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Hybrid linesearch algorithm for pseudomonotone equilibrium problem and fixed points of Bregman quasi asymptotically nonexpansive multivalued mappings

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Abstract. In this paper, we introduce a linesearch algorithm for solving fixed points of Bregman quasi asymptotically nonexpansive multivalued mappings and pseudomonotone equilibrium problem in reflexive Banach space. Using the linesearch method, we prove a strong convergence of the iterative scheme to a common point in the set of solutions of some equilibrium problem and common fixed point of the finite family of Bregman quasi asymptotically nonexpansive multivalued mappings with out imposing Bregman Lipschitz condition on the bifunction g as used by many authors in the extragradient method. Our results improve and generalize many recent results in the literature.

Keywords: Bregman-quasi-asymptotically multivalued nonexpansive mappings, strong convergence, pseudomonotone equilibrium problem, Legendre function.

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1. Introduction

Let E be a real Banach space, E^* be the dual of E and C be a nonempty closed and convex subset of E. Recall that a map $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. $T : C \to C$ is called asymptotically nonexpansive if there exists $\{\mu_n\} \subset [0, +\infty)$ such that $\mu_n \to 0$ as $n \to +\infty$ and $\|T^n x - T^n y\| \leqslant (1 + \mu_n)\|x - y\|$ for all $x, y \in C$. Fixed point theory for nonlinear mappings find its

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application in many areas such as theory of differential equation, signal processing and image recovery, see for example Byrne [12].

A bifunction $g : C \times C \to \mathbb{R}$ is said to be an equilibrium bifunction if $g(x, x) = 0$ for all $x \in C$. The equilibrium problem with respect to g and C is to find $z^* \in C$ such that

$$g(z^*, y) \geqslant 0 \quad for\ all\ y \in C. \tag{1}$$

The set of solution of equilibrium problem is denoted by $EP(g)$, i.e.

$$EP(g) = \{z^* \in C : g(z^*, y) \geqslant 0 \text{ for all } \ y \in C\}.$$

Equilibrium problem (1) was introduced by Blum and Oettli [6]. Various problems in linear and nonlinear programming, physics, engineering, economics, transportation, etc can be reformulated as equilibrium problems, see for example [6, 17, 26, 40, 42]. The problem of finding a common points of the set of equilibrium and the set of fixed points of nonlinear mappings have an attractive subject of researches, and many methods have been developed and investigated for solving this problem. Tada and Takahashi [33] introduced the following algorithm for equilibrium problem and fixed point of nonexpansive mapping in real a Hilbert space as follows:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily,} \\ z_n \in K \text{ such that,} \\ f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geqslant 0 \quad for\ all\ y \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S z_n, \ n \geqslant 0, \end{cases} \tag{2}$$

where K is a nonempty closed convex subset of H, S is nonexpansive map, $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfies $\liminf_{n\to+\infty} r_n > 0$. Other results involving fixed point and equilibrium problems include for example [13, 25, 28, 35, 36, 40, 42] and the references contained therein. From algorithm (2) to compute z_n at each step, one needs to solve the following regularized subproblem:

$$z_n \in K \text{ such that } g(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geqslant 0 \quad for\ all\ y \in K. \tag{3}$$

Observe that if g is monotone, the subproblem (3) is strongly convex problem and its unique solution exists. However if the monotone bifunction g is replaced with pseudomonotone bifunction, then the subproblem (3) is not strongly monotone and therefore the unique solution of (3) may not be guaranteed, see for example Dang [16]. Motivated by this, several algorithms for pseudomonotone equilibrium problems have been investigated. Anh [3] introduced extragradient method for pseudomonotone equilibrium problems and fixed points of nonexpansive mappings in a real Hilbert space H. He studied the following iterative algorithm:

$$\begin{cases} x_0 \in K \text{ arbitrarily,} \\ y_n = \operatorname{argmin}\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in K\}, \\ u_n = \operatorname{argmin}\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in K\}, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)S u_n, \ n \geqslant 0, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are real sequences satisfying some conditions. Strong convergence the iterative scheme (4) was obtained by imposing Lipschitz-type condition on the bifunction g, i.e.

$$g(x,y)+g(y,x) \geqslant g(x,z)-c_1\|x-y\|^2-c_2\|y-x\|^2 \quad for\ all\ x,y,z \in C,$$

where c_1 and c_2 are constants. Various authors implemented extragradient method to solve pseudomonotone equilibrium problems (for example, see [19–21, 32] and the references contained therein). Recently, Eskandani et al. [18] introduced a hybrid extragradient method for solving pseudomonotone equilibrium problems and fixed points of multi-valued Bregman relatively nonexpansive mappings in reflexive Banach spaces as follows:

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ w_n^i = \text{argmin}\{\lambda_n g_i(x_n,y)+D_f(w,x_n): w\in C\} \quad i=1,2,\ldots,N, \\ z_n^i = \text{argmin}\{\lambda_n g_i(w_n^i,z)+D_f(z,x_n): z\in C\} \quad i=1,2,\ldots,N, \\ i_n \in \text{Argmin}\{D_f(z_n^i,x_n) \quad i=1,2,\ldots,N\} \ \bar{z}_n := z_n^{i_n}, \\ y_n = \nabla f^*(\beta_{n,0}\nabla f(\bar{z}_n)+\sum_{r=1}^{M}\beta_{n,r}\nabla f(z_{n,r})),\ z_{n,r}\in T_r\bar{z}_n, \\ x_{n+1} = \overleftarrow{Proj}_C^f(\nabla f^*(\alpha_n\nabla f(u_n)+(1-\alpha_n)\nabla f(y_n))) \end{cases} \tag{5}$$

The authors proved strong convergence of the iterative scheme (5) to common point in the set of solutions of finite family of pseudomonotone equilibrium problems and set of fixed points of finite family of multi-valued Bregman relatively nonexpansive mappings by imposing Bregman Lipschitz-type condition on the bifunctions g_i:

$$g_i(x,y)+g_i(y,x) \geqslant g_i(x,z)-c_1D_f(x,y)-c_2D_f(y,x) \quad for\ all\ x,y,z\in C.$$

However the Bregman Lipschitz-type condition on g_i is not easy to obtain since the two unkown constants c_1, c_2 are difficult to approximate. Inspired and motivated by the work of Eskandani et al., in this paper, we propose and study a linesearch algorithm for finding solution in the set of pseudomonotone equilibrium problem and set of fixed points of finite family of Bregman-quasi-asymptotically nonexpansive multi-valued mapping. Strong convergence of the iterative scheme is established with out Bregman Lipschitz-type condition on the bifunctions g_i.

2. Preliminaries

In this paper, we assume $f: E \to (-\infty,+\infty]$ to be a proper and convex function, i.e. dom$f \neq \emptyset$ and $f(\alpha x+(1-\alpha)y) \leqslant \alpha f(x)+(1-\alpha)f(y)$ forall $x,y\in E,\ \alpha\in(0,1)$, where dom $f = \{x \in E : f(x) < +\infty\}$. The Fenchel conjugate of f is a function $f^*: E^* \to (-\infty,+\infty]$ defined by

$$f^*(\zeta) = \sup\{\langle x,\zeta\rangle - f(x) : x\in E\}. \tag{6}$$

From (6), we can easily obtain $f^*(\zeta)+f(x) \geqslant \langle x,\zeta\rangle$ for every $x\in E$ and $\zeta\in E^*$ which is called Fenchel inequality. The subdifferential of f is the mapping $\partial f: E \to 2^{E^*}$ defined by

$$\partial f(x) = \{\zeta\in E^* : f(y) \geqslant f(x)+\langle y-x,\zeta\rangle \quad for\ all\ y\in E\}\ for\ all\ x\in E.$$

It is known that if the function f is proper, lower semicontinuous and convex, then for each $x \in dom\ f$ the subdifferential $\partial f(x)$ is a nonempty closed convex set. Moreover $\zeta \in \partial f(x)$ if and only if $f(x) + f^*(\zeta) = \langle x, \zeta \rangle\ \ for\ all\ x \in E$, see [5]. Furthermore, see for example [34], if $f : E \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is also proper, convex and weak* lower semicontinuous function. The function f is called coercive if $\lim\limits_{\|x\| \to +\infty} f(x) = +\infty$ and it is called strongly coercive if $\lim\limits_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$.

Let $x \in$ int dom$f$ and $y \in E$. The right-hand derivative of f at x in the direction y is defined as

$$f^\circ(x, y) = \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}. \tag{7}$$

The function f is said to have Gâteaux derivative at x if $\lim\limits_{t \to 0} \frac{f(x+ty)-f(x)}{t}$ exists for all $y \in E$. In this case the gradient ∇f of f at x is bounded linear functional defined by $f^\circ(x, y) = \langle y, \nabla f(x) \rangle$. f is said to be Gâteaux differentiable if its Gâteaux derivative exists at each $x \in$ int dom$f$. f is said to be Fréchet differentiable at x if the limit in (7) is attained uniformly in $\|y\| = 1$. f is said to be uniformly Fréchet differentiable on a subset C of E when the limit in (7) is attained uniformly for every $x \in C$ and $\|y\| = 1$. It is well known that (see for example [2]), if f is uniformly Fréchet differentiable on bounded subset of E, then f is uniformly continuous on bounded set of E.

The function f is called essentially smooth if ∂f is both single-valued and bounded on its domain. When f is strictly convex on every convex subset of dom ∂f and $(\partial f)^{-1}$ is locally bounded on its domain, f is called essentially strictly convex. f is said to be Legendre if it is both essentially smooth and essentially strictly convex. We know that if the subdifferential of f ∂f is single-valued, then it coincides with its gradient i.e. $\partial f = \nabla f$ (see for example [27]).

