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A class of $(2m-1)$-weakly amenable Banach algebras

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Abstract. Let $\mathfrak{A}$ be a Banach space and λ be a non-zero fixed element of $\mathfrak{A}^*$(dual space of $\mathfrak{A}$) with non-zero kernel. Defining algebra product in $\mathfrak{A}$ as $a \cdot b = \lambda(a)b$ for $a, b \in \mathfrak{A}$, we show that $\mathfrak{A}$ is a $(2m-1)$-weakly amenable Banach algebra but not $2m$-weakly amenable for any $m \in \mathbb{N}$. Furthermore, we show the converse of the statement [2, Proposition 1.4.(ii)] "for a non-unital Banach algebra $\mathfrak{A}$, if $\mathfrak{A}$ is weakly amenable then $\mathfrak{A}^\#$ is weakly amenable" does not hold.

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1. Introduction and preliminaries

Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. A derivation $D : \mathfrak{A} \longrightarrow X$ is a linear map satisfying the equation $D(ab) = D(a) \cdot b + a \cdot D(b)$. For any $x \in X$, define $\delta_x : \mathfrak{A} \longrightarrow X$ by $\delta_x(a) = a{\cdot}x - x{\cdot}a$. Then δ_x is a bounded derivation from $\mathfrak{A}$ to X. This derivation is called inner derivation. The spaces of all bounded derivations and all inner derivations from $\mathfrak{A}$ to X are denoted by $\mathcal{Z}^1(\mathfrak{A}, X)$ and $\mathcal{N}^1(\mathfrak{A}, X)$, respectively. Then the first (topological) cohomology group of $\mathfrak{A}$ with coefficients in X is the quotient space $\mathcal{Z}^1(\mathfrak{A}, X)/\mathcal{N}^1(\mathfrak{A}, X)$ and denoted by $\mathcal{H}^1(\mathfrak{A}, X)$ (for reviewing these concepts one may see a standard text such as [1]). Much studies have been devoted to the calculation of cohomology groups $\mathcal{H}^1(\mathfrak{A}, X)$ and the higher dimensions cohomology groups $\mathcal{H}^n(\mathfrak{A}, X)$.

A Banach algebra $\mathfrak{A}$ is called amenable if $\mathcal{H}^1(\mathfrak{A}, X^*) = (0)$ for every Banach $\mathfrak{A}$-bimodule X. The Banach algebra $\mathfrak{A}$ is said to be weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = (0)$.

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The notions of n-weakly amenable ($n \in \mathbb{N}$) and permanently weakly amenable were introduced by Dales, Ghahramani and Gronbæk [2]. A Banach algebra $\mathfrak{A}$ is said to be n-weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = (0)$ and is permanently weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = (0)$ for each $n \in \mathbb{N}$. They considered the weakly amenable Banach subalgebra $E\hat{\bigotimes}E^*$ (tensor algebra of E) of $B(E)$ for a Banach space E and proved it is $(2m-1)$-weakly amenable but not $2m$-weakly amenable for any $m \in \mathbb{N}$. Their work along with Zhang's work [3] were a motivation to recognize some more such Banach algebras.

In this article, we define a special product on a given Banach space such that it becomes a $(2m-1)$-weakly amenable Banach algebra but not $2m$-weakly amenable for any $m \in \mathbb{N}$.

2. Main results

Lemma 2.1 Let $\mathfrak{A}$ be a Banach space and λ be a fixed non-zero element of $\mathfrak{A}^*$ with non-zero kernel. Define algebra product on $\mathfrak{A}$ as $a{\cdot}b = \lambda(a)b$ for $a, b \in \mathfrak{A}$. Then $\mathfrak{A}$ is a weakly amenable Banach algebra.

