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Equivariant homologies for operator algebras; a survey

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Abstract. This is a survey of a variety of equivariant (co)homology theories for operator algebras. We briefly discuss a background on equivariant Hochschild cohomology. We discuss a notion of equivariant L^2-cohomology and equivariant L^2-Betti numbers for subalgebras of a von Neumann algebra. For graded C^*-algebras (with grading over a group) we elaborate on a notion of graded L^2-cohomology and its relation to equivariant L^2-cohomology.

Keywords: Equivariant Hochschild cohomology, equivariant L^2-cohomology, group action, graded L^2-Betti number, graded algebra.

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1. Introduction

Homology theories are generally regarded as a method of constructing algebraic invariants of other structures, with a wide range of applications in geometry, analysis and algebra.

The first historical instance of a homology theoretical consideration is probably the Euler polyhedron formula (now called Euler characteristics), rigorously proved by Cauchy in 1811 (for any convex polyhedron). It was Riemann who first's defined the genus (and connectedness numerical invariants) in 1857 and Betti who showed the independence of homology numbers from the choice of basis in 1871.

Poincarè invented the fundamental group and initiated the study of the so called algebraic varieties and their homology theory. He was then led to Betti numbers through solving certain vector field problems (to determine the number of holes in the configuration space). One could make a simple observation of how holes (and so Betti numbers) are significant by examining the behavior of solutions of differential equations, say on

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the torus compared to the sphere. Poincarè [19] published his ideas of analysis situs, where he introduced the notion of homology classes. To classify possible configurations of orientable cycles, he used Betti numbers (as refinements of the Euler characteristics). He also noticed that classifying non-orientable cycles requires torsion coefficients.

The history goes on with great names, such as Alexander (with theory of knots) Veblen, Brouwer, as well as van Kampen and Lefschetz. It was Noether who first recognized that the Betti numbers and torsion coefficients are numerical invariants of isomorphism classes of finitely-generated abelian groups.

Algebraic counterparts of the theory came later. As one of the first instances of such algebraic homology theories, the Hochschild (co)homology was introduced by Hochschild [13] for algebras over a field, and extended to algebras over more general rings by Cartan and Eilenberg around 1956. Cyclic (co)homology which generalizes the de Rham (co)homology of manifolds were independently introduced by Tsygan (homology) and Connes (cohomology) in the 80's. These are naturally related to the de Rham theory, Hochschild (co)homology and K-theory. The later development are, among others, due to Karoubi, Wodzicki and Loday. The origin of the equivariant homology theory (motivated by group extensions) goes back to Whitehead in 1950. In algebraic topology, this provides an invariant for classification of topological spaces with a (given) group action. This is motivated by gauge theory in physics, which is a field theory where the Lagrangian is invariant under certain Lie groups of local transformations. It is shown that all homology theories have very natural and successful equivariant counterparts.

The main focus of the present survey is a notion of equivariant L^2-cohomology for operator algebras. This is discussed in more details in Section 3. We briefly review equivariant Hochschild cohomology in somewhat more technical details and return to it again in the last section, where we discuss equivariant L^2-cohomology and equivariant L^2-Betti numbers.

2. Classical equivariant theories

2.1 *Topological background*

The main idea in singular cohomology is to give a functor from the category of topological spaces and continuous maps to the category of graded rings and homomorphisms. When there is also a group action on the topological space (with appropriate continuity properties) one may ask about the existence of a functor which encodes both the topology and the algebra (the action). Equivariant cohomology is supposed to do this.

The origin of equivariant cohomology goes back to 1959 when Borel defined equivariant singular cohomology using the so called Borel construction [5]. In 1950, Cartan [7] studied the action of a compact connected Lie group on a manifold and constructed a differential complex out of the differential forms on the manifold and the Lie algebra of the acting group (which gives the real equivariant singular cohomology in modern terms). When the group is trivial, this reduces to the de Rham complex of smooth differential forms which computes the real singular cohomology of the manifold. The equivariant cohomology proved later to be a strong computational tool, when it was used in the equivariant integration formula of Atiyah-Bott [2] and Berline-Vergne [4].

For a manifold M on which the circle S^1 acts, the Atiyah-Bott fixed point formula describes the equivariant index of an elliptic operator on M in terms of local data near the fixed points of the action. An application of this formula was a geometric interpretation of the Weyl formula for the characters of irreducible representations of compact Lie groups. The character formula has continuous analogues: the formulae for the Fourier transforms of coadjoint orbits, which are linked to representation theory via Kirillov's orbit method. For compact groups, this is the Harish-Chandra formula; for non-compact semi-simple groups, Rossmann gave a fixed point formula in the case of discrete series characters. Then Berline and Vergne (1983) used the equivariant forms to find a geometric interpretation of Rossmann formula. The cohomological tool behind their computation was a deformation of the de Rham complex with the use of vector fields [4]. A similar approach was used by Witten in 1982 with a motivation in suppersymmetry and Morse theory [25].

In turn, the equivariant integration formula is used to show that the stationary phase approximation formula is exact for a symplectic action (Atiyah-Bott 1984), to calculate the number of rational curves in a quintic threefold (Kontsevich 1995, Ellingsrud-Strømme 1996), and to calculate the characteristic numbers of a compact homogeneous space (Tu 2010) [22]. Following Tu, we refer the reader to the expository articles by Bott [1, 2] and Vergne [23] for further applications.

2.2 *Equivariant Hochschild cohomology*

The equivariant Hochschild cohomology theory for Banach and operator algebras is developed by Jensen [14, 15]. Jensen introduces this cohomology first in a purely algebraic setting and then adapt it to the context of Banach and operator algebras. Here we briefly describe Jensen construction. First we need to give some necessary definitions. We start with the notion of action of a locally compact group on a C^*-algebra.

We say that a group G acts on an algebra $\mathcal{A}$ if there is a homomorphism α from G into the group of automorphisms of $\mathcal{A}$, that is, a map $\alpha : G \to \mathrm{Aut}(\mathcal{A}); g \mapsto \alpha_g$ with the following properties:

(i) for each $g \in G$, the map α_g is bijective and linear;
(ii) $\alpha_g(ab) = \alpha_g(a)\alpha_g(b)$ $(a, b \in \mathcal{A})$;
(iii) $\alpha_{gh} = \alpha_g\alpha_h$ $(g, h \in G)$.