For a Legendre function f, the following results are well known, (see for example [5, 7])

(i) f is Legendre if and only if f^* is Legendre;
(ii) $(\partial f)^{-1} = \partial f^*$;
(iii) f is essentially smooth if and only if f^* is essentially strictly convex;
(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$ = dom∇f^* = int domf^* and ran∇f^* = dom∇f = int dom$f$.

In case E is smooth and strictly convex, the function $f(x) = \frac{1}{p}\|x\|^p$, where $p \in (1, +\infty)$ is Legendre (see for example [39]).

For a proper convex and Gâteaux differentiable function $f : E \to (-\infty, +\infty]$ ([8, 14]), the Bregman distance corresponding to f between x and y is the function $D_f : dom\ f \times int\ dom\ f \to \mathbb{R}$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle \ \ for\ all\ \ x \in dom\ f,\ y \in int\ dom\ f. \tag{8}$$

It follows from (8) that $D_f(x, y) \geqslant 0$ for every $x \in dom\ f,\ y \in int\ dom\ f$ and if f is strictly convex, then $D_f(x, y) = 0$ if and only if $x = y$ (see, [9]). Moreover, Bregman distance satisfies the three point identity:

$$D_f(x, y) + D_f(y, z) = D_f(x, z) + \langle x - y, \nabla f(z) - \nabla f(y) \rangle \tag{9}$$

for all $x \in dom\ f$ and $y, z \in int\ dom\ f$.

Let $C \subset int\ dom\ f$, then the Bregman projection with respect to f of $x \in int\ dom\ f$ onto the nonempty closed convex subset C of E is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Remark 1 *When E is smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for every $x \in E$, then we have $\nabla f(x) = 2J(x)$ for all $x \in E$ and hence*

$$D_f(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 = \phi(x, y) \quad for\ all\ x, y \in E,$$

where J is the normalized duality mapping defined by $J(x) = \{\zeta \in E^* : \langle x, \zeta\rangle = \|x\|^2 = \|\zeta\|^2\}$ and ϕ is the Lyapunov functional introduced by Alber [1]. Therefore, the Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ which is defined by

$$\phi(\Pi_C(x), x) = \inf\{\phi(y, x) : y \in C\}.$$

If $E = H$ a Hilbert space, then the Bregman projection $P_C^f(x)$ reduces to the metric projection $P_C(x)$ of H onto C.

Let $B_r = \{h \in E : \|h\| \leqslant r\}$ for all $r > 0$. The function f is bounded if $f(B_r)$ is bounded for all $r > 0$ and f is uniformly convex on bounded subsets of E [41] if the function $\rho_r : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_r(t) = \inf_{x,y \in B_r,\ \|x-y\|=t,\ \alpha \in (0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)},$$

satisfies $\rho_r(t) > 0$ for all $r, t > 0$, where ρ_r is called the *gauge of uniform convexity of* f. The *gauge of uniform smoothness of* f is the function $\sigma_r : [0, +\infty) \to [0, +\infty)$ defined by

$$\sigma_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} \frac{\alpha f(x + (1-\alpha) t y) + (1-\alpha) f(x - t\alpha y) - f(x)}{\alpha(1-\alpha)},$$

where $S_E = \{h \in E : \|h\| = 1\}$. The function f is said to be uniformly smooth on E if $\sigma_r(t) > 0$ for all $r, t > 0$.

The modulus of total convexity of f at $x \in$int dom$f$ is the function $v_f(x, .) : [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(x, t) = \inf\{D_f(y, x) : y \in \mathrm{dom} f,\ \|y - x\| = t\}.$$

The function f is totally convex at x if $v_f(x, t) > 0$ for all $t > 0$ and f is totally convex if it is totally convex at each point $x \in$dom$f$. Let B be a bounded subset of E. For $t > 0$, define a functional on B, v_f : int dom$f \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(B, t) = \inf\{v_f(x, t) : x \in B \cup \mathrm{dom} f\}.$$

f is totally convex on bounded set B if $v_f(B, t) > 0$ for any bounded subset D of E and $t > 0$, where $v_f(., t)$ is the total convexity of the function f on the set B. Furthermore the function f is totally convex on bounded sets (see for example [10, 41]) if and only if f is uniformly convex on bounded sets.

Let $f : E \to (-\infty, +\infty]$ be a proper, convex, Legendre and Gâteaux differentiable function. Associated with f is the function $V_f : E \times E^* \to [0, +\infty)$ (see [1, 14]) defined by

$$V_f(x, \zeta) = f(x) - \langle x, \zeta \rangle + f^*(\zeta) \quad for\ all\ x \in E,\ \zeta \in E^*. \tag{10}$$

From Definition (10), it is obvious that $V_f(.,.) \geqslant 0$, $V_f(x, \zeta) = D_f(x, \nabla f^*(\zeta))$ $for\ all\ x \in E,\ \zeta \in E^*$ and $D_f(x, y) = V_f(x, \nabla f(y))$. Furthermore, $V_f(x, .)$ is convex for any $x \in E$. Thus, for $t \in (0, 1)$ and $x, y \in E$, we obtain

$$D_f(z, \nabla f^*(t\nabla f(x) + (1-t)\nabla f(y))) \leqslant tD_f(z, x) + (1-t)D_f(z, y). \tag{11}$$

Also, by subdifferential inequality ([23]), we get

$$V_f(x, \zeta) + \langle \vartheta, \nabla f^*(\zeta) - x \rangle \leqslant V_f(x, \zeta + \vartheta) \quad for\ all\ x \in E,\ \zeta,\ \vartheta\ \in E^*. \tag{12}$$

Definition 2.1 Let $T : C \to 2^C$ be a multivalued mapping. For $p \in C$, we define $Tp,\ T^2p,\ T^3p,\ T^4p, \ldots, T^np,\ n \geqslant 1$ as follows:

$$\begin{aligned}
Tp &= \{p_1 \in C : p_1 \in Tp\},\\
T^2p &= T(Tp) := \bigcup_{p_1 \in T(p)} Tp_1,\\
T^3p &= T(T^2p) := \bigcup_{p_2 \in T^2(p)} Tp_2,\\
&\vdots\\
T^np &= T(T^{n-1}p) := \bigcup_{p_{n-1} \in T^{n-1}(p)} Tp_{n-1},\ n \geqslant 1
\end{aligned}$$

A point $x \in C$ is called a fixed point of multivalued mapping T if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$, i.e. $F(T) = \{x \in C : x \in Tx\}$. A point $x \in C$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C such as $x_n \rightharpoonup x$ and $\lim_{n \to +\infty} d(x_n, Tx_n) = 0$ (see [29]). The set of asymptotic fixed points of T is denoted by $\widetilde{F(T)}$

Definition 2.2 Let $f : E \to (-\infty, +\infty]$ be convex and Gâteaux differentiable function. A multivalued mapping $T : C \to 2^C$ is said to be Bregman relatively nonexpansive if $F(T) \neq \emptyset$, $F(T) = \widetilde{F(T)}$ and $D_f(u, w) \leqslant D_f(u, x)$ $for\ all\ u \in F(T),\ w \in Tx$. T is called Bregman-quasi nonexpansive if $F(T) \neq \emptyset$ and $D_f(u, w) \leqslant D_f(u, x)$ for all $u \in F(T),\ w \in Tx$. T is Bregman quasi-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [0, +\infty)$ such that $k_n \to 0$ as $n \to +\infty$ and $D_f(u, w) \leqslant (1 + k_n)D_f(u, x)$ $for\ all\ w \in T^nx$ and $u \in F(T)$. T is said T is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x$, $w_n \in Tx_n$ and $w_n \to y$, then $y \in Tx$. A Bregman quasi-asymptotically nonexpansive multivalued mapping is said to be uniformly $L-$Lipschitz continuous if

$$\|s_n - d_n\| \leqslant L\|x - y\| \quad for\ all\ x, y \in C,\ s_n \in T^n x,\ d_n \in T^n y \quad for\ all\ n \geqslant 1.$$

Definition 2.3 A bifunction $g : C \times C \to \mathbb{R}$ is said to be

(1) γ-strongly monotone on C if there exists $\gamma > 0$ such that

$$g(x, y) + g(y, x) \leqslant -\gamma||x - y||^2 \quad for\ all \quad x, y \in C,$$

(2) Monotone if

$$g(x, y) + g(y, x) \leqslant 0 \quad for\ all \quad x, y \in C,$$

(3) Pseudomonotone if

$$g(x, y) \geqslant 0 \quad \Rightarrow \quad g(y, x) \leqslant 0 \quad for\ all\ x, y \in C.$$

It is clear from Definition 2.3, that (1) implies (2) and (2) implies (3). To solve the equilibrium problem, we assume the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x, x) = 0$ for every $x \in C$;
(A2) $g(x, .)$ is convex and subdifferentiable on C;
(A3) g is pseudomonotone on C with respect to $EP(g, C)$;
(A4) g is jointly continuous on $\triangle \times \triangle$ where $\triangle$ is an open convex set containing C in the sense that if $x, y \in \triangle$ and $\{x_n\}$, $\{y_n\}$ are two sequences in $\triangle$ such that $x_n \rightharpoonup x$, $y_n \rightharpoonup y$, then $g(x_n, y_n) \to g(x, y)$.

In the sequel we will need the following lemmas:

Lemma 2.4 [38] Assume the bifunction g satisfies (A1)-(A4), then the set $EP(g, C)$ of solutions of the equilibrium problems is closed and convex.