Proof. Clearly, $\mathfrak{A}$ is a non-commutative Banach algebra and the module actions can be formulated as $a{\cdot}\Lambda =< a, \Lambda > \lambda$ and $\Lambda{\cdot}a = \lambda(a)\Lambda$ for $a \in \mathfrak{A}$ and $\Lambda \in \mathfrak{A}^*$. Let $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ be a bounded derivation. Then,

$$\begin{aligned} D(ab) &= D(a){\cdot}b + a{\cdot}D(b), \\ D(\lambda(a)b) &= \lambda(b)D(a) + < a, D(b) > \lambda. \end{aligned} \tag{1}$$

Now, let $a = b$, then (1) implies $< a, D(a) >= 0$, and therefrom $< a + b, D(a + b) >= 0$, and this in turn yields $< a, D(b) >= - < b, D(a) >$. Since $\lambda \neq 0$ there exists $e_0 \in \mathfrak{A}$ as left identity. Indeed, $e_0{\cdot}a = \lambda(e_0)a = 1a = a$ for each $a \in \mathfrak{A}$. Hence,

$$\begin{aligned} D(e_0a) &= D(e_0){\cdot}a + e_0{\cdot}D(a) \\ &= D(e_0){\cdot}a + < e_0, D(a) > \lambda \\ &= D(e_0){\cdot}a - < a, D(e_0) > \lambda \\ &= D(e_0){\cdot}a - a{\cdot}D(e_0), \quad D(e_0) \in \mathfrak{A}^*. \end{aligned}$$

■

For referring to the mentioned algebra in Lemma 2.1 we call it **Scalar algebra**.

Lemma 2.2 Let $\mathfrak{A}$ be a Scalar algebra. Then $\mathfrak{A}$ is a left ideal in its second dual algebra $\mathfrak{A}^{**}$. Furthermore, $\mathfrak{A}$ is Arens regular.

Proof. Let $\mathfrak{A}$ be produced by $\lambda \in \mathfrak{A}^*$. We equip $\mathfrak{A}^{**}$ with first Arens product and identify $a \in \mathfrak{A}$ with $\hat{a} \in \mathfrak{A}^{**}$, where ˆ is the canonical embedding of $\mathfrak{A}$ into $\mathfrak{A}^{**}$. Hence,

$$\psi\Box a = \psi\Box\hat{a} =< \lambda, \psi > \hat{a} =< \lambda, \psi > a.$$

For Arens regularity let $\eta \in \mathfrak{A}^*$, and Φ, $\Psi \in \mathfrak{A}^{**}$. Then, we have first Arens product

$$< \eta, \Phi\Box\Psi > =< \Psi \cdot \eta, \Phi >=<< \eta, \Psi > \lambda, \Phi >=< \eta, \Psi >< \lambda, \Phi >$$

second Arens product

$$< \eta, \Phi\Diamond\Psi > =< \eta \cdot \Phi, \Psi >=<< \lambda, \Phi > \eta, \Psi >=< \lambda, \Phi >< \eta, \Psi > .$$

■

Remark 1 *In the general case, $\mathfrak{A}$ is not a right ideal in $\mathfrak{A}^{**}$. For this, let $\mathfrak{A}$ be a non-reflexive Banach algebra. Choose $\psi \in \mathfrak{A}^{**}\backslash\mathfrak{A}$ and $a \in \mathfrak{A}$ for which $\lambda(a) \neq 0$. Then $a\Box\psi = \hat{a}\Box\psi = \lambda(a)\psi \notin \mathfrak{A}$.*

Theorem 2.3 The Scalar algebra $\mathfrak{A}$ is $(2m-1)$-weakly amenable.

Proof. The Scalar algebra $\mathfrak{A}$ is weakly amenable and has left identity. It is also a left ideal in its second dual algebra, so by $[3, Theorem\ 3]$ it is $(2m-1)$-weakly amenable. ■

In continuation, we show that every Scalar algebra is not 2-weakly amenable. Then immediately by [2, proposition 1.2] it is not 2m-weakly amenable for any $m \in \mathbb{N}$.

Lemma 2.4 Let $\mathfrak{A}$ be a Scalar algebra produced by $\lambda \in \mathfrak{A}^*$. Then $I = \{\lambda\}^\perp$ is a closed two sided ideal in $\mathfrak{A}^{**}$ with codimension one and $\mathfrak{A}^{**} \cong I \bigoplus \mathbb{C}$.