We denote this action by $G \curvearrowright_\alpha \mathcal{A}$. Also, the fixed point algebra of $\mathcal{A}$ with respect to α is the set $\mathcal{A}^\alpha = \{a \in A | \alpha_g(a) = a \ (g \in G)\}$.

Let $G \curvearrowright_\alpha \mathcal{A}$. Suppose X is an $\mathcal{A}$-bimodule. Then, X is said to be an equivariant (α, β)-G-$\mathcal{A}$-module if there is a map $\beta : G \to \mathrm{Lin}(X); g \mapsto \beta_g$ with the following properties:

(i) for each $g \in G$ the map β_g is a linear bijective,
(ii) $\beta_g(a \cdot x) = \alpha_g(a) \cdot \beta_g(x)$ and $\beta_g(x \cdot a) = \beta_g(x) \cdot \alpha_g(a)$ $(a \in \mathcal{A}, x \in X)$,
(iii) $\beta_{gh} = \beta_g\beta_h$ $(g, h \in G)$.

We denote this action by $G \curvearrowright_\beta X$. Next let us recall the definition of an equivariant cochain.

Let $G, \mathcal{A}, X, \alpha$ and β be as above. For each $n \geqslant 1$ define

$$C^n_G(\mathcal{A}, X) = \{T \in L^n(A, X) | T(\alpha_g(a_1), ..., \alpha_g(a_n)) = \beta_g(T(a_1, ..., a_n)) \ (g \in G)\}.$$

The elements of $C^n_G(\mathcal{A}, X)$ are called the (α, β)-equivariant n-cochains (of $\mathcal{A}$ with coefficients in X).

The equivariant zero-cochains are defined separately as follows

$$C^0_G(\mathcal{A}, X) = \{x \in X | \beta_g(x) = x \ (g \in G)\}.$$

Note that the usual coboundary operator $\delta^n : C^n(\mathcal{A}, X) \to C^{n+1}(\mathcal{A}, X)$ maps $C^n_G(\mathcal{A}, X)$ into $C^{n+1}_G(\mathcal{A}, X)$. We denote the restriction map by δ^n_G.

The equivariant cocycle and coboundary groups are defined, respectively, by

$$Z^n_G(\mathcal{A}, X) = \ker \delta^n_G, \quad B^n_G(\mathcal{A}, X) = \operatorname{ran} \delta^{n-1}_G$$

and the equivariant cohomology groups are defined as the quotients

$$H^n_G(\mathcal{A}, X) = Z^n_G(\mathcal{A}, X)/B^n_G(\mathcal{A}, X).$$

Suppose $\mathcal{A}$ is a Banach algebra with G a group acting on it. Then, $\mathcal{A}$ is said to be G-contractible if $H^1_G(\mathcal{A}, X) = 0$ for all Banach G-$\mathcal{A}$-modules X. Also $\mathcal{A}$ is said to be G-amenable if $H^1_G(\mathcal{A}, X) = 0$ for all dual G-$\mathcal{A}$-modules X. Jensen also calculates some low-dimensional equivariant Hochschild cohomologies. For example, he shows that If $\mathcal{A}$ is a unital abelian C^*-algebra and α is an action of a group G on $\mathcal{A}$, then $H^2_G(\mathcal{A}, \mathcal{A}) = 0$ [14, Corollary II.1.10]. He also verifies the reduction dimension formula [14, Corollary II.2.6] as $H^{n+p}_G(\mathcal{A}, X) \simeq H^n_G(\mathcal{A}, BL^p(\mathcal{A}, X))$.

The relation between ordinary and equivariant cohomology is also investigated. Let $\mathcal{A}$ be a Banach algebra and G be a locally compact group acting on $\mathcal{A}$. Then the map $H^n_G(\mathcal{A}, X) \to H^n(\mathcal{A}, X)$ is injective, whenever G is compact (amenable) and X is a Banach (dual) module [14, Proposition II.3.2]. Also if the action α of G on $\mathcal{A}$ is inner and the Banach algebra generated by $\{\alpha_g | g \in G\}$ is amenable and the module actions on X have norm at most 1, then the map $H^n_G(\mathcal{A}, X) \to H^n(\mathcal{A}, X)$ is also surjective. In particular, if G is an amenable group that acts innerly on a Von Neumann algebra $\mathcal{M}$, then $H^n_G(\mathcal{M}, \mathcal{M})$ and $H^n(\mathcal{M}, \mathcal{M})$ are isomorphic [14, Lemma II.3.10].

If $\mathcal{A}$ is a UHF C^*-algebra and G has product action on $\mathcal{A}$, then $H^n_G(\mathcal{A}, X) = 0$ for every dual module X [14, Corollary III.2.8]. For a II_1-factor $\mathcal{M}$ with a discrete group action, $\mathcal{M}$ is injective whenever $H^1_G(\mathcal{M}, X) = 0$ for every dual module X [14, Theorem III.3.7].

Next, a relative equivariant cohomology theory could be defined. Let $\mathcal{A}$ be a Banach algebra, G a group acting on $\mathcal{A}$, and X a Banach or dual G-$\mathcal{A}$-module. If $\mathcal{B}$ is a G-invariant Banach subalgebra of $\mathcal{A}$, then

$$Z^n_G(\mathcal{A}, \mathcal{B}; X) = \{T \in C^n_G(\mathcal{A}, \mathcal{B}; X) | \delta^n_G(T) = 0\},$$

and

$$B^n_G(\mathcal{A}, \mathcal{B}; X) = \{\delta^{n-1}_G(T) | T \in C^{n-1}_G(\mathcal{A}, \mathcal{B}; X)\}.$$

Here $C^n_G(\mathcal{A}, \mathcal{B}; X)$ is the set of relative equivariant n-cochains in the sense of [14, Definition II.4.1]. Then, define the relative equivariant cohomology group to be the quotient of the cocycle and the coboundary groups, that is

$$H^n_G(\mathcal{A}, \mathcal{B}; X) = Z^n_G(\mathcal{A}, \mathcal{B}; X)/B^n_G(\mathcal{A}, \mathcal{B}; X).$$

Proposition 2.1 [14, Proposition II.4.4] Suppose $\mathcal{A}$ is a Banach algebra, G is a group acting on $\mathcal{A}$, and $\mathcal{B}$ is a G-invariant Banach subalgebra of $\mathcal{A}$. If (i) or (ii) is satisfied, then $H^n_G(\mathcal{A}, \mathcal{B}; X) = H^n_G(\mathcal{A}, X)$.

(i) $\mathcal{B}$ is G-contractible, X is Banach G-$\mathcal{A}$-module, and G is discrete.