Lemma 2.5 [37] Let C be a nonempty subset of E and $f : C \to \mathbb{R}$ be a convex and subdifferentiable function, then f is minimized at $x \in C$ if and only if $0 \in \partial f(x) + N_C(x)$, where $N_C(x)$ is the normal cone to C at $x \in C$, i.e.

$$N_C(x) = \{\zeta \in E^* : \langle y - x, \zeta\rangle \leqslant 0 \quad for\ all\ y \in C\}.$$

Lemma 2.6 [15] Let E be a reflexive Banach space and $f : E \to \mathbb{R}$, $g : E \to \mathbb{R}$ are two convex functions such that $dom\ f \cap dom\ g \neq \emptyset$ and f is continuous, then

$$\partial(f + g) = \partial f(x) + \partial g(x) \quad for\ all\ x \in E.$$

Lemma 2.7 [22] Let $f : E \to (-\infty, +\infty]$ be a Legendre function and C be a nonempty, closed and convex subset of int dom f. Also, let $T : C \to 2^C$ be a closed Bregman quasi-asymptotically nonexpansive multivalued mapping. Then $F(T)$ is closed and convex.

Lemma 2.8 [21] Assume $g : \triangle \times \triangle \to \mathbb{R}$ satisfies conditions (A2) and (A4). Let $\{x_n\}$, $\{z_n\}$ be two sequences in $\triangle$ such that $x_n \rightharpoonup \bar{x}$, $z_n \rightharpoonup \bar{z}$ where $\bar{x}$, $\bar{z} \in \triangle$. Then $\partial_2 g(z_n, x_n) \subseteq \partial_2 g(\bar{z}, \bar{x})$.

Lemma 2.9 [4] Let $f : E \to (-\infty, +\infty]$ be convex and continuous on int'dom f. Then for all $x \in$ int dom f, $\partial f(x)$ is nonempty and bounded.

Lemma 2.10 [11] Let $f : E \to (-\infty, +\infty]$ be a Legendre function, then f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets.

Lemma 2.11 [11] Let E be a reflexive Banach space with the dual E^* and let $f : E \to (-\infty, +\infty]$ be lower semicontinuous at $x \in$ int dom f. Then the following are equivalent:

(1) f is totally convex at x;
(2) There exists a convex and lower semicontinuous function $\varphi : [0, +\infty) \to [0, +\infty]$ with int dom $\varphi \neq \emptyset$, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$ such that

$$\langle y - x, y^* - x^* \rangle \geqslant \varphi(\|y - x\|) \text{ for all } x^* \in \partial f(x),\ y^* \in \partial f(y);$$

(3) There exists a nondecreasing function $\theta : [0, +\infty) \to [0, +\infty]$ with $\lim\limits_{t\to 0}\theta(t) = 0$ such that

$$\|y - x\| \leqslant \theta(\|y^* - x^*\|) \text{ for all } x^* \in \partial f(x),\ y^* \in \partial f(y).$$

Remark 2 *It is known that when $f : E \to (-\infty, +\infty]$ is Gâteaux differentiable at $x \in$int dom f, then $\nabla f(x) = \partial f(x)$ a singleton set. Moreover in this paper we will assume $\varphi(t) = ct^2$ and $\theta(t) = ct$ for $c > 1$ and $t > 0$.*

Lemma 2.12 [10] Let C be a nonempty closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$. Then

(i) $z = P_C^f(x)$ if and only if $\langle y - z, \nabla f(x) - \nabla f(y) \rangle \leqslant 0$ for all $y \in C$;
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leqslant D_f(y, x)$ for all $y \in C$.

Recall that a function $f : E \to (-\infty, +\infty]$ is called sequentially consistent [10] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded, then

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \text{ implies } \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Lemma 2.13 [9] The function $f : E \to (-\infty, +\infty]$ is totally convex on bounded sets if and only if f is sequentially consistent.

Lemma 2.14 [24] Let E be a Banach space and $f : E \to \mathbb{R}$ be Gâteaux differentiable function and uniformly convex on bounded subsets of E. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E, then

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \text{ if and only if } \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Lemma 2.15 [31] Let $f : E \to (-\infty, +\infty]$ be totally convex and Gâteaux differentiable function. Suppose $x \in E$ and the sequence $\{D_f(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.

Lemma 2.16 [23] Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of E^* and $x \in E$. If the sequence $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.17 [24] Let $r > 0$ be a constant and $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then

$$f\left(\sum_{i=1}^{N} \beta_i x_i\right) \leqslant \sum_{i=1}^{N} \beta_i f(x_i) - \beta_i \beta_j \rho_r(\|x_i - x_j\|),$$

where $\beta_i \in (0,1)$ for $i \in \{1,2,3,\dots,N\}$ with $\sum_{i=1}^{N} \beta_i = 1$, $x_i \in B_r(0)$ and ρ_r is the gauge of the uniform convexity of f.

Lemma 2.18 [30] Let $f : E \to (-\infty,+\infty]$ be a uniformly Fréchet differentiable function and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to strong topology of E^*.

Lemma 2.19 [41] Let $f : E \to (-\infty,+\infty]$ be a strongly coercive function. If ∇f is uniformly continuous on bounded subsets of E, then $f^* : E^* \to (-\infty,+\infty]$ is uniformly convex on bounded subsets of E^*.

Lemma 2.20 [41] Let $f : E \to (-\infty,+\infty]$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:

(i) f is strongly coercive and uniformly convex on bounded subsets of E;
(ii) f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of $\mathrm{dom} f^* = E^*$.

Lemma 2.21 [18] Let E be a reflexive Banach space and C be a nonempty closed and convex subset of E. Also, let $A :\to E^*$ be a mapping and $f : E \to \mathbb{R}$ be a Legender function. Then

$$\overleftarrow{Proj}_C^f (\nabla f^*[\nabla f(x) - \lambda A(y)]) = \underset{w \in C}{argmin}\{\lambda\langle w - y, A(y) + D_f(w,x)\},$$

for every $x \in E,\ y \in C$ and $\lambda \in (0,\infty)$.

Lemma 2.22 Let C be a nonempty closed convex subset of real reflexive Banach space E, $f : E \to (-\infty,+\infty]$ be strongly coercive and Fréchet differentiable with the Fenchel congugate $f^* : E^* \to (-\infty,+\infty]$ such that dom $f^* = E^*$ and $g : C \times C \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous on the second argument. Then for every $x \in C$ and $\eta \in (0,+\infty)$ there exists a unique $z \in C$ such that

$$z = \underset{y \in C}{argmin}\left\{\eta g(x,y) + D_f(y,x)\right\}.$$

Proof. Since f is Fréchet differentiable, it is continuous and consequently lower semicontinuous. Then the result follows from Lemma 3.1 in Eskandani et al. [18]. ■

3. Main Results

In this section we present a linesearch algorithm as follows:

Algorithm 1

Step 0: Let $\alpha,\ h,\ \sigma \in (0,1)$, $c = \frac{1}{d},\ d \in (h,1)$, $0 < \alpha_n < a < 1,\ \eta_n \in (\eta, 1],\ 0 < \eta < 1$ and $\beta_{n,0} + \sum_{i=1}^{N} \beta_{n,i} = 1$.
Step 1: Let $x_1 \in C_1 = C$.
Step 2: Set

$$y_n = \underset{y \in C}{argmin}\Big\{\eta_n g(x_n, y) + D_f(y, x_n)\Big\}. \tag{13}$$

Step 3: If $y_n = x_n$, then set $z_n = x_n$. Otherwise find the smallest nonnegative integer m such that

$$\begin{cases} g(z_{n,m}, x_n) - g(z_{n,m}, y_n) \geqslant \frac{\alpha}{\eta_n} D_f(y_n, x_n), \\ z_{n,m} = (1-\sigma^m)x_n + \sigma^m y_n. \end{cases} \tag{14}$$

Set $\sigma_n = \sigma^m$ and $z_n = z_{n,m}$ and go to step 4.
Step 4: Select $w_n \in \partial_2 g(z_n, x_n)$ and compute

$$u_n = P_C^f \nabla f^*(\nabla f(x_n) - \gamma_n w_n), \quad \gamma_n = \begin{cases} \frac{hg(z_n,x_n)}{||w_n||^2}, & y_n \neq x_n \\ 0, & Otherwise \end{cases}.$$

Step 5: Compute

$$\begin{cases} v_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)(\beta_{n,0}\nabla f(u_n) + \sum_{i=1}^{N} \beta_{n,i} \nabla f(s_{n,i}))), \\ C_{n+1} = \{p \in C_n : D_f(p, v_n) \leqslant D_f(p, x_n) + \tau_n\}, \\ x_{n+1} = P_{C_{n+1}}^f x_1,\ n \geqslant 1, \end{cases} \tag{15}$$

where $s_{n,i} \in T_i^n u_n$ and $\tau_n = (1-\alpha_n)M_n^N \sup_{q\in\Omega} D_f(q, x_n)$ and $M_n^N = \sum_{i=1}^{N} \beta_{n,i} k_{n,i}$.
Step 6: Set $n = n+1$ and go to step 2.

In the following result, we show that the linesearch $z_{n,m}$ and γ_n are well defined.

Lemma 3.1 Let C be a nonempty closed convex subset of a reflexive Banach space E. Also, let $f : E \to (-\infty, +\infty]$ be Legendre function and $g : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). Assume $y_n \neq x_n$ for some $n \in \mathbb{N}$. Then

(1) There exists $m \in \mathbb{N}$ such that inequality (14) hold;
(2) $g(z_n, x_n) > 0$;
(3) $0 \notin \partial_2 g(z_n, x_n)$.