Proof. We equip $\mathfrak{A}^{**}$ with the first Arens product and define $\Phi : \mathfrak{A}^{**} \longrightarrow \mathbb{C}$ by $\Phi(\psi) =< \lambda, \psi >$. It is easy to see that Φ is a non-zero character on $\mathfrak{A}^{**}$. Indeed, for any ψ_1 and ψ_2 in $\mathfrak{A}^{**}$, we have

$$\begin{aligned}
\Phi(\psi_1\Box\psi_2) &=< \lambda, \psi_1\Box\psi_2 > \\
&=< \psi_2\cdot\lambda, \psi_1 > \\
&=<< \lambda, \psi_2 > \lambda, \psi_1 > \\
&= \Phi(\psi_1)\Phi(\psi_2).
\end{aligned}$$

Clearly, $ker\Phi = \{\lambda\}^\perp$ is a closed two sided ideal in $\mathfrak{A}^{**}$. Hence, $\frac{\mathfrak{A}^{**}}{I} \cong \mathbb{C}$. For short exact sequence $0 \longrightarrow I \xrightarrow{\imath} \mathfrak{A}^{**} \xrightarrow{\Phi} \mathbb{C} \longrightarrow 0$, define $\theta : \mathbb{C} \longrightarrow \mathfrak{A}^{**}$ by $\theta(z) = z\hat{e_0}$ where e_0 is a (fixed) left identity of $\mathfrak{A}$. Then θ is a continuous homomorphism and $\Phi \circ \theta = \imath_{\mathbb{C}}$. Hence, the extension splits strongly; that is, $\mathfrak{A}^{**} \cong I \bigoplus \mathbb{C}$. ■

Proposition 2.5 The Scalar algebra $\mathfrak{A}$ is not 2-weakly amenable.

Proof. We specify dual module actions as

$$\begin{aligned}
&a\cdot\psi = \hat{a}\Box\psi = \lambda(a)\psi, \\
&\psi\cdot a = \psi\Box\hat{a} =< \lambda, \psi > \hat{a}, \quad a \in \mathfrak{A}, \psi \in \mathfrak{A}^{**}.
\end{aligned}$$

Let $D : \mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ be a continuous derivation. Then, for given $a, b \in \mathfrak{A}$, we have

$$\begin{aligned}
D(a\cdot b) &= D(a)\cdot b + a\cdot D(b), \\
D(\lambda(a)b) &=< \lambda, D(a) > \hat{b} + \lambda(a)D(b).
\end{aligned}$$

Consequently, $<\lambda, D(a)>=0$. So, $D(a) \in \{\lambda\}^{\perp} = I$. It is easy to see that every bounded linear operator $D : \mathfrak{A} \longrightarrow I = \{\lambda\}^{\perp}$ is a bounded derivation. Now, define $D : \mathfrak{A} \longrightarrow I$ by $D(a) = f(a)\psi$ for some non-zero $f \in \mathfrak{A}^*$ and non-zero $\psi \in I$. If D is an inner derivation, then

$$\begin{aligned} D(a) = f(a)\psi &= \delta_w(a) \quad \text{for some } w \in \mathfrak{A}^{**} \cong I \oplus \mathbb{C}e \\ f(a)\psi &= a\cdot(V+ce) - (V+ce)\cdot a \quad \text{for some } V \in I, c \in \mathbb{C} \\ &= a\cdot V - V\cdot a \\ &= \lambda(a)V - 0 \\ &= \lambda(a)V. \end{aligned}$$

Choose $f \in \mathfrak{A}^*$ such that $f(a) = 0$ but $\lambda(a) \neq 0$ for some $a \in \mathfrak{A}$. Therefore, $V = 0$ and thereof $D = 0$. This contradiction completes the proof. ■

Now, we turn our attention to the relation between weak amenability of a non-unital Banach algebra and its unitization. By [2, Proposition1.4] if a non-unital Banach algebra is $(2n-1)$-weakly amenable, then so is its unitization. By a Scalar algebra, we show its converse is not true in general. In fact, we have the following proposition.