(ii) $\mathcal{B}$ is G-amenable and X is a dual G-$\mathcal{A}$-module.

This could be used to show that $H^1_{\mathbb{Z}}(\mathcal{A},\mathcal{A}) \simeq \mathbb{C}$ where the action of $\mathbb{Z}$ on $\mathcal{A}$ is defined by $\alpha_n = \alpha^n$ for all n in $\mathbb{Z}$.

If we define an action of $\mathbb{Z}_p$ (the cyclic group of order p) on $\mathcal{A}$, for p in $\mathbb{N}$, by $\alpha_i = \alpha^i_{\theta_p}$, $i \in \mathbb{Z}_p$ where θ_p is a primitive p-th root of unity, then

(i) $H^n_{\mathbb{Z}_2}(\mathcal{A},\mathcal{A}) \simeq \mathbb{C}$ for all $n \geqslant 0$,
(ii) $H^n_{\mathbb{Z}_p}(\mathcal{A},\mathcal{A}) \simeq \mathbb{C}$ for $n = 0, 1$,
(iii) $H^n_{\mathbb{Z}_p}(\mathcal{A},\mathcal{A}) = 0$ for $n > 1$.

For each discrete group G, that if $\mathcal{A}$ is G-contractible, then $H^n_G(\mathcal{A}, X) = 0$ for all n and all Banach G-$\mathcal{A}$-modules X. For the compact case we have the following proposition.

Proposition 2.2 [15, Lemma III.1.1] If $\mathcal{A}$ is a contractible Banach algebra and G is a compact group acting on $\mathcal{A}$, then $\mathcal{A}$ is G-contractible. Moreover, $H^n_G(\mathcal{A}, X) = 0$ for all $n \geqslant 1$ and all Banach G-$\mathcal{A}$-modules X.

The next proposition deals with the converse of the above proposition.

Proposition 2.3 [15, Corollary III.1.7] Suppose $\mathcal{A}$ is a unital Banach algebra and G is a group acting on $\mathcal{A}$.

(i) If $\mathcal{A}$ is G-contractible, then $\mathcal{A}$ is contractible.
(ii) If the group G is compact, then $\mathcal{A}$ is contractible if and only if $\mathcal{A}$ is G-contractible.

Suppose $\mathcal{A}$ is a finite-dimensional C^*-algebra, α is an action of a group G on $\mathcal{A}$, and $(\mathcal{A}, \alpha, X, \beta)$ is a Banach or dual equivariant module. Then $H^n_G(\mathcal{A}, X) = 0$ for all $n \geqslant 1$. This leads to the fact that $\mathcal{A}$ is contractible if and only if it is G-contractible.

If $\mathcal{A}$ is amenable as a Banach algebra and G is amenable as a group, then $\mathcal{A}$ is G-amenable [15, Lemma III.2.1]. In the case where G is a discrete we have that

(i) If $\mathcal{A}$ is G-amenable, then $\mathcal{A}$ is amenable.
(ii) If G is amenable as a group, then $\mathcal{A}$ is G-amenable if and only if $\mathcal{A}$ is amenable.

If $\mathcal{A}$ is a UHF C^*-algebra and G has product action on $\mathcal{A}$, then $\mathcal{A}$ is G-amenable [15, Corollary III.2.8].

For a II_1-factor $\mathcal{M}$ the following theorem holds.

Theorem 2.4 [15, Theorem III.3.7]. Suppose $\mathcal{M}$ is a II_1-factor and G a discrete group acting on $\mathcal{M}$. If $\mathcal{M}$ is not amenable (as a von Neumann algebra), there is a normal dual $\mathcal{M}$-module X that is also a dual G-$\mathcal{M}$-module such that $H^1_G(\mathcal{M}, X) \neq 0$. Consequently, if $\mathcal{M}$ is G-amenable, then $\mathcal{M}$ is amenable.

We return to this theory in Section 4, where we give more details needed for equivariant L^2-cohomology.

3. L^2-cohomology

3.1 *Basic notions*

Throughout the rest of the paper, for an algebra $\mathcal{A}$ we denote its opposite algebra by $\mathcal{A}^\circ$ and the enveloping algebra associated with $\mathcal{A}$ by $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^\circ$.

Also, $\mathcal{A}_1$ denotes the closed unit ball of a C^*-algebra $\mathcal{A}$. The weak (respectively, strong) operator topology is denoted by WOT (respectively, SOT) and UWOT denotes the ultraweak operator topology on $B(\mathcal{H})$, the space of bounded linear operators on a Hilbert space $\mathcal{H}$.

Let X be a Banach $\mathcal{A}$-bimodule. Then X is said to be dual if it is a dual of a Banach space and for each $a \in \mathcal{A}$, the maps $X \to X; x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are weak* continuous. If in addition, $\mathcal{A}$ admits a weak* topology (for example whenever $\mathcal{A}$ is a von Neumann algebra), and for every $x \in X$ the maps $\mathcal{A} \to X; a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weak* continuous, then X is called a normal dual module.

Let X be a Banach $\mathcal{A}$-bimodule. Let $BL^0(\mathcal{A}, X) = X$ and for each $n \in \mathbb{N}$, denote by $BL^n(\mathcal{A}, X)$ the space of all bounded n-linear maps from $\mathcal{A}^n$ into X.

For a C^*-algebra $\mathcal{A}$, a map $\tau : \mathcal{A}^+ \to [0, \infty]$ is called tracial (or a trace) if $\tau(ab) = \tau(ba)$, $\tau(a + b) = \tau(a) + \tau(b)$ and $\tau(\lambda a) = \lambda\tau(a)$, for all $a, b \in \mathcal{A}^+$ and $\lambda \geqslant 0$. A trace τ is said to be faithful if $a = 0$ whenever $\tau(a^*a) = 0$, for $a \in \mathcal{A}$. Each faithful trace on $\mathcal{A}$ induces a norm $||.||_\tau$ on $\mathcal{A}$ defined by $||a||_\tau^2 = \tau(a^*a)$ $(a \in \mathcal{A})$. A trace τ on a von Neumann algebra $\mathcal{M}$ is said to be normal if $a_i \uparrow a$ in the SOT of $\mathcal{M}$, then $\tau(a_i) \uparrow \tau(a)$.

Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. A closed densely defined operator $T : Dom(T) \subseteq \mathcal{H} \to \mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $Tu = uT$ for all unitary elements in the commutant $\mathcal{M}'$. The set of all operators affiliated with $\mathcal{M}$ is denoted by $\mathcal{U}(\mathcal{M})$. Linear operations and multiplication on $\mathcal{U}(\mathcal{M})$ is defined as follows. For each $x, y \in \mathcal{U}(\mathcal{M})$ one can form the natural sum and product of x and y but these may not be closed operators. Their closure are elements of $\mathcal{U}(\mathcal{M})$ called the strong sum and strong product denoted by $x + y$ and xy (here the finiteness of $\mathcal{M}$ is crucial). The involution operation is defined obviously. With these definitions $\mathcal{U}(\mathcal{M})$ becomes a $*$-algebra containing $\mathcal{M}$ as a $*$-subalgebra. This algebra can obtained in the following way.

Theorem 3.1 [3, Theorem 1] If $\mathcal{M}$ is a finite von Neumann algebra, then there exists a unital $*$-algebra R containing $\mathcal{M}$ as a $*$-subalgebra such that

(i) R is a regular ring,
(ii) the relations $x, y, z \in R$ and $x^*x + y^*y + z^*z = 1$ imply $x, y, z \in \mathcal{M}$.

These conditions determine R uniquely up to a $*$-isomorphism that leaves fixed the elements of $\mathcal{M}$. Note that a ring R is called (von Neumann) regular if, for each element $x \in R$, there exists an element $y \in R$ such that $x = xyx$.

As an example, if $\mathcal{M}$ is an abelian von Neumann algebra, then the algebra $\mathbb{M}_2(\mathcal{M})$ of 2×2 matrices over $\mathcal{M}$ is a finite von Neumann algebra and $\mathcal{U}(\mathbb{M}_2(\mathcal{M}))$ is identified with $\mathbb{M}_2(\mathcal{U}(\mathcal{M}))$ [3, Lemma 2].

If T is a closed densely defined operator on $\mathcal{H}$ with the polar decomposition $T = v|T|$, then $T \in \mathcal{U}(\mathcal{M})$ if and only if $v \in \mathcal{M}$ and $|T| = \int_0^\infty \lambda \, dE_\lambda \in \mathcal{U}(\mathcal{M})$. In this case, the spectral projections E_λ belong to $\mathcal{M}$.

For a finite von Neumann algebra $\mathcal{M}$, the set $\mathcal{U}(\mathcal{M})$ is an involutive algebra under the natural operations. The interested reader will find more details in [3, 10].

Nelson in [18] showed that if $\mathcal{M} \subseteq B(\mathcal{H})$ is a tracial von Neumann algebra with a faithful, normal, finite trace τ, then $\mathcal{U}(\mathcal{M})$ carries a translation-invariant topology which is called the measure topology and is generated by the family $\{N(\epsilon, \delta)\}_{\epsilon,\delta>0}$ of neighborhoods of 0 of the form

$$N(\epsilon, \delta) = \{a \in \mathcal{M}; \text{for some projection}\, p \in \mathcal{M}, \|ap\| \leqslant \epsilon \,\text{and}\, \tau(1 - p) \leqslant \delta\}.$$

He also showed that one can identify $\mathcal{U}(\mathcal{M})$ with the closure of $\mathcal{M}$ in this topology [18, Theorem 4]. More precisely, each $T \in \mathcal{U}(\mathcal{M})$ with the spectral decomposition $v \int_0^\infty \lambda \, dE_\lambda$ is the limit of a sequence $T_n = v \int_0^n \lambda \, dE_\lambda \in \mathcal{M}$ in the measure topology.

This fact is used in Section 4 to extend an action of a group G on $\mathcal{M}$ to an action of G on $\mathcal{U}(\mathcal{M})$.

Let $\mathcal{A}$ be a unital algebra with a unit 1. Then, $\mathcal{A}^e$ is an $\mathcal{A}$-bimodule with the actions

$$a \cdot (b \otimes c^\circ) = ab \otimes c^\circ \quad \text{and} \quad (b \otimes c^\circ) \cdot a = b \otimes (ca)^\circ \qquad (a, b, c \in \mathcal{A}). \tag{1}$$

If $\mathcal{A}$ is a Banach algebra, these actions extend by linearity and continuity.

In general, there is a one to one correspondence between bimodules over $\mathcal{A}$ and left (right) modules over $\mathcal{A}^e$. Indeed, if X is an $\mathcal{A}$-bimodule, then it is a left $\mathcal{A}^e$-module via $(a \otimes b^\circ) \cdot x = a \cdot x \cdot b$ (or it is a right $\mathcal{A}^e$-module via $x \cdot (a \otimes b^\circ) = b \cdot x \cdot a$) for $a, b \in \mathcal{A}$ and $x \in X$. Conversely, if X is a left $\mathcal{A}^e$-module, then it is an $\mathcal{A}$-bimodule with the actions $a \cdot x = (a \otimes 1^\circ) \cdot x$ and $x \cdot b = (1 \otimes b^\circ) \cdot x$ for every $a, b \in \mathcal{A}$ and $x \in X$. The right module actions are defined similarly.

For a von Neumann algebra $\mathcal{M}$, its enveloping von Neumann algebra $\mathcal{M}\bar{\otimes}\mathcal{M}^\circ$ is denoted by $\mathcal{M}^e$. Since $\mathcal{U}(\mathcal{M}^e)$ is a left $\mathcal{M}^e$-module with the multiplication, by the above argument, $\mathcal{U}(\mathcal{M}^e)$ is a $\mathcal{M}$-bimodule with the module actions

$$m \cdot T = (m \otimes 1^\circ)T, \quad T \cdot n = (1 \otimes n^\circ)T \qquad (m, n \in \mathcal{M},\ T \in \mathcal{U}(\mathcal{M}^e)). \tag{2}$$

The idea of embedding a finite von Neumann algebra $\mathcal{M}$ into the regular ring of affiliated operators $\mathcal{U}(\mathcal{M})$ goes back to the works of Murray and von Neumann in their 1936 paper [17]. In 1956, Yuzo Utumi gave a construction for embedding $\mathcal{M}$ in a regular ring Q, which is called its maximal ring of right quotients [24]. Then, Roos (1968) showed that Q and $\mathcal{U}(\mathcal{M})$ are the same. This detected a deep connection between ring theory and operator theory. Since then the theory of regular rings of operators has ripened and their algebraic properties proved neatly and efficiently.