Proof. (1) By contradiction assume for each $m \in \mathbb{N}$

$$g(z_{n,m}, x_n) - g(z_{n,m}, y_n) < \frac{\alpha}{\eta_n} D_f(y_n, x_n). \tag{16}$$

Since $z_{n,m} = (1-\sigma^m)x_n + \sigma^m y_n$, it follows that $z_{n,m} \to x_n$ as $m \to +\infty$ and using (A4), we have from (16) that

$$g(x_n, x_n) - g(x_n, y_n) \leqslant \frac{\alpha}{\eta_n} D_f(y_n, x_n).$$

By (A1), we get

$$0 \leqslant g(x_n, y_n) + \frac{\alpha}{\eta_n} D_f(y_n, x_n). \tag{17}$$

Since y_n is the solution of (13), we have

$$\eta_n g(x_n, y_n) + D_f(y_n, x_n) \leqslant \eta_n g(x_n, y) + D_f(y, x_n) \ for\ all\ y \in C.$$

In particular, for $x_n \in C$, we get

$$\eta_n g(x_n, y_n) + D_f(y_n, x_n) \leqslant 0. \tag{18}$$

By (17) and (18), we obtain $\frac{1-\alpha}{\eta_n} D_f(y_n, x_n) \leqslant 0$. Since $\eta_n \leqslant 1$, we obtain $(1-\alpha)D_f(y_n, x_n) \leqslant 0$. By our assumption f is Legendre, then f strictly convex and consequently $D_f(y_n, x_n) > 0$ when $y_n \neq x_n$. This implies $(1-\alpha) \leqslant 0$ which is not possible because $\alpha \in (0, 1)$. Therefore (1) holds.

(2) By (A1) and (A2), we have

$$0 = g(z_n, z_n) = g(z_n, (1-\sigma_n)x_n + \sigma_n y_n) \leqslant (1-\sigma_n)g(z_n, x_n) + \sigma_n g(z_n, y_n). \tag{19}$$

From (14) and (19), we obtain

$$g(z_n, x_n) \geqslant \sigma_n(g(z_n, x_n) - g(z_n, y_n)) \geqslant \frac{\alpha\sigma_n}{\eta_n} D_f(y_n, x_n) > 0,$$

since $y_n \neq x_n$. Therefore (2) holds.

(3) The proof can be found in Tran et al. [38] (Lemma 4.5). ■

Lemma 3.2 Let C be a nonempty closed convex subset of a reflexive Banach space E, $f : E \to (-\infty, +\infty]$ be convex, strongly coercive and Legendre. If $g : \triangle \times \triangle \to \mathbb{R}$ satisfies (A1)-(A4) and y_n is defined as in (13), then we have

$$\langle y - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \leqslant \eta_n g(x_n, y) - \eta_n g(x_n, y_n).$$

for any $n \in \mathbb{N}$ and $y \in C$

Proof. Let $n \geqslant 1$ and $y \in C$. From (13), Lemma 2.5 and Lemma 2.6, we obtain

$$0 \in \eta_n \partial_2 g(x_n, y_n) + \nabla_1 D_f(y_n, x_n) + N_C(y_n).$$

Therefore there exists $w \in \partial_2 g(x_n, y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$0 = \eta_n w + \nabla f(y_n) - \nabla f(x_n) + \bar{w}. \tag{20}$$

Since $w \in \partial_2 g(x_n, y_n)$, we have

$$g(x_n, y) \geqslant g(x_n, y_n) + \langle y - y_n, w\rangle. \tag{21}$$

Using (20) and definition of $N_C(y_n)$, we get $\langle y - y_n, -\eta_n w - \nabla f(y_n) + \nabla f(x_n)\rangle \leqslant 0$ so that

$$\eta_n\langle y - y_n, w\rangle \geqslant \langle y - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle. \tag{22}$$

Combining (21) and (22), we obtain

$$\eta_n g(x_n, y) - \eta_n g(x_n, y_n) \geqslant \langle y - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle.$$

■

Now we prove the following strong convergence theorem.

Theorem 3.3 Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to (-\infty, +\infty]$ be Legendre, uniformly Fréchet differentiable, strongly coercive, totally convex and bounded on bounded subsets of E. For each $i = 1, 2, 3, \ldots, N$, let $T_i : C \to 2^C$ be closed Bregman quasi asymptotically nonexpansive multivalued mappings with sequences $\{k_{n,i}\}$ and $g : \triangle \times \triangle \to \mathbb{R}$ satisfies conditions (A1)-(A4) such that $\Omega = \cap_{i=1}^N F(T_i) \cap EP(g, C) \neq \emptyset$. If $\beta_{n,i} \in (\mu, 1 - \mu)$ for some $\mu \in (0, 1)$, then the sequence generated by linesearch algorithm converges strongly to $u^* = P^f_\Omega x_1$.

Proof. We start by showing $\Omega = \cap_{i=1}^N F(T_i) \cap EP(g, C)$ is closed and convex. By Lemma 2.4, $EP(g, C)$ is closed and convex and it follows from Lemma 2.7 that $\cap_{i=1}^N F(T_i)$ is closed and convex so that $\Omega = \cap_{i=1}^N F(T_i) \cap EP(g, C)$ is closed and convex.

Next we show C_n forall $n \geqslant 1$ is closed and convex.
Observe $C_1 = C$ is closed and convex. Assume C_n is closed and convex for some $n > 1$. Using the three point identity (9), it is easy to see $D_f(p, v_{n,}) \leqslant D_f(p, x_n) + \tau_n$ if and only if

$$\langle p, \nabla f(x_n) - \nabla f(v_n)\rangle \leqslant \langle v_n, \nabla f(x_n) - \nabla f(v_n)\rangle + D_f(v_n, x_n) + \tau_n.$$

Therefore we obtain C_{n+1} is closed and convex.

We now show that $\Omega \subset C_n$ for all $n \geqslant 1$. It is clear that $\Omega \subset C = C_1$. Suppose $\Omega \subset C_n$ for some $n > 1$. Let $h_n = \nabla f^*(\beta_{n,0}\nabla f(u_n) + \sum_{i=1}^N \beta_{n,i}\nabla f(s_{n,i})))$. then for any $p \in \Omega \subset C_n$ and using (10), we have

$$\begin{aligned} D_f(p, v_n) &= D_f\Big(p, \nabla f^*(\alpha_n\nabla f(x_n) + (1 - \alpha_n)\nabla f(h_n))\Big) \\ &= V_f(p, \alpha_n\nabla f(x_n) + (1 - \alpha_n)\nabla f(h_n)) \\ &= f(p) - \langle p, \alpha_n\nabla f(x_n) + (1 - \alpha_n)\nabla f(h_n)\rangle \\ &\quad + f^*(\alpha_n\nabla f(x_n) + (1 - \alpha_n)\nabla f(h_n)) \\ &\leqslant f(p) - \alpha_n\langle p, \nabla f(x_n)\rangle - (1 - \alpha_n)\langle p, \nabla f(h_n)\rangle \\ &\quad + \alpha_n f^*(\nabla f(x_n)) + (1 - \alpha_n)f^*(\nabla f(h_n)) \\ &= \alpha_n V_f(p, \nabla f(x_n)) + (1 - \alpha_n)V_f(p, \nabla f(h_n)) \\ &= \alpha_n D_f(p, x_n) + (1 - \alpha_n)D_f(p, h_n). \end{aligned} \tag{23}$$

and using (11),

$$D_f(p, h_n) = D_f(p, \nabla f^*(\beta_{n,0}\nabla f(u_n) + \sum_{i=1}^N \beta_{n,i}\nabla f(s_{n,i})))) \leqslant \beta_{n,0}D_f(p, u_n) + \sum_{i=1}^N \beta_{n,i}D_f(p, s_{n,i}).$$

Since $s_{n,i} \in T_i^n u_n$ and $T_i,\ i = 1,2,3,\dots,N$ are Bregman quasi asymptotically nonexpansive multivalued mappings, we get

$$\begin{aligned} D_f(p,h_n) &\leqslant \beta_{n,0} D_f(p,u_n) + \sum_{i=1}^{N} \beta_{n,i}(1+k_{n,i}) D_f(p,u_n) \\ &= \beta_{n,0} D_f(p,u_n) + \sum_{i=1}^{N} \beta_{n,i} D_f(p,u_n) + \sum_{i=1}^{N} \beta_{n,i} k_{n,i} D_f(p,u_n) \\ &= D_f(p,u_n) + \sum_{i=1}^{N} \beta_{n,i} k_{n,i} D_f(p,u_n). \end{aligned} \tag{24}$$

But from the definition of u_n, Lemma 2.12(ii) and (12), we have

$$\begin{aligned} D_f(p,u_n) &= D_f(p, P_C^f \nabla f^*(\nabla f(x_n) - \gamma_n w_n)) \\ &\leqslant D_f(p, \nabla f^*(\nabla f(x_n) - \gamma_n w_n)) \\ &= V_f(p, \nabla f(x_n) - \gamma_n w_n) \\ &\leqslant V_f(p, \nabla f(x_n) - \gamma_n w_n + \gamma_n w_n) - \langle \nabla f^*(\nabla f(x_n) - \gamma_n w_n) - p, \gamma_n w_n \rangle \\ &= D_f(p,x_n) + \langle \nabla f^*(\nabla f(x_n) - \gamma_n w_n) - p, -\gamma_n w_n \rangle \\ &= D_f(p,x_n) + \langle \nabla f^*(\nabla f(x_n) - \gamma_n w_n) - \nabla f^*(\nabla f(x_n)), -\gamma_n w_n \rangle - \langle x_n - p, \gamma_n w_n \rangle. \end{aligned}$$

By Lemma 2.11(3) and Remark *2*, we obtain

$$\begin{aligned} D_f(p,u_n) &\leqslant D_f(p,x_n) + \|\nabla f^*(\nabla f(x_n) - \gamma_n w_n) - \nabla f^*(\nabla f(x_n))\| \|\gamma_n w_n\| - \gamma_n \langle x_n - p, w_n \rangle \\ &\leqslant D_f(p,x_n) + \theta\Big(\|\nabla f(\nabla f^*(\nabla f(x_n) - \gamma_n w_n)) - \nabla f(\nabla f^*(\nabla f(x_n)))\| \Big) \|\gamma_n w_n\| \\ &\quad -\gamma_n \langle x_n - p, w_n \rangle \\ &= D_f(p,x_n) + c\gamma_n^2 \|w_n\|^2 - \gamma_n \langle x_n - p, w_n \rangle. \end{aligned} \tag{25}$$

