 2.4, $u^3u_x, u_{2x}u_x, u_{3x}u, u_{4x}, \alpha u_{2x}, \alpha uu_x, \beta u_{2x}$ and βuu_x are divergences, so they must be excluded from $\mathcal{Q}$. Also, the third and fourth terms in list (9) are multiples of each other, therefore u^2u_{2x} and u_x^2u are divergence-equivalent and one of them must be removed. For each divergence-equivalent pair, the corresponding term in $\mathcal{Q}$ with the lowest order is our choice and the other is omitted. Therefore, $\mathcal{Q}$ is shrunk to the following

$$\mathcal{Q} = \{u^5, u_x^2u, \alpha u^3, \alpha^2u, \alpha\beta u, \beta^2u, \beta u^3\}.$$

Now, taking a linear combination of the terms in $\mathcal{Q}$, we form the primitive density of rank 5 for the KS equation as

$$\rho = c_1u^5 + c_2u_x^2u + c_3\alpha u^3 + c_4\alpha^2u + c_5\alpha\beta u + c_6\beta^2u + c_7\beta u^3, \quad (10)$$

where $c_1, \ldots, c_7$ are arbitrary scalars. We will specify c_i's in the next subsection.

4.2 *Determining the Actual Density*

A trivial conservation law will fit one of two possible cases. The first case occurs when the density ρ and the flux J vanish independently for solutions of the given PDE. The second case occurs when the conservation law (2) holds identically for u, without u being a solution of the PDE [14]. It is important to note that conservation laws are valuable that are not trivial and are not equivalent. For this reason, we compute densities that are not equivalent or divergent. To determine the unknown coefficients in (10), we calculate $D_t\rho$ and replace u_t and its differentials with their equivalent representations using (1).

$$D_t\rho = (5c_1u^4 + c_2u_x^2 + 3c_3\alpha u^2 + c_4\alpha^2 + c_5\alpha\beta + c_6\beta^2 + 3c_7\beta u^2)u_t + 2c_2u_xuu_{xt}.$$

Let $E = -D_t\rho$. Then, using (1), we have

$$E = (5c_1u^4 + c_2u_x^2 + 3c_3\alpha u^2 + c_4\alpha^2 + c_5\alpha\beta + c_6\beta^2 + 3c_7\beta u^2)(\alpha uu_x + \beta u_{2x} + \kappa u_{4x}) + 2c_2u_xu(\alpha u_x^2 + \alpha uu_{2x} + \beta u_{3x} + \kappa u_{5x}).$$

According to (2), E is required to be exact. So, by the Theorem 2.4, $\mathcal{L}_{u(x)}E = 0$. This equation gives a system of linear equations as

$$c_1 = c_2 = 0, \qquad c_3 = -\frac{\beta}{\alpha}c_7, \quad (11)$$

and c_4, c_5, c_6 and c_7 are arbitrary. Assuming $c_4 = c_5 = c_6 = c_7 = 1$ and substituting (11) in (10), the actual density is obtained as follows:

$$\rho = (\alpha^2 + \alpha\beta + \beta^2)u.$$

4.3 *Computing the Flux*

After constructing the density ρ, we are ready to compute the corresponding flux using the fact that $J = \mathrm{Div}^{-1}(E)$, by Theorem 2.8. Substituting (11) in E, we have

$$E = (\alpha^2 + \alpha\beta + \beta^2)(\alpha u u_x + \beta u_{2x} + \kappa u_{4x}).$$

The integrand $\mathcal{I}_{u(x)}E$ can be calculated by (5) as

$$\mathcal{I}_{u(x)}E = \sum_{k=1}^{4} \big(\sum_{j=0}^{k-1} u_{jx}(-D_x)^{k-(j+1)} \big) \frac{\partial f}{\partial u_{kx}} = \\ \alpha^3 u^2 + \alpha^2\beta u^2 + \alpha\beta^2 u^2 + \alpha^2\beta u_x \alpha\beta^2 u_x + \beta^3 u_x + \alpha^2\kappa u_{3x} + \alpha\beta\kappa u_{3x} + \beta^2\kappa u_{3x}.$$

Using (4), we obtain the corresponding flux of rank 5 for equation as follows:

$$J = \mathrm{Div}^{-1}(E) = \tfrac{1}{2}\alpha^3 u^2 + \tfrac{1}{2}\alpha^2\beta u^2 + \tfrac{1}{2}\alpha\beta^2 u^2 + \alpha\beta^2 u_x + \beta^3 u_x + \alpha^2\kappa u_{3x} + \alpha^2\beta u_x + \\ \beta^2\kappa u_{3x} + \alpha\beta\kappa u_{3x}.$$

So a conservation law of rank 5 for the Kuramoto-Sivashinsky equation is constructed as follows:

$$D_t\rho + \mathrm{Div}J = D_t((\alpha^2 + \alpha\beta + \beta^2)u) + \mathrm{Div}\big(\tfrac{1}{2}\alpha^3 u^2 + \tfrac{1}{2}\alpha^2\beta u^2 + \tfrac{1}{2}\alpha\beta^2 u^2 + \\ \alpha\beta^2 u_x + \beta^3 u_x + \alpha^2\kappa u_{3x} + \alpha^2\beta u_x + \beta^2\kappa u_{3x} + \alpha\beta\kappa u_{3x}) = 0.$$

Additional conservation laws for the Kuramoto-Sivashinsky equation of rank 1, 3 and 7 are as follows:

$$\rho_1 = u,$$
$$J_1 = \tfrac{1}{2}\alpha u^2 + \beta u_x + \kappa u_{3x},$$
$$\rho_3 = (\alpha + \beta)u,$$
$$J_3 = \tfrac{1}{2}\alpha^2 u^2 + \tfrac{1}{2}\alpha\beta u^2 + \alpha\kappa u_{3x} + \beta\kappa u_{3x} + \alpha\beta u_x + \beta^2 u_x,$$
$$\rho_7 = (\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)u + (\alpha + \beta)(2u u_x^2 + u^2 u_{2x}),$$
$$J_7 = \tfrac{\kappa(\alpha+\beta)}{3}(40u u_{2x} u_{3x} + 26u u_x u_{4x} - 20u_x^2 u_{3x}) + (\alpha\beta^3 + \alpha^3\beta + \alpha^2\beta^2 + \beta^4)u_x + \\ \kappa(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)u_{3x} + \tfrac{1}{2}(\alpha^2\beta^2 + \alpha\beta^3 + \alpha^3\beta + \alpha^4)u^2 + \\ \tfrac{7}{3}\kappa(\alpha + \beta)u^2 u_{5x} + (\beta\alpha + \beta^2)u^2 u_{3x} + (2\beta\alpha + 2\beta^2)u u_x u_{2x} + \\ +(\alpha^2 + \alpha\beta)(3u^2 u_x^2 + u^3 u_{2x}).$$

5. Conclusions

In this paper, we consider the Kuramoto-Sivashinsky equation which appears in some physical phenomena. This equation is uniform in rank and admits scaling symmetry. The scaling symmetry was obtained by the concept of weight of variables and the density is constructed by using this symmetry. The corresponding flux is computed by one dimensional homotopy operator. We obtain the conservation laws of rank 1, 3, 5 and 7 for the Kuramoto-Sivashinsky equation by the scaling method.

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