3.2 L^2-cohomology for von Neumann algebras

Finding a suitable homology in the context of von Neumann algebras goes back to the works of Johnson, Kadison and Ringrose (1971-72) and then to Sinclair and Smith (1995). Their goal was to find a powerful invariant to distinguish von Neumann algebras.

The theory of L^2-homology for von Neumann algebras is introduced by Connes and Shlyakhtenko [9] following the works of Gaboriau (2002) in the field of ergodic equivalence relations, where he showed that all the L^2-Betti numbers are the same for those discrete groups that can generate isomorphic ergodic measure-preserving equivalence relations. Connes and Shlyakhtenko used the theory of correspondences together with the algebraic description of L^2-Betti numbers given by Lück, to define the k-th L^2-homology of a von Neumann algebra $\mathcal{M}$ by

$$H_k^{(2)}(\mathcal{M}) = H_k(\mathcal{M}, \mathcal{M}\bar{\otimes}\mathcal{M}^\circ),$$

where H_k means the algebraic Hochschild homology. Then the corresponding L^2-Betti numbers of $\mathcal{M}$ are defined to be $\beta_k^{(2)}(\mathcal{M}) = \dim_{\mathcal{M}\bar{\otimes}\mathcal{M}^\circ} H_k^{(2)}(\mathcal{M})$. The dimension function used here is the extended dimension function of Lück, that is explained in more details in the next section.

The correspondences between von Neumann algebras was introduced by A. Connes as a tool to study of II_1 factors ([7, 8]). Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras. A correspondence between $\mathcal{M}$ and $\mathcal{N}$ is a Hilbert space $\mathcal{H}$ with a pair of commuting normal representations $\pi_{\mathcal{M}}$ and $\pi_{\mathcal{N}^\circ}$ of $\mathcal{M}$ and $\mathcal{N}^\circ$, respectively. The set of equivalence classes is denoted by $\mathrm{Corr}(\mathcal{M},\mathcal{N})$ and $\mathrm{Corr}(\mathcal{M})$ is nothing but $\mathrm{Corr}(\mathcal{M},\mathcal{M})$. The standard form of $\mathcal{M}$ yields an element $L^2(\mathcal{M})$ of $\mathrm{Corr}(\mathcal{M})$ called the identity correspondence of $\mathcal{M}$. A correspondence $\mathcal{H}$ on $\mathcal{M}$ carries a bimodule structure as follows

$$a \cdot h \cdot b = \pi_{\mathcal{M}}(a)\pi_{\mathcal{M}^\circ}(b^\circ)h \qquad (a, b \in \mathcal{M}).$$

$\mathrm{Corr}(\mathcal{M})$ plays the role of unitary representations and there is a dictionary between a discrete group Γ and a II_1 factor $\mathcal{M}$ as follows [9].

Discrete group	II_1 factor $\mathcal{M}$
Unitary Representation	$\mathcal{M}-\mathcal{M}$ Hilbert Bimodule
Trivial Representation	$L^2(\mathcal{M})$
Regular Representation	Coarse Correspondence
Amenability	$L^2(\mathcal{M}) \subset_{\text{weakly}} L^2(\mathcal{M})\bar{\otimes}L^2(\mathcal{M}^\circ)$
Property T	$L^2(\mathcal{M})$ isolated

The coarse correspondence is given by the bimodule $L^2(\mathcal{M})\bar{\otimes}L^2(\mathcal{M}^\circ)$ with bimodule actions

$$a \cdot (x \otimes y^\circ) = ax \otimes y^\circ \text{ and } (x \otimes y^\circ) \cdot a = x \otimes (ya)^\circ, \ (a \in \mathcal{M}, x \otimes y^\circ \in \mathcal{M} \otimes \mathcal{M}^\circ).$$

Connes and Shlyakhtenko investigate the relation between L^2-Betti numbers of a discrete group Γ with L^2-Betti numbers of its group algebra and showed that $\beta_k^{(2)}(\Gamma) = \beta_k^{(2)}(\mathbb{C}\Gamma, \tau)$, where τ is the standard trace on the group algebra $\mathbb{C}\Gamma$ [9, Proposition 2.3]. For a II_1-factor $\mathcal{M}$ they computed its L^2-Betti numbers. They showed that the zeroth Betti number is 0 and each L^2-Betti number of $\mathcal{M}$ is a limit of L^2-Betti numbers of its sub-complexes [9, Lemma 2.2]. They also showed that for a von Neumann algebra $(\mathcal{M}, \tau)$ with a diffuse center, $\beta_1^{(2)}(\mathcal{M}, \tau) = 0$ [9, Corollary 3.5]. In particular, if $\mathcal{M}$ is commutative, then $\beta_k^{(2)}(\mathcal{M}) = 0$, for every $k \geqslant 1$ [9, Corollary 5.4] (the same result is obtained by A. Thom for all von Neumann algebras with a diffuse center [21, Theorem 2.2]). Finally they give connections to free probability theory including the inequality between the microstates and microstates-free entropy.

In Section 4 we will combine the Jensen equivariant theory and Connes-Shlyakhtenko L^2-theory to give a numerical invariant in the context of dynamical systems of operator algebras. To do this, we first introduce a notion of equivariant L^2-cohomology and equivariant L^2-Betti numbers for a subalgebra of a von Neumann algebra. Then, considering a grading of a group G on a C^*-algebra $\mathcal{A}$, we give a definition of the graded L^2-cohomology and the associated L^2-Betti numbers for $\mathcal{A}$. We investigate the relation between graded L^2-cohomology of $\mathcal{A}$ and the equivariant L^2-cohomology of $\mathcal{A}$, when the group G is abelian.

3.3 *The dimension function*

Here we give a brief review of the generalized Murray-von Neumann dimension function. The interested reader may find more details in [16, Chapter 6].

Let R be a unital ring and X be an R-module. The dual module X^* of X is $\mathrm{Hom}_R(X, R)$, where the R-multiplications are given by $(fr)(x) = f(x)r$ and $(rf)(x) =$

$rf(x)$, for $f \in X^*, x \in X$ and $r \in R$. If Y is an R-submodule of X, then the closure of Y in X is defined to be

$$\overline{Y} = \{x \in X | f(x) = 0 \text{ for all } f \in X^* \text{ with } Y \subseteq \ker(f)\}.$$

For an R-module X, define the submodule $\mathrm{T}X$ and the quotient module $\mathrm{P}X$ by

$$\mathrm{T}X := \{x \in X | f(x) = 0 \text{ for all } f \in X^*\} = \overline{\{0\}}$$

and

$$\mathrm{P}X := X/\mathrm{T}X.$$

Note that $\mathrm{TP}X = 0, \mathrm{PP}X = \mathrm{P}X, X^* = (\mathrm{P}X)^*$, and that $\mathrm{P}X = 0$ is equivalent to $X^* = 0$. If X is a finitely generated (f.g.) R-module, then $\mathrm{P}X$ is finitely generated projective and $X \cong \mathrm{T}X \oplus \mathrm{P}X$.