Since $w_n \in \partial_2 g(z_n,x_n)$, we have $\langle x_n - p, w_n \rangle \geqslant g(z_n,x_n) - g(z_n,p)$. By pseudomonotonicity property of g and $p \in EP(g,C)$, we get $g(z_n,p) \leqslant 0$. Hence,

$$\gamma_n \langle x_n - p, w_n \rangle \geqslant \gamma_n (g(z_n,x_n) - g(z_n,p)) \geqslant \gamma_n g(z_n,x_n) = \frac{\gamma_n^2 \|w_n\|^2}{h}.$$

Therefore, from (25), we have

$$\begin{aligned} D_f(p,u_n) &\leqslant D_f(p,x_n) + \frac{\gamma_n^2 \|w_n\|^2}{d} - \frac{\gamma_n^2 \|w_n\|^2}{h} \\ &= D_f(p,x_n) - (\frac{1}{h} - \frac{1}{d}) \gamma_n^2 \|w_n\|^2 \\ &\leqslant D_f(p,x_n). \end{aligned} \tag{26}$$

Putting (26) in (24), we obtain

$$\begin{aligned}D_f(p,h_n) &\leqslant D_f(p,x_n) - (\frac{1}{h}-\frac{1}{d})\gamma_n^2\|w_n\|^2\\&\quad+\sum_{i=1}^{N}\beta_{n,i}k_{n,i}D_f(p,x_n) - \sum_{i=1}^{N}\beta_{n,i}k_{n,i}(\frac{1}{h}-\frac{1}{d})\gamma_n^2\|w_n\|^2\\&\leqslant D_f(p,x_n) - (\frac{1}{h}-\frac{1}{d})\gamma_n^2\|w_n\|^2 + M_n^N\sup_{p\in\Omega}D_f(p,x_n) \qquad (27)\\&\leqslant D_f(p,x_n) + M_n^N\sup_{p\in\Omega}D_f(p,x_n).\end{aligned}$$

Using (27) in (23), we have

$$\begin{aligned}&\leqslant \alpha_n D_f(p,x_n) + (1-\alpha_n)[D_f(p,x_n) + M_n^N\sup_{p\in\Omega}D_f(p,x_n)]\\&= D_f(p,x_n) + (1-\alpha_n)M_n^N\sup_{p\in\Omega}D_f(p,x_n)\\&= D_f(p,x_n) + \tau_n. \qquad (28)\end{aligned}$$

This implies $p \in C_{n+1}$ and $\Omega \subset C_{n+1}$.

Next we show the sequence $\{x_n\}$ is Cauchy.

Now, $x_{n+1} = P^f_{C_{n+1}}x_1 \in C_{n+1} \subset C_n \quad for\ all\ n \geqslant 1$. Therefore,

$$D_f(x_n,x_1) = D_f(P^f_{C_n}x_1,x_1) \leqslant D_f(x_{n+1},x_1). \tag{29}$$

This shows that $\{D_f(x_n,x_1)\}$ is decreasing sequence. Also, by Lemma 2.12(ii), we obtain

$$D_f(x_n,x_1) \leqslant D_f(p,x_1) - D_f(p,x_n) \leqslant D_f(p,x_1) \quad \text{for all } n \geqslant 1. \tag{30}$$

By (29) and (30), we conclude that $\lim_{n\to+\infty} D_f(x_n,x_1)$ exists. Since $\{D_f(x_n,x_1)\}$ is bounded and the function f is totally convex and uniformly Fréchet differentiable, it follows from Lemma 2.15 that the sequence $\{x_n\}$ is bounded. Observe that

$$D_f(x_{n+1},x_n) = D_f(x_{n+1},P^f_{C_n}x_1) \leqslant D_f(x_{n+1},x_1) - D_f(x_n,x_1). \tag{31}$$

Then it follows that $\lim_{n\to+\infty} D_f(x_{n+1},x_n) = 0$. Due to assumption that the function f is totally convex on bounded sets, then f is sequentially consistent (by Lemma 2.13) and therefore,

$$\lim_{n\to+\infty}\|x_{n+1}-x_n\| = 0. \tag{32}$$

Now, using (31) and for any $m,\ n \in \mathbb{N}$ with $m > n$, we obtain

$$D_f(x_m,x_n) = D_f(x_m,P^f_{C_n}x_1) \leqslant D_f(x_m,x_1) - D_f(x_n,x_1),$$

so that $\lim_{n,m\to+\infty} D_f(x_m, x_n) = 0$ and consequently $\lim_{m,n\to+\infty} \|x_m - x_n\| = 0$. This implies $\{x_n\}$ is Cauchy sequence in E. Since E is reflexive and C is closed, there exists some $u^* \in C$ such that $\lim_{n\to+\infty} \|x_n - u^*\| = 0$.

Next we prove that $u^* \in \cap_{i=1}^N F(T_i) \cap EP(g, C)$. Since $x_{n+1} \in C_{n+1}$, then from (15), we have $D_f(x_{n+1}, v_n) \leqslant D_f(x_{n+1}, x_n) + \tau_n$. From the definition of τ_n, we get $\tau_n \to 0$ as $n \to +\infty$. It follows that $D_f(x_{n+1}, v_n) \to 0$ as $n \to +\infty$ and essential consistency of f guarantees that

$$\|x_{n+1} - v_n\| \to 0 \text{ as n} \to +\infty. \tag{33}$$

From (32) and (33), we obtain

$$\|v_n - x_n\| \to 0 \text{ as n} \to +\infty. \tag{34}$$

By Lemma 2.18, ∇f is uniformly norm-to-norm continuous on bounded subsets of E. Thus, from (34), we get

$$\|\nabla f(v_n) - \nabla f(x_n)\| \to 0 \text{ as n} \to +\infty. \tag{35}$$

From definition of v_n and condition $0 < \alpha_n < a < 1$, we have

$$(1 - \alpha_n)(\nabla f(h_n) - \nabla f(x_n)) = \nabla f(v_n) - \nabla f(x_n),$$

so that

$$\|\nabla f(h_n) - \nabla f(x_n)\| = \frac{1}{1-\alpha_n}\|\nabla f(v_n) - \nabla f(x_n)\| \leqslant \frac{1}{1-a}\|\nabla f(v_n) - \nabla f(x_n)\|.$$

It follows from (35) that

$$\|\nabla f(h_n) - \nabla f(x_n)\| \to 0 \text{ as n} \to +\infty. \tag{36}$$

By Lemma 2.20, ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*, hence we have from (36) that

$$\|h_n - x_n\| \to 0 \text{ as n} \to +\infty. \tag{37}$$

Since $\{x_n\}$ is bounded and ∇f is bounded on bounded subsets of E, then there exists a real number say $l > 0$ such that $D_f(p, x_n) \leqslant l \;\; for\ all\ n \geqslant 1$. Therefore, from (26), we have that $D_f(p, u_n)$ is bounded and consequently $D_f(p, s_{n,i}),\ i = 1, 2, 3, \ldots, N$ are bounded. The function f is totally convex and strongly coercive which is bounded on bounded subsets of E. Now, by Lemma 2.10 and Lemma 2.20, we conclude that ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^* and therefore is bounded. Hence, by Lemma 2.16, $\{u_n\}$ and $\{s_{n,i}\},\ i = 1, 2, 3, \ldots, N$ are bounded.

Let $r = \max_{1\leqslant i\leqslant N}\sup_{n\geqslant 1}\{\|\nabla f(u_n)\|,\ \|\nabla f(s_{n,i})\|\}$. Then by (10), we have

$$\begin{aligned}
D_f(p,h_n) &= D_f(p,\nabla f^*(\beta_{n,0}\nabla f(u_n)+\sum_{i=1}^{N}\beta_{n,i}\nabla f(s_{n,i}))))\\
&= V_f(p,\beta_{n,0}\nabla f(u_n)+\sum_{i=1}^{N}\beta_{n,i}\nabla f(s_{n,i}))\\
&= f(p)-\langle p,\beta_{n,0}\nabla f(u_n)+\sum_{i=1}^{N}\beta_{n,i}\nabla f(s_{n,i})\rangle + f^*(\beta_{n,0}\nabla f(u_n)+\sum_{i=1}^{N}\beta_{n,i}\nabla f(s_{n,i})).
\end{aligned}$$

By our assumption, f is uniformly Fr$\acute{e}$chet differentiable and bounded on bounded subsets of E, therefore by Lemma 2.18 ∇f is uniformly continuous on bounded subsets which implies by Lemma 2.19 f^* is uniformly convex. Hence, from Lemma 2.17, we have

$$\begin{aligned}
D_f(p,h_n) &\leqslant f(p)-\beta_{n,0}\langle p,\nabla f(u_n)\rangle-\sum_{i=1}^{N}\beta_{n,i}\langle p,\nabla f(s_{n,i})\rangle+\beta_{n,0}f^*(\nabla f(u_n))\\
&\quad+\sum_{i=1}^{N}\beta_{n,i}f^*(\nabla f(s_{n,i}))-\beta_{n,0}\beta_{n,i}\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|)\\
&= \beta_{n,0}V_f(p,\nabla f(u_n))+\sum_{i=1}^{N}\beta_{n,i}V_f(p,\nabla f(s_{n,i}))-\beta_{n,0}\beta_{n,i}\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|)\\
&= \beta_{n,0}D_f(p,u_n)+\sum_{i=1}^{N}\beta_{n,i}D_f(p,s_{n,i})-\beta_{n,0}\beta_{n,i}\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|).
\end{aligned}$$