Proposition 2.6 Let $\mathfrak{A}$ be a Scalar algebra produced by $\lambda \in \mathfrak{A}^*$ and $I = ker\lambda$. Then $I^{\#} \cong \mathfrak{A}$ is weakly amenable but I is not.

Proof. $I = ker\lambda$ is a closed two sided ideal in $\mathfrak{A}$ and we have the short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\lambda} \mathbb{C} \longrightarrow 0.$$

Clearly, λ is a non-zero character on $\mathfrak{A}$. Hence, $\dfrac{\mathfrak{A}}{I} \cong \mathbb{C}$. Define $\theta : \mathbb{C} \longrightarrow \mathfrak{A}$ by $\theta(z) = ze_0$, where e_0 is a (fixed) left identity of $\mathfrak{A}$. Obviously, θ is a continuous homomorphism and $\lambda \circ \theta = \imath_{\mathbb{C}}$. Therefore, the extension splits strongly; that is, $\mathfrak{A} \cong I \oplus \mathbb{C} = I^{\#}$. As we saw in Lemma 2.1, $I^{\#}(=\mathfrak{A})$ is weakly amenable. If I is weakly amenable then I^2 is dense in I, but $I^2 = (0)$, a contradiction. Thus, I is not weakly amenable. ■

Let us consider some examples.

***Example* 2.7** Suppose $\mathfrak{A} = \left\{ a = \begin{pmatrix} 0 & 0 \\ a_1 & a_2 \end{pmatrix},\ a_1, a_2 \in \mathbb{C} \right\}$, the subspace of $M_{2\times 2}$ normed by $\|a\| = |a_1| + |a_2|$. It is easy to see that $\mathfrak{A}$ with respect to matrix multiplication is a non-commutative Banach algebra. Meanwhile, if we define $\lambda : \mathfrak{A} \longrightarrow \mathbb{C}$ by $\lambda(a) = a_2$ then $\mathfrak{A}$ is a Scalar algebra. $\mathfrak{A}$ has left identity. In fact every element as $\begin{pmatrix} 0 & 0 \\ a_1 & 1 \end{pmatrix}, (a_1 \in \mathbb{C})$ is a left identity for $\mathfrak{A}$. But $\mathfrak{A}$ has no right bounded approximate identity. If otherwise, let $\{e_\alpha\}$, where $e_\alpha = \begin{pmatrix} 0 & 0 \\ e_{1\alpha} & e_{2\alpha} \end{pmatrix},\ e_{1\alpha}, e_{2\alpha} \in \mathbb{C}$ be a bounded net in $\mathfrak{A}$ such that $ae_\alpha \longrightarrow a,\ (a \in \mathfrak{A})$. Then $\|ae_\alpha - a\| = |a_2 e_{1\alpha} - a_1| + |a_2|\,|e_{2\alpha} - 1|$ leads to contradiction. Obviously, $\mathfrak{A}$ is a Scalar algebra. Thus, by Theorem 2.3 and Proposition 2.5, $\mathfrak{A}$ is $(2m-1)$-weakly amenable but not $2m$-weakly amenable for any $m \in \mathbb{N}$. This example can be extended to higher dimensions square matrices space.

***Example* 2.8** Let G be a locally compact group. $M(G)$ the space of all complex regular Borel measures on G along with $\|\mu\| = |\mu|(G)$ is a Banach space. The mapping λ :

$M(G) \longrightarrow \mathbb{C}$ defined by $\lambda(\mu) = \mu(G)$ is a bounded non-zero linear functional with non-zero kernel. We define a product in $M(G)$ by $\mu{\cdot}\nu = \lambda(\mu)\nu = \mu(G)\nu$. With this product, $M(G)$ is a Scalar algebra. So $M(G)$ is $(2m-1)$-weakly amenable Banach algebra but not $2m$-weakly amenable for any $m \in \mathbb{N}$.

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