Let $\mathcal{M}$ be a finite von Neumann algebra with a fixed normal, faithful state τ. The tracial functional τ_n on $M_n(\mathcal{M})$ given by $\tau_n(\{a_{ij}\}_{i,j=1}^n) = \sum_{i,j=1}^n \tau(a_{ii})$ is positive and faithful (but not a state, since it takes the value n on the unit matrix). If P is a finitely generated projective $\mathcal{M}$-bimodule, then it is isomorphic to $\mathcal{M}^n A$ for some idempotent matrix $A \in M_n(\mathcal{M})$. Now the Murray-von Neumann dimension of P is defined to be

$$\dim_{\mathcal{M}}(P) = \tau_n(A) \in [0, \infty).$$

This definition is independent of the choice of the matrix A. If X is an arbitrary $\mathcal{M}$-bimodule, then the dimension of X is defined by

$$\dim'_{\mathcal{M}}(X) = \sup\{\dim_{\mathcal{M}}(P) | P \text{ is a f.g. projective submodule of } X\} \in [0, \infty].$$

By [16, Theorem 6.7], $\dim'$ is the only dimension function that extends the Murray-von Neumann dimension and satisfies the following properties;

(i) (Extension Property) If P is a f.g. projective $\mathcal{M}$-module, then

$$\dim'_{\mathcal{M}}(P) = \dim_{\mathcal{M}}(P);$$

(ii) (Additivity) If $0 \to X_0 \to X_1 \to X_2 \to 0$ is an exact sequence of modules, then $\dim'_{\mathcal{M}}(X_1) = \dim'_{\mathcal{M}}(X_0) + \dim'_{\mathcal{M}}(X_2)$;

(iii) (Continuity) If Y is a submodule of the f.g. module X, then

$$\dim'_{\mathcal{M}}(Y) = \dim'_{\mathcal{M}}(\overline{Y});$$

(iv) If X is f.g., $\dim'_{\mathcal{M}}(X) = \dim_{\mathcal{M}}(\mathrm{P}X)$ and $\dim'_{\mathcal{M}}(\mathrm{T}X) = 0$.

4. Equivariant L^2-cohomology

4.1 *Equivariant L^2-cohomology*

In this section we give a new concept of equivariant L^2-cohomology for a tracial $*$-algebra.

Let $\mathcal{A}$ be an algebra. If $G \curvearrowright_\alpha \mathcal{A}$, then $\mathcal{A}$ provided with the module actions of left and right multiplication with $\beta_g = \alpha_g$ is a Banach equivariant G-$\mathcal{A}$-module. On the other hand, let $\mathcal{M}$ be a von Neumann algebra and let $G \curvearrowright_\alpha \mathcal{M}$. By the module structure of $\mathcal{M}^e$ described in (1) and by an argument similar to [14, I.1.4 (a)], G acts on $\mathcal{M}^e$ via the map $\tilde{\alpha}_g : \mathcal{M}^e \to \mathcal{M}^e$ defined by $\tilde{\alpha}_g(a \otimes b^\circ) = \alpha_g(a) \otimes \alpha_g^\circ(b) \quad (a, b \in \mathcal{M}, g \in G)$, extended by linearity and continuity. Hence $\mathcal{M}^e$ is an equivariant $(\alpha, \tilde{\alpha})$-G-$\mathcal{M}$-module.

Let $\mathcal{M}$ be a von Neumann algebra with a finite, faithful normal trace τ and let $G \curvearrowright_\alpha \mathcal{M}$. This trace τ is called invariant under this action if for each $g \in G$ and $a \in \mathcal{M}$ we have $\tau(\alpha_g(a)) = \tau(a)$.

Let us observe that if $\mathcal{M}$ is a von Neumann algebra with a finite, faithful, normal trace τ and $G \curvearrowright_\alpha \mathcal{M}$ with the extra property that τ is invariant under the action α, then there is an action $\beta : G \to Lin(\mathcal{U}(\mathcal{M}^e))$ which makes $\mathcal{U}(\mathcal{M}^e)$ an equivariant (α, β)-G-$\mathcal{M}$-module.

To see this, by the actions described in (1), $\mathcal{M}^e$ is a Banach $\mathcal{M}$-bimodule and as above, we have an action $\tilde{\alpha}_g : \mathcal{M}^e \to \mathcal{M}^e \quad (g \in G)$, making $\mathcal{M}^e$ into an equivariant $\mathcal{M}$-module. Since $\mathcal{M}^e$ is dense in $\mathcal{U}(\mathcal{M}^e)$ in the measure topology, for every $T \in \mathcal{U}(\mathcal{M}^e)$ there is a sequence $(T_n) \subseteq \mathcal{M}^e$ such that $T_n \to T$ in the measure topology. Now we define an action $\beta : G \to Lin(\mathcal{U}(\mathcal{M}^e))$ by $\beta_g(T) = \lim_n \tilde{\alpha}_g(T_n)$ $(T \in \mathcal{U}(\mathcal{M}^e), g \in G)$, where the limit is taken in the measure topology. If $T = 0$, then there is a sequence $(T_n) \subseteq \mathcal{M}^e$ such that $T_n \to 0$. By the definition of neighborhoods of 0 in the measure topology, for all $\epsilon, \delta > 0$, $T_n \in N(\epsilon, \delta)$ for n large enough. Since τ is invariant under α, the finite, faithful, normal trace $\tau \bar{\otimes} \tau^\circ$ defined by $\tau \bar{\otimes} \tau^\circ(a \otimes b^\circ) = \tau(a)\tau^\circ(b^\circ)$ is invariant under $\tilde{\alpha}$. This shows that $T_n \in N(\epsilon, \delta)$ if and only if $\tilde{\alpha}_g(T_n) \in N(\epsilon, \delta)$, so that $\tilde{\alpha}_g(T_n) \to 0$. Thus $\beta_g(T) = 0$. This argument justifies that the action β is well-defined.