Since $T_i,\ i=1,2,3,\dots,N$ are Bregman quasi multivalued mappings and $s_{n,i}\in T_i^n u_n$, we obtain

$$\begin{aligned}
D_f(p,h_n) &\leqslant \beta_{n,0}D_f(p,u_n)+\sum_{i=1}^{N}\beta_{n,i}(1+k_{n,i})D_f(p,u_n)-\beta_{n,0}\beta_{n,i}\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|)\\
&= D_f(p,u_n)+\sum_{i=1}^{N}\beta_{n,i}k_{n,i}D_f(p,u_n)-\beta_{n,0}\beta_{n,i}\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|). \qquad (38)
\end{aligned}$$

Using (26), (27) in (38), $\beta_{n,0}$ and $\beta_{n,i}\in(\mu,1-\mu)$ for $i=1,\dots,N$ and $\mu\in(0,1)$, we get

$$\begin{aligned}
&D_f(p,h_n)\\
&\leqslant D_f(p,x_n)-(\frac{1}{h}-\frac{1}{d})\gamma_n^2\|w_n\|^2+M_n^N\sup_{p\in\Omega}D_f(p,x_n)-\beta_{n,0}\beta_{n,i}\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|)\\
&= D_f(p,x_n)-(\frac{1}{h}-\frac{1}{d})\gamma_n^2\|w_n\|^2+\tau_n-\beta_{n,0}\beta_{n,i}\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|)\\
&\leqslant D_f(p,x_n)-(\frac{1}{h}-\frac{1}{d})\gamma_n^2\|w_n\|^2+\tau_n-\mu^2\rho_r^*(\|\nabla f(u_n)-\nabla f(s_{n,i})\|). \qquad (39)
\end{aligned}$$

Now,

$$\begin{aligned} D_f(p,x_n) - D_f(p,h_n) &= f(h_n) - f(x_n) + \langle p - h_n, \nabla f(h_n)\rangle - \langle p - x_n, \nabla f(x_n)\rangle \\ &= f(h_n) - f(x_n) + \langle p - x_n, \nabla f(h_n)\rangle + \langle x_n - h_n, \nabla f(h_n)\rangle - \langle p - x_n, \nabla f(x_n)\rangle \\ &= f(h_n) - f(x_n) + \langle p - x_n, \nabla f(h_n) - \nabla f(x_n)\rangle + \langle x_n - h_n, \nabla f(h_n)\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} D_f(p,x_n) - D_f(p,h_n) \leqslant |f(h_n) - f(x_n)| + \|p - x_n\| \|\nabla f(h_n) - \nabla f(x_n)\| \\ + \|x_n - h_n\| \|\nabla f(h_n)\|. \end{aligned}$$

Hence it follows from (36) and (37) that

$$D_f(p,x_n) - D_f(p,h_n) \to 0 \text{ as n} \to +\infty. \tag{40}$$

Using (40), it follows from (39) that

$$\lim_{n\to+\infty} \gamma_n \|w_n\| = 0 \; and \; \lim_{n\to+\infty} \rho_r^*(\|\nabla f(u_n) - \nabla f(s_{n,i})\| = 0 \;\; for \; all \; i \in \{1,2,3,\ldots,N\}.$$

By the property of ρ_r^* we deduce $\lim_{n\to+\infty}(\|\nabla f(u_n) - \nabla f(s_{n,i})\| = 0$ for all $i \in \{1,2,3,\ldots,N\}$. As ∇f^* is norm-to-norm uniformly continuous on bounded subsets of E^*, we get

$$\lim_{n\to+\infty} \|u_n - s_{n,i}\| = 0 \;\; for \; all \; i \in \{1,2,3,\ldots,N\}. \tag{41}$$

Thus, we obtain from Lemma 2.14

$$\lim_{n\to+\infty} D_f(u_n, s_{n,i}) = 0 \;\; for \; all \; i \in \{1,2,3,\ldots,N\}. \tag{42}$$

Now,

$$\begin{aligned} D_f(u_n,h_n) &= D_f(u_n, \nabla f^*(\beta_{n,0}\nabla f(u_n) + \sum_{i=1}^{N} \beta_{n,i}\nabla f(s_{n,i})))) \\ &\leqslant \beta_{n,0} D_f(u_n,u_n) + \sum_{i=1}^{N} \beta_{n,i} D_f(u_n, s_{n,i}) \\ &= \sum_{i=1}^{N} \beta_{n,i} D_f(u_n, s_{n,i}). \end{aligned}$$

It follows from (42) that $D_f(u_n,h_n) \to 0$ as $n \to +\infty$. The fact that f is totally convex on bounded subsets of E, then by Lemma 2.13, f is sequentially consistent and so

$$\|u_n - h_n\| \to 0 \text{ as n} \to +\infty. \tag{43}$$

Since $\|x_n - u_n\| \leqslant \|x_n - h_n\| + \|u_n - h_n\|$, then from (37) and (43), we get

$$\|u_n - x_n\| \to 0 \text{ as n} \to +\infty. \tag{44}$$

Therefore, from (32) and (44), we get

$$\|u_n - u_{n+1}\| \to 0 \text{ as n} \to +\infty. \tag{45}$$

Also, from $\lim_{n\to+\infty} \|x_n - u^*\| = 0$ and (44), we obtain

$$\lim_{n\to+\infty} \|u_n - u^*\| = 0. \tag{46}$$

Similarly, from (41) and (46), we have

$$\lim_{n\to+\infty} \|s_{n,i} - u^*\| = 0 \quad for\ all\ i \in \{1, 2, 3, \ldots, N\}. \tag{47}$$

For each $i = 1, 2, 3, \ldots, N$, let $\{d_{n,i}\}$ be a sequence generated as follows:

$$d_{2,i} \in T_i(s_{1,i}) \subset T_i^2(u_1),\ d_{3,i} \in T_i(s_{2,i}) \subset T_i^3(u_2),\ \ \ldots,$$
$$d_{n,i} \in T_i(s_{n-1,i}) \subset T_i^n(u_{n-1}), d_{n+1,i} \in T_i(s_{n,i}) \subset T_i^{n+1}(u_n), \ldots.$$

By our assumption, T_i is L_i-Lipschitz continuous for each $i = 1, 2, 3, \ldots, N$, then

$$\begin{aligned}\|d_{n+1,i} - s_{n,i}\| &\leqslant \|d_{n+1,i} - s_{n+1,i}\| + \|s_{n+1,i} - u_{n+1}\| + \|u_{n+1} - u_n\| + \|u_n - s_{n,i}\| \\ &\leqslant (1 + L_i)\|u_n - u_{n+1}\| + \|s_{n+1,i} - u_{n+1}\| + \|u_n - s_{n,i}\|.\end{aligned}$$

Using (41) and (45), we obtain

$$\lim_{n\to+\infty} \|d_{n+1,i} - s_{n,i}\| = 0 \quad for\ all\ i \in \{1, 2, 3, \ldots, N\},$$

and consequently from (47), we get

$$\lim_{n\to+\infty} \|d_{n+1,i} - u^*\| = 0 \quad for\ all\ i \in \{1, 2, 3, \ldots, N\}. \tag{48}$$

Since $d_{n+1,i} \in T_i(s_{n,i})$, $d_{n+1,i} \to u^*$ as $n \to +\infty$ and $s_{n,i} \to u^*$ as $n \to +\infty$ for each $i \in \{1, 2, 3, \ldots, N\}$, then by closedness of T_i we have that $u^* \in T_i(u^*)$ for each $i \in \{1, 2, 3, \ldots, N\}$, i.e. $u^* \in \cap_{i=1}^N F(T_i)$.

We now show $u^* \in EP(g, C)$. Let $H(y) = \eta_n g(x_n, y) + D_f(y, x_n)$. Since $D_f(.,.)$ is continuous, it follows from Lemma 2.6 that

$$\partial H(y) = \partial(\eta_n g(x_n, y) + D_f(y, x_n)) = \eta_n \partial_2 g(x_n, y) + \nabla_1 D_f(y, x_n).$$

Let $p_1^*, p_2^* \in C,\ q_1^* \in \partial_2 g(x_n, p_1^*)$ and $q_2^* \in \partial_2 g(x_n, p_2^*)$, then

$$\langle p_1^* - y, q_1^* \rangle \geqslant g(x_n, p_1^*) - g(x_n, y)\ for\ all\ y \in C, \tag{49}$$

and

$$\langle p_2^* - y, q_2^*\rangle \geqslant g(x_n, p_2^*) - g(x_n, y) \ for\ all\ y \in C. \tag{50}$$

Setting $y = p_2^*$ in (49), $y = p_1^*$ in (50) and combining the two inequalities, we obtain

$$\langle p_1^* - p_2^*, q_1^* - q_2^*\rangle \geqslant 0. \tag{51}$$

Observe $\nabla_1 D_f(p_1^*, x_n) = \nabla f(p_1^*) - \nabla f(x_n)$ and $\nabla_1 D_f(p_2^*, x_n) = \nabla f(p_2^*) - \nabla f(x_n)$. Since f is uniformly Fréchet differentiable and totally convex, then in view of Lemma 2.11(2) and Remark *2*, we get

$$\langle p_1^* - p_2^*, \nabla f(p_1^*) - \nabla f(p_2^*)\rangle \geqslant \varphi(\|p_1^* - p_2^*\|) = c\|p_1^* - p_2^*\|^2. \tag{52}$$

From (51) and (52), we have

$$\langle p_1^* - p_2^*, \eta_n q_1^* - \eta_n q_2^* + \nabla f(p_1^*) - \nabla f(p_2^*)\rangle \geqslant c\|p_1^* - p_2^*\|^2,$$

which implies

$$\langle p_1^* - p_2^*, \eta_n q_1^* + \nabla f(p_1^*) - \nabla f(x_n) - (\eta_n q_2^* + \nabla f(p_2^*) - \nabla f(x_n))\rangle \geqslant c\|p_1^* - p_2^*\|^2. \tag{53}$$