For every $a, b \in \mathcal{M}$, by the actions of $\mathcal{M}$ on $\mathcal{U}(\mathcal{M}^e)$ described in (2), we have

$$\begin{aligned}\beta_g(a \cdot T) = \beta_g((a \otimes 1^\circ)T) &= \lim_n \tilde{\alpha}_g((a \otimes 1^\circ)T_n) \\ &= \lim_n (\alpha_g(a) \otimes 1^\circ)\tilde{\alpha}_g(T_n) \qquad (\tilde{\alpha} \text{ is equivariant}) \\ &= (\alpha_g(a) \otimes 1^\circ) \cdot \lim_n \tilde{\alpha}_g(T_n) \\ &= \alpha_g(a) \cdot \beta_g(T).\end{aligned}$$

A similar argument shows that $\beta_g(T \cdot b) = \beta_g(T) \cdot \alpha_g(b)$.

If $g, h \in G$, then

$$\begin{aligned}\beta_{gh}(T) &= \lim_n \tilde{\alpha}_{gh}(T_n) = \lim_n \tilde{\alpha}_g(T_n)\tilde{\alpha}_h(T_n) \\ &= \lim_n \tilde{\alpha}_g(T_n) \lim_n \tilde{\alpha}_h(T_n) = \beta_g(T)\beta_h(T),\end{aligned}$$

thus β is a homomorphism.

Finally, we observe that β_g is bijective. For each $g \in G$ and $T \in \mathcal{U}(\mathcal{M}^e)$, $\beta_g(T)$ has a dense domain $D \subseteq \mathcal{L}^2(\mathcal{M} \bar{\otimes} \mathcal{M}^\circ)$. We show that for such g, $\beta_{g^{-1}}(T)$ with domain D' is the inverse of $\beta_g(T)$, that is, $\beta_{g^{-1}}(T)\beta_g(T)\xi = T\xi$, for all $\xi \in D \cap Dom(T)$ with $\beta_g(T)\xi \in D'$ and $\beta_g(T)\beta_{g^{-1}}(T)\eta = T\eta$, for all $\eta \in D' \cap Dom(T)$ with $\beta_{g^{-1}}(T)\eta \in D$. If $(T_n)_n$ is a sequence in $\mathcal{M}^e$ converging to T in the measure topology, then

$$\beta_g(T)\beta_{g^{-1}}(T) = \lim_n \tilde{\alpha}_g(T_n) \lim_n \tilde{\alpha}_{g^{-1}}(T_n) = \lim_n \tilde{\alpha}_{gg^{-1}}(T_n) = \lim_n (T_n) = T,$$

and similarly, $\beta_{g^{-1}}\beta_g(T) = T$. Hence $\mathcal{U}(\mathcal{M}^e)$ is an equivariant (α, β)-G-$\mathcal{M}$-module. We will use this fact in Definition 4.2.

Consider a tracial $*$-algebra $(\mathcal{A}, \tau)$ satisfying in the following two conditions:

(i) $\tau(a^*a) \geqslant 0$ for all $a \in \mathcal{A}$,
(ii) $\forall b \in \mathcal{A}\, \exists C > 0 : \ \tau(a^*b^*ba) \leqslant C\tau(a^*a) \quad (a \in \mathcal{A})$.

Suppose that $\mathcal{M} = W^*(\mathcal{A}) \subseteq B(\mathcal{L}^2(\mathcal{A}, \tau))$ be its enveloping von Neumann algebra. We adapt the following definition due to Thom [21, Definition 3.6].

Definition 4.1 With the above assumptions, the k-th L^2-cohomology of $\mathcal{A}$ is defined by

$$H^{(2),k}(\mathcal{A}) = H^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)),$$

where $H^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e))$ is the k-th Hochschild cohomology of $\mathcal{A}$ with coefficients in $\mathcal{U}(\mathcal{M}^e)$.

Note that if $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{U}(\mathcal{M})$ by [21, Corollary 3.7] we have an isomorphism of right $\mathcal{U}(\mathcal{M}^e)$-modules

$$H^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)) \cong H_k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e))' \cong (H_k(\mathcal{A}, \mathcal{M}^e) \otimes_{\mathcal{M}^e} \mathcal{U}(\mathcal{M}^e))'.$$

Combining this idea with the definition of equivariant Hochschild cohomology due to Jensen ([14, Definition I.1.7]) we arrive at the following (apparently new) notion:

Definition 4.2 Let $\mathcal{M}$ be a von Neumann algebra with a finite, faithful, normal trace τ and let $G \curvearrowright_\alpha \mathcal{M}$. Suppose that τ is invariant under the action α. Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{M}$ invariant under the action α. We denote the restricted action on $\mathcal{A}$ again by α.

The k-th equivariant L^2-cohomology of $\mathcal{A}$ is defined by

$$H_G^{(2),k}(\mathcal{A}) = H_G^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)),$$

where $H_G^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e))$ is the k-th equivariant Hochschild cohomology of $\mathcal{A}$ with coefficients in $\mathcal{U}(\mathcal{M}^e)$.

Also, the k-th equivariant L^2-Betti number of $\mathcal{A}$ is defined by

$$b_G^{(2),k}(\mathcal{A}) = \dim_{\mathcal{U}(\mathcal{M}^e)^\beta} H_G^{(2),k}(\mathcal{A}).$$

Here $\mathcal{U}(\mathcal{M}^e)^\beta$ denotes the fixed point algebra of $\mathcal{U}(\mathcal{M}^e)$ with respect to the action β and the dimension function is the generalized dimension function described above.

The algebra $\mathcal{U}(\mathcal{M}^e)$ above is a right $\mathcal{U}(\mathcal{M}^e)$-module with its usual multiplication. This gives $C^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)$ a left $\mathcal{U}(\mathcal{M}^e)$-module structure with the action

$$\xi \cdot T(a_1, ..., a_k) = T(a_1, ..., a_k)\xi, \quad (\xi \in \mathcal{U}(\mathcal{M}^e), T \in C^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)).$$

The restriction of this left action to $C_G^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)$ is an action of the fixed point algebra $\mathcal{U}(\mathcal{M}^e)^\beta$ on $C_G^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)$. That is, if $\xi \in \mathcal{U}(\mathcal{M}^e)^\beta$ and $T \in C_G^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)$, then $\xi \cdot T \in C_G^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)$. The critical fact here is that this action commutes with δ_G^k and hence it induces an action of $\mathcal{U}(\mathcal{M}^e)^\beta$ on $H_G^k(\mathcal{A}, \mathcal{U}(\mathcal{M}^e)$. This allows us to take the dimension over $\mathcal{U}(\mathcal{M}^e)^\beta$ in the above definition.

4.2 *Grading versus dual action*

Graded C^*-algebras and their relation with Fell bundles studied by Exel [11].

Definition 4.3 [11, Definition 3.1]. Let G be a discrete group and let $\mathcal{A}$ be a C^*-algebra. $\mathcal{A}$ is said to be G-graded if there is a collection of linearly independent closed subspaces $(\mathcal{A}_g)_{g\in G}$ of $\mathcal{A}$ with the following conditions

(i) $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$,
(ii) $\mathcal{A}_g^* \subseteq \mathcal{A}_{g^{-1}}$,
(iii) $\mathcal{A}$ is the closure of the direct sum $\oplus_{g\in G}\mathcal{A}_g$.

This grading is called topological grading if there exists a conditional expectation of $\mathcal{A}$ to $\mathcal{A}_e$. Here e is the identity element of G.

Similarly one can give a definition of a graded von Neumann algebras as follows.

Let G be a discrete group and let $\mathcal{M}$ be a von Neumann algebra. A G-grading on $\mathcal{M}$ is a collection of linearly independent closed subspaces $(\mathcal{M}_g)_{g\in G}$ of $\mathcal{M}$ with the following conditions:

(i) $\mathcal{M}_g\mathcal{M}_h \subseteq \mathcal{M}_{gh}$ for all $g, h \in G$,
(ii) $\mathcal{M}_g^* \subseteq \mathcal{M}_{g^{-1}}$,
(iii) $\mathcal{M}$ is the UWOT closure of the direct sum $\oplus_{g\in G}\mathcal{M}_g$.

The homology of graded rings is well studied. A good source is [12, Section 20]. For the rest of this section, we suppose that G is a discrete group and G° its opposite group.

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ be a G-graded *-subalgebra of $\mathcal{M}$. This grading of $\mathcal{A}$ induces a $(G \otimes G^\circ)$-grading on $\mathcal{A} \otimes \mathcal{A}^\circ$ with homogeneous components

$$(\mathcal{A} \otimes \mathcal{A}^\circ)_g = \{\sum_i a_i \otimes b_i^\circ | a_i \in \mathcal{A}, b_i^\circ \in \mathcal{A}^\circ \text{ and } \deg(a_i) + \deg(b_i^\circ) = g\}.$$

Let $(P = \{P_{k,*}\}, d) \to (\mathcal{A}, 0)$ be a graded projective resolution of $\mathcal{A}$ with the augmentation map σ in the sense of [12, Section 20], that is, a long exact sequence

$$... \xrightarrow{d} P_{1,*} \xrightarrow{d} P_{0,*} \xrightarrow{\sigma} \mathcal{A} \to 0. \tag{3}$$

Then, $\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{A}^\circ}(P_{k,*}, \mathcal{M} \otimes \mathcal{M}^\circ)$ is a graded module of linear maps and

$$0 \to \mathrm{Hom}_{\mathcal{A}^e}(P_{0,*}, \mathcal{M}^e) \xrightarrow{\delta} \mathrm{Hom}_{\mathcal{A}^e}(P_{1,*}, \mathcal{M}^e) \xrightarrow{\delta} ... \tag{4}$$

is a cochain complex of graded modules with the coboundary maps δ defined by $\delta(T) = -(-1)^{\deg(T)}Td$. We use the standard notation $\mathrm{Hom}_{\mathcal{A}^e}^{k,*}(P, \mathcal{M}^e)$ instead of $\mathrm{Hom}_{\mathcal{A}^e}(P_{k,*}, \mathcal{M}^e)$. Thus the cochain complex (4) has the form $(\mathrm{Hom}_{\mathcal{A}^e}^{k,*}(P, \mathcal{M}^e), \delta)$.

Now, let us introduce the L^2-cohomology of graded von Neumann algebras.

Definition 4.4 For every $k \geqslant 0$, the k-th graded L^2-cohomology of $\mathcal{A}$ is defined by

$$H_{grd}^{(2),k}(\mathcal{A}) := H^{k,*}(\mathrm{Hom}_{\mathcal{A}^e}(P, \mathcal{M}^e), \delta) = \mathrm{Ext}_{\mathcal{A}^e}^k(\mathcal{A}, \mathcal{M}^e),$$

and the k-th graded L^2-Betti number of $\mathcal{A}$ is defined by

$$\beta^k_{grd}(\mathcal{A}) = \dim_{\mathcal{A}^e} H^{(2),k}_{grd}(\mathcal{A}).$$

An important fact about graded C^*-algebras is the relation between grading on the C^*-algebra and the action induced by this grading on the C^*-algebra. Raeburn in [20, Theorem 3] showed that if G is an abelian group and if $\mathcal{A}$ is a topologically G-graded C^*-algebra, then there is a strongly continuous action α of the dual group $\widehat{G}$ on $\mathcal{A}$ such that for each $a \in \mathcal{A}_g$, $\alpha_\gamma(a) = \gamma(g)a$ and the conditional expectation $F : \mathcal{A} \to \mathcal{A}_e$ is defined by $F(a) = \int_{\widehat{G}} \alpha_\gamma(a) d\gamma$ for all $a \in \mathcal{A}$.

Conversely, let α be a $\widehat{G}$-action on a C^*-algebra $\mathcal{A}$. For every $g \in G$ let $\mathcal{A}_g = \{a \in \mathcal{A} | \alpha_\gamma(a) = \gamma(g)a, \gamma \in \widehat{G}\}$ and put $\mathcal{A}_\alpha = \overline{\oplus_{g\in G}\mathcal{A}_g}$. Then the collection $(\mathcal{A}_g)_{g\in G}$ is a grading on the closed subalgebra $\mathcal{A}_\alpha$ of $\mathcal{A}$.

4.3 *Concluding Remarks*

The idea of equivariant L^2-cohomology presented here is at its very beginning and deserves further development. There are a few important features that needs to be further investigated. The first is finding conditions under which the equivariant L^2-Betti numbers vanish. Second, it is nice to have a relation between equivariant L^2-cohomology and the L^2-cohomology of the corresponding crossed-product (or its subalgebras). Third, it is desirable to have a relation between equivariant L^2-cohomology and graded L^2-cohomology, as actions and gradings are interrelated as shown above. This is not an easy task, as the two notions are defined here via two different approaches. Summing up, the equivariant L^2-cohomology theory and its counterparts sound as a promising field of research with many non trivial and challenging problems.

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