Now, for $p^* \in C$, define $\mathcal{M}_n(p^*) = \eta_n q^* + \nabla f(p^*) - \nabla f(x_n)$ where $q^* \in \partial_2 g(x_n, p^*)$. Then $\mathcal{M}_n(p^*) \subseteq \partial H(p^*)$ for any $p^* \in C$. Therefore, it follows from (53) that

$$\langle p_1^* - p_2^*, t^n(p_1^*) - t^n(p_2^*)\rangle \geqslant c\|p_1^* - p_2^*\|^2 \ for\ all\ t^n(p_1^*) \in \mathcal{M}_n(p_1^*),\ t^n(p_2^*) \in \mathcal{M}_n(p_2^*). \tag{54}$$

Hence $\mathcal{M}_n$ is multivalued strongly monotone. Since

$$y_n = \underset{y\in C}{argmin}\left\{\eta_n g(x_n, y) + D_f(y, x_n)\right\},$$

then by Lemma 2.5 we have $0 \in \partial H(y_n) + N_C(y_n)$, i.e. $-\mathcal{M}_n(y_n) \subseteq N_C(y_n)$. Hence,

$$\langle y - y_n, t^n(y_n)\rangle \geqslant 0 \ for\ all\ y \in C,\ t^n(y_n) \in \mathcal{M}_n(y_n). \tag{55}$$

Let $p_1^*,\ p_2^*$ be replaced by $x_n,\ y_n$ in (54) respectively and $y = x_n$ in (55), then

$$\langle x_n - y_n, t^n(x_n)\rangle \geqslant \langle x_n - y_n, t^n(y_n)\rangle + c\|x_n - y_n\|^2 \geqslant c\|x_n - y_n\|^2 \tag{56}$$

for any $t^n(x_n) \in \mathcal{M}_n(x_n) \subseteq \partial H(x_n)$, $t^n(y_n) \in \mathcal{M}_n(y_n) \subseteq \partial H(y_n)$. The fact that $\nabla_1 D_f(x_n, x_n) = 0$, then $t^n(x_n) \in \eta_n \partial_2 g(x_n, x_n)$. Since $\lim_{n\to+\infty} \|x_n - u^*\| = 0$, we obtain from Lemma 2.8, $\partial_2 g(x_n, x_n) \subseteq \partial_2 g(u^*, u^*)$. As $0 < \eta \leqslant \eta_n \leqslant 1$ and using Lemma 2.9, it follows that $\{t^n(x_n)\}$ is bounded. Thus, from (56) and boundedness of $\{x_n\}$, we conclude that $\{y_n\}$ is bounded. Consequently boundedness of z_n follows from its definition and boundedness of $\{x_n\}$ and $\{y_n\}$. Then there exists a point say $\bar{z} \in C$ such that $z_n \rightharpoonup \bar{z}$. Therefore since $w_n \in \partial_2 g(z_n, x_n)$, then by Lemma 2.8 and 2.9 we have that $\{w_n\}$ is bounded. From the definition of γ_n, we get $g(z_n, x_n) = \frac{\gamma_n\|w_n\|\|w_n\|}{h}$. Since $\lim\limits_{n\to+\infty} \gamma_n\|w_n\| = 0$ and $0 < h < d < 1$, then

$$\lim_{n\to+\infty} g(z_n, x_n) = 0. \tag{57}$$

On the other hand, by condition (A1) and convexity of $g(z_n, .)$, we have

$$0 = g(z_n, z_n) = g(z_n, (1-\sigma_n)x_n + \sigma_n y_n) \leqslant (1-\sigma_n)g(z_n, x_n) + \sigma_n g(z_n, y_n).$$

Therefore, from the algorithm and (57), we obtain

$$\frac{\alpha\sigma_n}{\eta_n} D_f(y_n, x_n) \leqslant \sigma_n[g(z_n, x_n) - g(z_n, y_n)] \leqslant g(z_n, x_n) \to 0,\ as\ n \to +\infty. \tag{58}$$

We consider the following two cases:
Case I. If $\limsup\limits_{n\to+\infty} \sigma_n > 0$, then there exists σ' and a subsequence denoted by σ_n of σ_n such that $\sigma_n \to \sigma'$. From (58) and condition $\eta_n \in (\eta, 1],\ 0 < \eta < 1$, we obtain

$$D_f(y_n, x_n) \to 0,\ as\ n \to +\infty,$$

and using Lemma 2.14 we obtain

$$\lim_{n\to+\infty} \|y_n - x_n\| = 0. \tag{59}$$

Case II: If $\lim\limits_{n\to+\infty} \sigma_n = 0$, then let m be the smallest positive integer such that

$$g(z_{n,m}, x_n) - g(z_{n,m}, y_n) \geqslant \frac{\alpha}{\eta_n} D_f(y_n, x_n)\ and\ z_{n,m} = (1-\sigma^m)x_n + \sigma^m y_n.$$

Therefore for $m - 1 < m$, we have

$$g(z_{n,m-1}, x_n) - g(z_{n,m-1}, y_n) < \frac{\alpha}{\eta_n} D_f(y_n, x_n). \tag{60}$$

Now from Lemma 3.2 and letting $y = x_n$, we have

$$-\eta_n g(x_n, y_n) \geqslant \langle x_n - y_n, \nabla f(x_n) - \nabla f(y_n) = D_f(x_n, y_n) + D_f(y_n, x_n) \geqslant D_f(y_n, x_n),$$

i.e.,

$$D_f(y_n, x_n) \leqslant -\eta_n g(x_n, y_n). \tag{61}$$

From (60) and (61), we get

$$g(z_{n,m-1}, x_n) - g(z_{n,m-1}, y_n) < -\alpha g(x_n, y_n). \tag{62}$$

We know that there exists a subsequence of $\{y_n\}$ denote it by $\{y_n\}$ such that $y_n \rightharpoonup y' \in C$ as $n \to +\infty$. Also since $x_n \to u^*$ as $n \to +\infty$, then from the definition of $z_{n,m-1}$, we obtain $z_{n,m-1} \to u^*$ as $n \to +\infty$ (because $\sigma^m = \sigma_n$). Therefore by conditions (A1) and

(A4), it follows from (62) that $(1-\alpha)g(u^*, y') \geqslant 0$. Since $\alpha \in (0,1)$, then $g(u^*, y') \geqslant 0$ and consequently from (61), we have $D_f(y_n, x_n) \to 0$ as $n \to +\infty$. Again by Lemma 2.14, we get $\lim_{n\to+\infty} \|y_n - x_n\| = 0$. Since $x_n \to u^*$ as $n \to +\infty$, we have $y_n \to u^*$ as $n \to +\infty$. Also from Lemma 3.2, we have

$$\eta_n g(x_n, y) - \eta_n g(x_n, y_n) \geqslant \langle y - y_n, \nabla f(x_n) - \nabla f(y_n)\rangle \quad for\ all\ y \in C.$$

Allowing $n \to +\infty$ by considering conditions (A1) and (A4), we obtain $g(u^*, y) \geqslant 0,\ for\ all\ y \in C$, which shows that $u^* \in EP(g, C)$. Thus, $u^* \in \cap_{i=1}^N F(T_i) \cap EP(g, C) = \Omega$.

Next we show $u^* = P_\Omega^f x_1$. Since $x_n = P_{C_n}^f x_1$, then by Lemma 2.12(i) we obtain

$$\langle x_n - y, \nabla f(x_1) - \nabla f(x_n)\rangle \geqslant 0 \quad for\ all\ y \in C_n.$$

Since $\Omega \subset C_n$, we have $\langle x_n - h, \nabla f(x_1) - \nabla f(x_n)\rangle \geqslant 0$ for all $h \in \Omega$. Allowing $n \to +\infty$ in the equation above, we get $\langle u^* - h, \nabla f(x_1) - \nabla f(u^*)\rangle \geqslant 0$ for all $h \in \Omega$. and also, by Lemma 2.12(i), we have $u^* = P_\Omega^f x_1$. This completes the proof. ■

4. Application

Observe that in equation (1) if we set $g(z^*, y) = \langle y - z^*, \mathcal{A}z^*\rangle$, where $\mathcal{A} : C \to E^*$, then the equilibrium problem reduces to classical variational inequality problem which is problem of finding $z^* \in C$ such that

$$\langle y - z^*, \mathcal{A}z^*\rangle \geqslant 0 \quad for\ all\ y \in C. \tag{63}$$

The set of solutions of problem (63) is denoted by $VI(\mathcal{A}, C)$. Therefore, in view of Lemma 2.21, the strongly convex problem (13) becomes $y_n = P_C^f \nabla f^*(\nabla f(x_n) - \eta_n \mathcal{A}(x_n))$. Also equation (14) becomes $\langle x_n - z_{n,m}, \mathcal{A}z_{n,m}\rangle \geqslant \frac{\alpha}{\eta_n} D_f(y_n, x_n)$. With this, we have the following linesearch algorithm for variational inequality problem.

Algorithm 2

Step 0: Let $\alpha,\ h,\ \sigma \in (0,1)$, $c = \frac{1}{d},\ d \in (h,1)$, $0 < \alpha_n < a < 1,\ \eta_n \in (\eta, 1],\ 0 < \eta < 1$ and $\beta_{n,0} + \sum_{i=1}^N \beta_{n,i} = 1$.
Step 1: Let $x_1 \in C_1 = C$.
Step 2: Set

$$y_n = P_C^f \nabla f^*(\nabla f(x_n) - \eta_n \mathcal{A}(x_n)). \tag{64}$$

Step 3: If $y_n = x_n$, then set $z_n = x_n$. Otherwise find the smallest nonnegative integer m such that

$$\begin{cases} \langle x_n - z_{n,m}, \mathcal{A}(z_{n,m})\rangle \geqslant \frac{\alpha}{\eta_n} D_f(y_n, x_n), \\ z_{n,m} = (1-\sigma^m)x_n + \sigma^m y_n. \end{cases} \tag{65}$$

Set $\sigma_n = \sigma^m$ and $z_n = z_{n,m}$ and go to step 4.
Step 4: Compute

$$u_n = P_C^f \nabla f^*(\nabla f(x_n) - \gamma_n \mathcal{A}(z_n)), \quad \gamma_n = \begin{cases} \frac{h\langle x_n - z_n, \mathcal{A}(z_n)\rangle}{||\mathcal{A}(z_n)||^2}, & y_n \neq x_n \\ 0, & Otherwise \end{cases}.$$

Step 5: Compute

$$\begin{cases} v_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)(\beta_{n,0}\nabla f(u_n) + \sum_{i=1}^N \beta_{n,i}\nabla f(s_{n,i}))), \\ C_{n+1} = \{p \in C_n : D_f(p, v_n) \leqslant D_f(p, x_n) + \tau_n\}, \\ x_{n+1} = P_{C_{n+1}}^f x_1, \ n \geqslant 1, \end{cases} \tag{66}$$

where $s_{n,i} \in T_i^n u_n$ and $\tau_n = (1-\alpha_n)M_n^N \sup_{q\in\Omega} D_f(q, x_n)$ and $M_n^N = \sum_{i=1}^N \beta_{n,i}k_{n,i}$.
Step 6: Set $n = n+1$ and go to step 2.

We observe that in order to apply Theorem 3.3 to variational inequality problem, conditions (A1)-(A4) must be satisfied. Clearly (A1) and (A2) are satisfied. condition (A3) now becomes $\mathcal{A} : C \to E^*$ is pseudomonotone on $VI(\mathcal{A}, C)$, that is; $\langle y - z^*, \mathcal{A}(y)\rangle \geqslant 0, \ \forall\, z^* \in VI(\mathcal{A}, C)$. Moreover if $\mathcal{A} : \triangle \to E^*$ is such that for every sequence $\{z_n\}$ in $\triangle$, $z_n \rightharpoonup z$ implies $\mathcal{A}z_n \to \mathcal{A}z$, then the corresponding function g is jointly continuous on $\triangle \times \triangle$ (i.e. condition A(4)).

Theorem 4.1 Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to (-\infty, +\infty]$ be Legendre, uniformly Fréchet differentiable, strongly coercive, totally convex and bounded on bounded subsets of E. For each $i = 1, 2, 3, \ldots, N$, let $T_i : C \to 2^C$ be closed Bregman quasi asymptotically nonexpansive multivalued mappings with sequences $\{k_{n,i}\}$ and $\mathcal{A} : \triangle \to E^*$ satisfies conditions (A1)-(A4) such that $\Omega = \cap_{i=1}^N F(T_i) \cap VI(\mathcal{A}, C) \neq \emptyset$. If $\beta_{n,i} \in (\mu, 1-\mu)$ for some $\mu \in (0,1)$, then the sequence generated by Algorithm 2 converges strongly to $u^* = P_\Omega^f x_1$.

If $T_i, \ i = 1, 2, 3, \ldots, N$ are Bregman quasi nonexpansive multivalued mappings, then Algorithm 1 reduces to the following algorithm.

Algorithm 3

Step 0: Let $\alpha, \ h, \ \sigma \in (0,1)$, $c = \frac{1}{d}, \ d \in (h, 1)$, $0 < \alpha_n < a < 1, \ \eta_n \in (\eta, 1], \ 0 < \eta < 1$ and $\beta_{n,0} + \sum_{i=1}^N \beta_{n,i} = 1$.
Step 1: Let $x_1 \in C_1 = C$.
Step 2: Set

$$y_n = \underset{y \in C}{argmin} \left\{ \eta_n g(x_n, y) + D_f(y, x_n) \right\}. \tag{67}$$

Step 3: If $y_n = x_n$, then set $z_n = x_n$. Otherwise find the smallest nonnegative integer m such that

$$\begin{cases} g(z_{n,m}, x_n) - g(z_{n,m}, y_n) \geqslant \frac{\alpha}{\eta_n} D_f(y_n, x_n), \\ z_{n,m} = (1-\sigma^m)x_n + \sigma^m y_n. \end{cases} \tag{68}$$

Set $\sigma_n = \sigma^m$ and $z_n = z_{n,m}$ and go to step 4.
Step 4: Select $w_n \in \partial_2 g(z_n, x_n)$ and compute

$$u_n = P_C^f \nabla f^*(\nabla f(x_n) - \gamma_n w_n), \quad \gamma_n = \begin{cases} \frac{hg(z_n,x_n)}{||w_n||^2}, & y_n \neq x_n \\ 0, & Otherwise \end{cases}.$$

Step 5: Compute

$$\begin{cases} v_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)(\beta_{n,0}\nabla f(u_n) + \sum_{i=1}^N \beta_{n,i}\nabla f(s_{n,i}))), \\ C_{n+1} = \{p \in C_n : D_f(p, v_n) \leqslant D_f(p, x_n)\}, \\ x_{n+1} = P_{C_{n+1}}^f x_1, \ n \geqslant 1, \end{cases} \tag{69}$$

where $s_{n,i} \in T_i(u_n)$.
Step 6: Set $n = n + 1$ and go to step 2.
Using Algorithm 3, Theorem 3.3 reduces to the following corollary.

Corollary 4.2 Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to (-\infty, +\infty]$ be Legendre, uniformly Fréchet differentiable, strongly coercive, totally convex and bounded on bounded subsets of E. For each $i = 1, 2, 3, \dots, N$, let $T_i : C \to 2^C$ be closed Bregman quasi nonexpansive multivalued mappings and $g : \triangle \times \triangle \to \mathbb{R}$ satisfies conditions (A1)-(A4) such that $\Omega = \cap_{i=1}^N F(T_i) \cap EP(g, C) \neq \emptyset$. If $\beta_{n,i} \in (\mu, 1-\mu)$ for some $\mu \in (0, 1)$, then the sequence generated by Algorithm 3 converges strongly to $u^* = P_\Omega^f x_1$.

As a direct consequence of Remark *1* and Theorem 3.3, we have the following algorithm and corollary.

Algorithm 4

Step 0: Let $\alpha, \ h, \ \sigma \in (0, 1)$, $c = \frac{1}{d}, \ d \in (h, 1)$, $0 < \alpha_n < a < 1, \ \eta_n \in (\eta, 1], \ 0 < \eta < 1$ and $\beta_{n,0} + \sum_{i=1}^N \beta_{n,i} = 1$.
Step 1: Let $x_1 \in C_1 = C$.
Step 2: Set

$$y_n = \underset{y \in C}{argmin} \left\{ \eta_n g(x_n, y) + \frac{1}{2}\phi(y, x_n) \right\}. \tag{70}$$

Step 3: If $y_n = x_n$, then set $z_n = x_n$. Otherwise find the smallest nonnegative integer m such that

$$\begin{cases} g(z_{n,m}, x_n) - g(z_{n,m}, y_n) \geqslant \frac{\alpha}{2\eta_n}\phi(y_n, x_n), \\ z_{n,m} = (1-\sigma^m)x_n + \sigma^m y_n. \end{cases} \tag{71}$$

Set $\sigma_n = \sigma^m$ and $z_n = z_{n,m}$ and go to step 4.
Step 4: Select $w_n \in \partial_2 g(z_n, x_n)$ and compute

$$u_n = \Pi_C J^{-1}(J(x_n) - \gamma_n w_n), \quad \gamma_n = \begin{cases} \frac{hg(z_n,x_n)}{||w_n||^2}, & y_n \neq x_n \\ 0, & Otherwise \end{cases}.$$

Step 5: Compute

$$\begin{cases} v_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)(\beta_{n,0}J(u_n) + \sum_{i=1}^N \beta_{n,i}J(s_{n,i}))), \\ C_{n+1} = \{p \in C_n : \phi(p, v_n) \leqslant \phi(p, x_n) + \tau_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ n \geqslant 1, \end{cases} \tag{72}$$

where $s_{n,i} \in T_i^n u_n$ and $\tau_n = (1-\alpha_n)M_n^N \sup_{q\in\Omega}\phi(p, x_n)$ and $M_n^N = \sum_{i=1}^N \beta_{n,i}k_{n,i}$.

Step 6: Set $n = n + 1$ and go to step 2.

Corollary 4.3 Let C be a nonempty closed convex subset of a $2-$uniformly convex and uniformly smooth Banach space E with the dual E^* and $J : E \to E^*$ be a normalized duality mapping. For each $i = 1, 2, 3, \ldots, N$, let $T_i : C \to 2^C$ be closed quasi-ϕ- asymptotically nonexpansive multivalued mappings with sequences $\{k_{n,i}\}$ and $g : \triangle \times \triangle \to \mathbb{R}$ satisfies conditions (A1)-(A4) such that $\Omega = \cap_{i=1}^N F(T_i) \cap EP(g, C) \neq \emptyset$. If $\beta_{n,i} \in (\mu, 1-\mu)$ for some $\mu \in (0, 1)$, then the sequence generated by algorithm 4 converges strongly to $u^* = P_\Omega^f x_1$.

Remark 3 *The results presented in this paper generalize among others the results of Eskandani et al. [18] and Joumande and Moradlou [21] in the following sense:*

(1) Apart from the fact that Theorem 3.3 considered a more general class of maps (Bregman quasi asymptotically nonexpansive multivalued) than those studied in Eskandani et al. [18] (Bregman relatively nonexpansive multivalued maps), the Bregman Lipschitz conditions there were dispensed with.

(2) Corollary 4.3 generalizes Theorem 4.1 of Joumande and Moradlou [21] from single valued relatively nonexpansive mapping to finite family of quasi-ϕ-asymptotically nonexpansive multivalued mappings.

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